Passivity Enforcement of Descriptor Systems
via Structured Perturbation of Hamiltonian Matrix Pencils

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(joint work with Peter Benner)
Motivation

Given: Continuous-time LTI descriptor system

\[ \Sigma : \begin{cases} E \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t) + Du(t), \end{cases} \]

- \( E, A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}, C \in \mathbb{R}^{m \times n}, D \in \mathbb{R}^{m \times m}, \)
- Descriptor vector \( x(t) \in \mathbb{R}^n \), input vector \( u(t) \in \mathbb{R}^m \), output vector \( y(t) \in \mathbb{R}^m \).
- Assumptions: \( \lambda E - A \) is regular and stable, \((E; A, B, C, D)\) is \( C \)-controllable and \( C \)-observable (i.e., \( \text{rank} (\begin{bmatrix} \alpha E - \beta A & B \end{bmatrix}) = n \) and \( \text{rank} (\begin{bmatrix} \alpha E^T - \beta A^T & C^T \end{bmatrix}) = n \) for all \((\alpha, \beta) \in \mathbb{C}^2 \setminus \{(0,0)\}). \)
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- **Assumptions:** \( \lambda E - A \) is regular and stable, \((E; A, B, C, D)\) is C-controllable and C-observable (i.e., \( \text{rank} \left( \begin{bmatrix} \alpha E - \beta A & B \end{bmatrix} \right) = n \) and \( \text{rank} \left( \begin{bmatrix} \alpha E^T - \beta A^T & C^T \end{bmatrix} \right) = n \) for all \((\alpha, \beta) \in \mathbb{C}^2 \setminus \{(0, 0)\}).\)

Transfer function

\[ G(s) := C(sE - A)^{-1}B + D \]
Motivation

System $\Sigma$ is obtained, e.g., by some model order reduction technique or interpolation using frequency response data. From the theory of our application, we know that the system should have passive behaviour. However this property might be lost during the modeling process!
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Goal

Find a passive system

$$\tilde{\Sigma} : \begin{cases} \tilde{E}\dot{x}(t) = \tilde{A}x(t) + \tilde{B}u(t), \\ y(t) = \tilde{C}x(t) + \tilde{D}u(t), \end{cases}$$

with transfer function $\tilde{G}$ such that $\|\tilde{G} - G\|$ is small in some system norm!
Passivity Enforcement in the Literature

- [Grivet-Talocia ’04]: passivity enforcement of standard state-space systems via perturbation of Hamiltonian matrices,
- [Schröder, Stykel ’07]: using structure-preserving algorithms to compute required eigenvalues and eigenvectors,
- [Wang, Zheng, Koh, Pang, Wong ’10]: passivity enforcement of descriptor systems using skew-Hamiltonian/Hamiltonian matrix pencils, explicit computation of spectral projectors to extract relevant subsystems,
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- [Wang, Zheng, Koh, Pang, Wong ’10]: passivity enforcement of descriptor systems using skew-Hamiltonian/Hamiltonian matrix pencils, explicit computation of spectral projectors to extract relevant subsystems,
- **Here**: structure-preserving computation of required eigenvalues and eigenvectors, no projectors needed.
1 Preliminaries

2 Computation of the System Decomposition

3 Passivity Enforcement of the Proper Part

4 Numerical Example

5 Conclusion and Outlook
Passivity Concepts

“Passivity is the inability of a system to generate energy.”
Passivity Concepts

Scattering representation

A descriptor system $\Sigma$ in scattering representation is called (strictly) passive if

$$\int_0^t \|u(\tau)\|^2 - \|y(\tau)\|^2 d\tau \geq 0 (> 0)$$

for all $t > 0$, all $u \in L_2([0, t], \mathbb{R}^m)$ and consistent initial conditions. This property is also often called contractivity.
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Admittance/impedance representation

A descriptor system $\Sigma$ in admittance/impedance representation is called (strictly) passive if

$$\int_0^t u(\tau)^T y(\tau) d\tau \geq 0 (> 0)$$

for all $t > 0$, all $u \in L_2([0, t], \mathbb{R}^m)$ and consistent initial conditions.
**Bounded Realness**

