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# A Note on Interpolation, Best Approximation, and the Saturation Property

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**Abstract** In this note, we prove that the well known saturation assumption implies that piecewise polynomial interpolation and best approximation in finite element spaces behave in similar fashion. That is, the error in one can be used to estimate the error in the other. We further show that interpolation error can be used as an a posteriori error estimate that is both reliable and efficient.

**Keywords** Saturation Property, Best Approximation, A Posteriori Error Estimation

**Mathematics Subject Classification (2000)** 65N30, 65N15, 65N50

## 1 Introduction

Let  $\mathcal{T}$  be a simplicial triangulation of some domain  $\Omega$  in arbitrarily many space dimensions. We assume that the elements in  $\mathcal{T}$  are shape regular but do not require that  $\mathcal{T}$  is quasiuniform. Associated with  $\mathcal{T}$  are two conforming, piecewise polynomial finite element spaces  $\mathcal{S}_p$  and  $\mathcal{S}_{2p}$  of polynomial degree  $p$  and  $2p$ . Moreover, we have the interpolation operators  $\mathcal{I}_p$  and  $\mathcal{I}_{2p}$  of the usual kind mapping the continuous functions into  $\mathcal{S}_p$  respectively  $\mathcal{S}_{2p}$ . As the points at which  $\mathcal{I}_p$  interpolates are contained in the set of nodes at which  $\mathcal{I}_{2p}$  interpolates,  $\mathcal{I}_p u = \mathcal{I}_p \mathcal{I}_{2p} u$  for all continuous functions  $u$ . With slight modification our results also hold for the usual families of tensor product finite element spaces that also satisfy this nested node property and for other norms and seminorms than that considered in this note as well.

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We consider a continuous function  $u$  in the Sobolev space  $W_r^1(\Omega)$ , where 1 is the differentiation order and  $r$  an arbitrary index greater than or equal to 1, that can in some sense be better approximated by the functions in  $\mathcal{S}_{2p}$  than by those in  $\mathcal{S}_p$ . More precisely, we assume that there is a sufficiently small constant  $\beta \geq 0$  to be quantified later, depending on this particular function  $u$ , such that

$$|u - \mathcal{I}_{2p}u|_{1,r} \leq \beta |u - \mathcal{I}_p u|_{1,r} \quad (1.1)$$

holds for the error measured in the  $W_r^1$ -seminorm. This is a saturation property. One can expect that for decreasing element sizes and sufficiently smooth functions  $u$  the constant  $\beta$  tends asymptotically to zero but will not make explicit use of this.

We will compare the interpolation error  $|u - \mathcal{I}_p u|_{1,r}$  with the error  $|u - \chi|_{1,r}$  for any  $\chi \in \mathcal{S}_p$ , and in particular with the best approximation error in this seminorm. Of course the best approximation error can be estimated by this interpolation error. The interesting observation proved in this note is the converse; that is, the interpolation error of the function  $u$  under consideration can be estimated by the best approximation error. The saturation property (1.1) thus implies that the interpolation and the approximation error can be estimated by each other. A local variant of this result even does not require that the function  $\chi$  is continuous across the elements.

There is a related result by Andreas Veeger [3], who has shown that the error of the global Ritz projection with respect to globally continuous finite element functions is essentially the same as the one of the element-wise Ritz projection. In contrast to the present work, no saturation assumption is used in this paper. The price is a significantly more technical argumentation. Moreover, Veeger's result seems to be tied to the  $H^1$ -seminorm and perhaps is less easily transferred to other cases.

## 2 The main result

The central result on which our estimate is based is the following simple observation that is standard in finite element theory and has been used at many places:

**Lemma 2.1** *For all elements  $t \in \mathcal{T}$  and all functions  $v \in \mathcal{S}_{2p}$ ,*

$$|\mathcal{I}_p v|_{1,r,t} \leq \theta |v|_{1,r,t}, \quad (2.1)$$

*with a constant  $\theta > 0$  that depends only on the polynomial degrees of the finite element spaces and the shape regularity of the elements in the triangulation.*

