Stabilizing solution:

where

\( a) There exists an optimal control (\( x(t) = u(t) = u^*) \))

subject to the differential-algebraic control system

\[ Ex(t) = Ax(t) + Bu(t), \quad E\dot{x}(t) = Ex(t) \quad \text{lim}_{t \to \infty} t = 0. \]

Here:

- \( \Phi - A \in \mathbb{R}^{n \times n} \) regular, \( R \in \mathbb{R}^{m \times n} \)
- state \( x \in C_1^2 (\mathbb{R}^R, \mathbb{R}^n) \) with \( E \in C_1^2 (\mathbb{R}^R, \mathbb{R}^n) \)
- control input \( u \in Ct \)
- \( Q = Q^T \in \mathbb{R}^{n \times n} \), \( R = R^T \in \mathbb{R}^{m \times m} \)

Lur'e Equations

Main tool: the Lur'e equation

\[ i_T \in X + A B_1 X + Q B_1 B_2 + S T \]

where a solution \((X, L, I) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m} \times \mathbb{R}^{n \times m} \) fulfills

\[ \text{rank}(I) = n \]

Notation:

\[ M = \lambda \in \mathbb{N} = \lambda + n \]

System Space:

\( V_{\text{sys}} \) is the smallest subspace in \( \mathbb{R}^{n \times m} \) in which the solution trajectories \((x, u)\) evolve.

Stabilizing solution: \((X, L, I) \in X \times L \times I \) if and only if

\[ \text{rank}(I) = n \]

Liu cost: \((i_T X + L X = 0) \)

Feasibility Condition

Define the space of consistent initial differential variables

\[ V_{\text{diff}} = \{ x_0 \in \mathbb{R}^n : \text{a solution } (x(t), u(t)) \text{ of the DAE with } E\dot{x}(t) = Ex(t) \}

and the optimal value function \( V: EVD \to \mathbb{R} \cup \{-\infty\} \) with

\[ V^*(Ex(t)) = \inf [f(x(u), u(t)) \in V_{\text{diff}}(Ex(t)) \text{ solves all DAEs} \]}

Feasibility Theorein:

The following statements are equivalent:

1) \( V^*(Ex(t)) \in V_{\text{diff}} \)

2) \( E\dot{x}(t) = Ex(t), \text{no uncontrollable modes on the imaginary} \]

Lur'e equation has a stabilizing solution \((X, L) = \) \( \text{stabilizing} \)

If the above are satisfied then it holds

\[ V^*(Ex(t)) = i_T X + L X \quad \text{then } \forall x_0 \in V_{\text{diff}}(Ex(t)) \text{ solves all DAEs} \]

Optimal Controls

Feasibility condition suggests the existence of a sequence of solution trajectories \( (x_k, u_k) \) with

\[ V^*(Ex(t)) = \text{lim}_{k \to \infty} V^*(x_0) \]

Questions:

1) Does there exist an optimal control, i.e., a solution \((x(t), u(t)) \text{ with } Ex(t) = Ex_0, \text{lim}_{t \to \infty} t = 0) \text{ such that} \]

\[ V^*(Ex(t)) = \text{lim}_{t \to \infty} \text{j(x(t), u(t))} \]

Answers:

- If \( \text{rank}(\text{R}(x)) \) is at most one then

\[ \text{rank}(\text{R}(x)) = n + m \quad \forall x \in \mathbb{R}^{n \times m} \]

Conclusions

We have solved the linear-quadratic optimal control problem for DAEs under very general assumptions. In particular, we do not need

- \[ V^*(x_0) = \]

- Invertibility of \( R \)

- Impulse-controllability.

References
