

- (1) Let  $F$  be a flat of a hyperplane arrangement  $\mathcal{H}$ . Prove that

$$\sum_f (-1)^{\dim f} = (-1)^{\dim F},$$

where we sum over all the faces  $f$  of the hyperplane arrangement induced by  $\mathcal{H}$  on  $F$ . (*Hint:* one way to prove this identity is by induction on both the dimension of  $F$  and the number of hyperplanes in the arrangement induced by  $\mathcal{H}$  on  $F$ .)

- (2) Let  $P$  be a finite poset with the relation  $\leq$  (i.e., a partially ordered set with unique meets and joins). We use the notations  $\geq$ ,  $<$ , and  $>$  in the usual sense. Recall the definition of the Möbius function:

$$\mu(x, y) = \begin{cases} 0 & \text{if } x > y, \\ 1 & \text{if } x = y, \\ -\sum_{x \leq z < y} \mu(x, z) & \text{if } x < y. \end{cases}$$

Prove the Möbius inversion theorem, namely that for two functions  $\phi$  and  $\psi$  defined on  $P$ ,

$$\phi(x) = \sum_{y \geq x} \psi(y) \iff \psi(x) = \sum_{y \geq x} \mu(x, y) \phi(y).$$

- (3) A *signed graph* is a graph equipped with a function  $E \rightarrow \{\pm\}$  that assigns a sign to each edge. A *proper  $k$ -coloring* of a signed graph  $\Sigma$  is a function  $x : V \rightarrow \{-k, -k+1, \dots, k-1, k\}$  such that  $x(i) \neq x(j)$  if  $ij$  is a positive edge and  $x(i) \neq -x(j)$  if  $ij$  is a negative edge of  $\Sigma$ . A proper coloring is *zero-free* if  $x(j) \neq 0$  for all  $j$ . Let  $\chi_\Sigma(2k+1)$  be the number of proper  $k$ -colorings and  $\chi_\Sigma^*(2k)$  be the number of proper zero-free  $k$ -colorings of  $\Sigma$ . Prove:

- (a)  $\chi_\Sigma(2k+1)$  and  $\chi_\Sigma^*(2k)$  are polynomials in  $k$ .  
 (b)  $\chi_\Sigma(2k+1)$  and  $\chi_\Sigma^*(2k)$  are the two constituents of the Ehrhart quasipolynomial of an inside-out polytope.

(*Hint:* To prove this Ehrhart style, consider the cube  $[-1, 1]^V$  together with the hyperplane arrangement that comes naturally with a signed graph. For part (b), contract this cube to  $[0, 1]^V$  and shift the hyperplanes accordingly.) This setup can be used to prove a signed analogue of every theorem we discussed in Tuesday's lecture. Prove as many as you feel like...

- (4) Compute  $A_{\mathcal{P}}(t)$ , where  $\mathcal{P}$  is the regular tetrahedron with vertices  $(0, 0, 0)$ ,  $(1, 1, 0)$ ,  $(1, 0, 1)$ , and  $(0, 1, 1)$ .  
 (5) Compute  $A_{\mathcal{P}}(t)$ , where  $\mathcal{P}$  is the rational triangle with vertices  $(0, 0)$ ,  $(\frac{1}{2}, \frac{1}{2})$ , and  $(1, 0)$ .  
 (6) Prove that for a rational pointed  $d$ -cone  $\mathcal{K}$ ,  $\alpha_{\mathcal{K}}(\mathbf{z})$  is a rational function that satisfies

$$\alpha_{\mathcal{K}}\left(\frac{1}{\mathbf{z}}\right) = (-1)^d \alpha_{\mathcal{K}}(\mathbf{z}).$$

(7) Suppose  $\Delta$  is a rational simplex. Prove that

$$\sum_{\substack{\mathcal{F} \subseteq \Delta \\ \dim \mathcal{F} > 0}} \omega_{\Delta}(\mathcal{F}) \sum_{\mathbf{v} \text{ a vertex of } \mathcal{F}} \sigma_{\mathcal{K}_{\mathbf{v}}(\mathcal{F})^{\circ}}(\mathbf{z}) = \sum_{\mathbf{v} \text{ a vertex of } \Delta} \sum_{\substack{\mathcal{F} \subseteq \mathcal{K}_{\mathbf{v}} \\ \dim \mathcal{F} > 0}} \omega_{\mathcal{K}_{\mathbf{v}}}(\mathcal{F}) \sigma_{\mathcal{F}^{\circ}}(\mathbf{z}).$$

(8) Prove that if  $\mathcal{P}$  is a rational convex polytope, then the quasipolynomial  $A_{\mathcal{P}}$  satisfies

$$A_{\mathcal{P}}(-t) = (-1)^{\dim \mathcal{P}} A_{\mathcal{P}}(t).$$

(9) Recall from the first exercise on the second problem set that to any permutation  $\pi \in S_d$  on  $d$  elements we can associate the simplex

$$\Delta_{\pi} := \text{conv} \{ \mathbf{0}, \mathbf{e}_{\pi(1)}, \mathbf{e}_{\pi(1)} + \mathbf{e}_{\pi(2)}, \dots, \mathbf{e}_{\pi(1)} + \mathbf{e}_{\pi(2)} + \dots + \mathbf{e}_{\pi(d)} \}.$$

Prove that for all  $\pi \in S_n$ ,  $A_{\Delta_{\pi}}(t) = \frac{1}{d!} t^d$ .