*Proof* We prove the estimate first for the reference element  $\hat{t}$  underlying the triangulation and denote by  $\hat{\mathcal{I}}_p$  the interpolation operator  $\mathcal{I}_p$  pulled back to  $\hat{t}$ . Then there exists a constant  $\hat{\theta}$  such that

$$|\hat{\mathcal{I}}_p \hat{v}|_{1,r,\hat{t}} \leq \hat{\theta} |\hat{v}|_{1,r,\hat{t}}$$

holds for all polynomials  $\hat{v}$  of order  $2p$  with vanishing mean value, because the seminorm on the right hand side is a norm on the space of these polynomials and since the space of these polynomials is finite dimensional. Because  $\hat{\mathcal{I}}_p \alpha = \alpha$  for all constant

functions  $\alpha$  and since the  $W_r^1$ -seminorm is invariant to the addition of a constant value to the functions under consideration, the estimate (2.1) thus holds for the reference element. Back transformation shows the proposition for the elements in  $\mathcal{T}$ , where in this last step the degree of deviation of the shape of the elements from that of the reference element but neither their size nor the polynomial degree  $p$  enter.  $\square$

The constant  $\theta$  depends on the polynomial degree  $p$ , which is assumed to be fixed here. It is basically determined by the corresponding constant  $\widehat{\theta}$  for the reference element, which can in the  $L_2$ -like case be calculated solving a little generalized eigenvalue problem. The local, element-wise estimate (2.1) implies the global estimate

$$|\mathcal{I}_p v|_{1,r} \leq \theta |v|_{1,r}, \quad v \in \mathcal{S}_{2p}, \quad (2.2)$$

on the given domain  $\Omega$ . This estimate is the only place where the particular structure of the norms or seminorms under consideration enters. It is the starting point for the proof of our main theorem, that can therefore be transferred to any other norm (like the derivative-free  $L_p$ -norms, for example) for which such an estimate can be proven, and to every function  $u$  that satisfies a corresponding saturation assumption.

**Theorem 2.1** *Assume that  $u$  is a continuous function in  $W_r^1(\Omega)$  that satisfies the saturation assumption (1.1) with a constant  $\beta = \beta(u)$  such that  $\theta\beta < 1$ . Then*

$$|u - \mathcal{I}_p u|_{1,r} \leq \frac{1 + \theta}{1 - \theta\beta} |u - \chi|_{1,r} \quad (2.3)$$

for all functions  $\chi$  in the finite element space  $\mathcal{S}_p$ . That is, the interpolation error can be estimated by the best approximation error.

*Proof* Since  $\mathcal{I}_p u = \mathcal{I}_p \mathcal{I}_{2p} u$  and  $\chi = \mathcal{I}_p \chi$ , we obtain from (2.2) for every  $\chi \in \mathcal{S}_p$

$$|\mathcal{I}_p u - \chi|_{1,r} = |\mathcal{I}_p(\mathcal{I}_{2p} u - \chi)|_{1,r} \leq \theta |\mathcal{I}_{2p} u - \chi|_{1,r}.$$

By the triangle inequality therefore

$$|\mathcal{I}_p u - u|_{1,r} \leq |u - \chi|_{1,r} + \theta \{ |\mathcal{I}_{2p} u - u|_{1,r} + |u - \chi|_{1,r} \}.$$

Inserting the saturation assumption (1.1) on the right hand side and resolving for the left hand side, the proposition already follows.  $\square$

There is a local version of Theorem 2.1 that holds under the somewhat stronger assumption that we have a local saturation property

$$|u - \mathcal{I}_{2p} u|_{1,r,t} \leq \beta |u - \mathcal{I}_p u|_{1,r,t} \quad (2.4)$$

that holds separately for each element  $t \in \mathcal{T}$ . The same kind of reasoning then yields

**Theorem 2.2** *Assume that  $u$  is a continuous function in  $W_r^1(\Omega)$  that satisfies the local saturation property (2.4) with a constant  $\beta = \beta(u)$  such that  $\theta\beta < 1$ . For each single element  $t \in \mathcal{T}$  and every polynomial  $\chi$  of order  $p$  then*

$$|u - \mathcal{I}_p u|_{1,r,t} \leq \frac{1 + \theta}{1 - \theta\beta} |u - \chi|_{1,r,t}. \quad (2.5)$$

The point here is that the function that is globally composed of the single polynomials  $\chi$  does not need to be continuous. The best approximation in  $\mathcal{S}_p$  can thus even be replaced by the function that is composed of the best local approximations by polynomials of order  $p$ , without regard to the continuity across the element boundaries.