**Bounded real transfer function**

The transfer function $G$ is called (strictly) bounded real, if

- $G$ has no poles with nonnegative real parts,
- $H(i\omega) := I - G(i\omega)G^H(i\omega)$ is positive semidefinite (positive definite) for all values $\omega \in \mathbb{R}$. 
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Positive Realness

Positive real transfer function

The transfer function $G$ is called (strictly) positive real, if

1. $G$ has no poles with positive (nonnegative) real parts,
2. $H(i\omega) := \frac{1}{2} \left( G(i\omega) + G^H(i\omega) \right)$ is positive semidefinite (positive definite) for any $i\omega$ that is not a pole of $G$ with $\omega \in \mathbb{R}$,
3. $i\omega$ or $\infty$ is a pole of $G$, then it is simple and the relevant residue matrix is positive semidefinite Hermitian.
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- $i\omega$ or $\infty$ is a pole of $G$, then it is simple and the relevant residue matrix is positive semidefinite Hermitian.
Laurent series expansion of \( G(s) \) at \( s = \infty \):

\[
G(s) = \sum_{k=-\infty}^{-1} M_k s^k + M_0 + \sum_{k=1}^{d} M_k s^k
\]

\[
= : G_{sp}(s)
\]

\[
= : G_p(s)
\]

\[
= : G_i(s)
\]

- \( G_{sp} \): strictly proper part, i.e., \( \lim_{\omega \to \infty} \| G_{sp}(i\omega) \| = 0 \),
- \( G_p \): proper part, i.e., \( \lim_{\omega \to \infty} \| G_p(i\omega) \| < \infty \),
- \( G_i \): improper part, i.e. \( \lim_{\omega \to \infty} \| G_i(i\omega) \| = \infty \).
Equivalent Passivity Conditions

Scattering representation

A descriptor system $\Sigma$ in scattering representation is (strictly) passive if and only if

- the proper part $G_p := G_{sp} + M_0$ is (strictly) bounded real,
- $M_k = 0$ for $k \geq 1$. 
Equivalent Passivity Conditions

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**Admittance/impedance representation**

A descriptor system $\Sigma$ in admittance/impedance representation is (strictly) passive if and only if

- the proper part $G_p := G_{sp} + M_0$ is (strictly) positive real,
- $M_1$ is symmetric positive semidefinite ($M_1 = 0$).
- $M_k = 0$ for $k \geq 2$. 
General Outline of Passivity Enforcement Procedure

Basic Steps

1. Decouple $G$ into its proper and improper part.
2. Check, if $M_k = 0$ for $k \geq 1$ (scattering representation) or $k \geq 2$ (admittance/impedance representation) $\Rightarrow$ if not, passivity cannot be enforced.
3. Enforce bounded/positive realness of the proper part.
4. Enforce positive semidefiniteness of $M_1$ for systems in admittance/impedance representation.
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Reduction of the System Pencil

Theorem

[Benner, Chu ’05]

For any regular matrix pencil $A - \lambda E$ there exist orthogonal matrices $U, V \in \mathbb{R}^{n\times n}$ such that

$$U(A - \lambda E)V = \begin{bmatrix}
A_{11} - \lambda E_{11} & A_{12} - \lambda E_{12} & A_{13} - \lambda E_{13} & A_{14} - \lambda E_{14} \\
0 & A_{22} & A_{23} - \lambda E_{23} & A_{24} - \lambda E_{24} \\
0 & 0 & A_{33} & A_{34} \\
0 & 0 & 0 & A_{44}
\end{bmatrix}
\begin{bmatrix}
n_1 \\
n_2 \\
n_3 \\
n_4
\end{bmatrix},$$

where $\text{rank}(E_{11}) = n_1$, $\text{rank}(E_{23}) = n_3$, $\text{rank}(A_{44}) = n_4$, and

$$\text{rank} \left( \begin{bmatrix}
A_{22} & A_{23} - \lambda E_{23} \\
0 & A_{33}
\end{bmatrix} \right) = n_2 + n_3 \quad \forall \lambda \in \mathbb{C}.$$
Distinction of Cases

[Benner, Chu ’05]

Case 1: $n_2 \neq n_3$

- There exist $M_k \neq 0$ with $k \geq 2$.
- System $\Sigma$ is non-passive and passivity cannot be enforced.
Distinction of Cases

Case 1: $n_2 \neq n_3$
- There exist $M_k \neq 0$ with $k \geq 2$.
- System $\Sigma$ is non-passive and passivity cannot be enforced.