### 3 Application to a posteriori error estimation

In this section, we consider the application of these results to a posteriori error estimation for elliptic boundary value problems. We seek a computable global upper bound for the error (making the estimator *reliable*) and both global and element-wise lower bounds for the error (making the estimator *efficient*), see the recent survey [4]. For the ease of presentation, we restrict ourselves to the most simple model problem, the Dirichlet problem for the Laplace equation. We emphasize, however, that the technique presented here does not rely on this particular setting and can easily be generalized to other situations, including non-selfadjoint, indefinite, and nonlinear problems, and even systems including saddle point problems. The same kind of result holds for all problems for which the approximate solution is quasi-optimal in the spaces under consideration and analogues of our two theorems can be proven.

Let  $u_h \in \mathcal{S}_p$  be the finite element approximation of the solution  $u \in H^1(\Omega)$  of the given boundary value problem for the Laplace equation. The approximate solution  $u_h$  is the best approximation of the solution  $u$  with respect to the energy norm, in the given case the  $H^1$ -seminorm, which is the norm here. Theorem 2.1 thus implies that

$$\frac{1 - \theta\beta}{1 + \theta} |u - \mathcal{I}_p u|_{1,2} \leq |u - u_h|_{1,2} \leq |u - \mathcal{I}_p u|_{1,2} \quad (3.1)$$

and Theorem 2.2 implies

$$\frac{1 - \theta\beta}{1 + \theta} |u - \mathcal{I}_p u|_{1,2;t} \leq |u - u_h|_{1,2;t} \quad (3.2)$$

for every single  $t \in \mathcal{T}$ . By these observations, the quantities

$$\eta_t = |u - \mathcal{I}_p u|_{1,2;t} \quad (3.3)$$

are both reliable and efficient local error estimators. As both the global and element-wise lower bounds remain true for general  $\chi \in \mathcal{S}_p$ , they are immune to perturbations in  $u_h$  due to incomplete solution of the linear system, numerical quadrature errors and other variational crimes. The upper bound in (3.1) is not, but such issues have been widely studied in the literature and need not be repeated here.

We now briefly describe a procedure for approximating the  $\eta_t$  that still depend on the unknown solution  $u$ . Our starting point is the representation

$$u - \mathcal{I}_p u = \sum_j \mathcal{F}_j(\partial^{p+1} u) \psi_j \quad (3.4)$$

of the interpolation error on a triangle  $t$ , where the  $\psi_j$  form a basis for the space of polynomials of degree  $p + 1$  that are zero at all nodes of  $\mathcal{S}_p$  on  $t$  and the coefficient functions  $\mathcal{F}_j$  depend in an explicitly known and computationally accessible way on potentially all derivatives of  $u$  of order  $p + 1$ , generically denoted  $\partial^{p+1} u$ . This error representation can be derived from Sobolev's counterpart (see [2], for example) of Taylor's theorem for weakly differentiable functions. Approximations to the derivatives  $\partial^{p+1} u$  on element  $t$  are given by constants computed by a superconvergent recovery procedure that we now summarize. The derivatives of order  $p$  of the

approximate solution  $u_h$ , denoted  $\partial^p u_h$ , are piecewise constant. The recovery operator  $\mathcal{R}\partial^p u_h$  consists of projecting these piecewise constant functions onto the space of continuous piecewise linear finite element functions using  $L_2$ -projection, followed by a smoothing step. This results in a globally superconvergent piecewise linear approximation of the order  $p$  derivatives  $\partial^p u$ . Then  $\partial\mathcal{R}\partial^p u_h$  is a piecewise constant approximation of  $\partial^{p+1}u$ . The local error indicators  $\eta_t$  are then approximated by

$$\eta_t \approx \left| \sum_j \mathcal{F}_j(\partial\mathcal{R}\partial^p u_h)\psi_j \right|_{1,2;t}. \quad (3.5)$$

They depend only on the computed solution  $u_h$ , the choice of norm, and on the shape and size of the finite elements. See [1] for a more detailed discussion.

## References

1. Bank, R.E., Xu, J., Zheng, B.: Superconvergent derivative recovery for Lagrange triangular elements of degree  $p$  on unstructured grids. *SIAM J. Numer. Anal.* **45**, 2032–2046 (2007)
2. Dupont, T., Scott, R.: Polynomial approximation of functions in Sobolev spaces. *Math. Comp.* **34**, 441–463 (1980)
3. Veerer, A.: Approximating gradients with continuous piecewise polynomial functions. arXiv:1402.3945 (2014)
4. Verfürth, R.: *A posteriori error estimation techniques for finite element methods*. Oxford University Press, Oxford (2013)