Case 2: $n_2 = n_3 = 0$
- We have $M_k = 0$ for $k \geq 1$ and

\[
G(s) = \begin{bmatrix} C_1 & C_4 \end{bmatrix} \left( s \begin{bmatrix} E_{11} & E_{14} \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} A_{11} & A_{14} \\ 0 & A_{44} \end{bmatrix} \right)^{-1} \begin{bmatrix} B_1 \\ B_4 \end{bmatrix} + D
\]

\[
= G_p(s).
\]
Distinction of Cases

[Benner, Chu ’05]

Case 3: $n_2 = n_3 \neq 0$

- $M_k = 0$ for $k \geq 2$.
- By only using orthogonal transformations we obtain $M_1 = N^{-1}M$.
- We can again obtain $G_p$ as

$$G_p(s) = [C_1 \quad C_2] \left( s \begin{bmatrix} E_{11} & E_{12} \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \right)^{-1} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} + D.$$
**Distinction of Cases**

**Case 3: \( n_2 = n_3 \neq 0 \)**

- \( M_k = 0 \) for \( k \geq 2 \).
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\]

**Discussion**

- **Pro:** only using **orthogonal transformations**, \( O(n^3) \) flops,
- **Con:** URV and RRQR decompositions with **delicate rank decisions**.
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5 Conclusion and Outlook
Let $G_p$ be a proper and stable with realization $(E; A, B, C, D)$.

Theorem

Let $\lim_{\omega \to \infty} \sigma_{\text{max}}(G(i\omega)) < 1$. Then $G_p$ is strictly bounded real if and only if the extended skew-Hamiltonian/Hamiltonian (sH/H) matrix pencil

$$\lambda \begin{bmatrix}
E & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & E^T & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
- \begin{bmatrix}
A & B & 0 & 0 \\
C & D & 0 & l_m \\
0 & 0 & -A^T & -C^T \\
0 & -l_m & -B^T & -D^T \\
\end{bmatrix},$$

has no finite, purely imaginary eigenvalues.
Bounded/Positive Realness and Hamiltonian Pencils

Let $G_p$ be a proper and stable with realization $(E; A, B, C, D)$.

**Theorem**

Let $D + D^T$ be nonsingular and $\lim_{\omega \to \infty} \lambda_{\min}(H(i\omega)) > 0$. Then $G_p$ is strictly positive real if and only if the extended skew-Hamiltonian/Hamiltonian ($sH/H$) matrix pencil

$$\lambda \begin{bmatrix} E & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & E^T & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} A & -B & 0 & B \\ C & -2(D + D^T) & B^T & 0 \\ 0 & C^T & -A^T & -C^T \\ C & 0 & B^T & 2(D + D^T) \end{bmatrix}$$

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\end{bmatrix}
$$

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**General Idea:** If the matrix pencils have purely imaginary eigenvalues, perturb these away from the imaginary axis to passify the proper subsystem.
Graphical Interpretation

Passivation Steps:

1. Choose new eigenvalues (displacement $\tilde{\omega}_j - \omega_j = \alpha |W|$), with tuning parameter $\alpha$, bandwidth of passivity violation $W$.
2. Compute perturbed $sH/H$ matrix pencil that has (approximately) the new eigenvalues.
3. Repeat 1+2 until passivity is enforced.
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Choice of the Perturbation

\[ G_p(s) = C_p(sE_p - A_p)^{-1}B_p + D \]
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\[ = [C_1 \quad C_2] \left( s \begin{bmatrix} E_{11} & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix} \right)^{-1} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} + D \]

\[ = C_1 (sE_{11} - A_{11})^{-1} B_1 + D - C_2 A_{22}^{-1} B_2 \]

\[ = G_{sp}(s) \]

\[ = M_0 \]
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\[ = C_1 (sE_{11} - A_{11})^{-1} B_1 + D - C_2 A_{22}^{-1} B_2 \]

= \text{free parameters} \quad \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}

\text{Which matrix should be perturbed?}
Choice of the Perturbation

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\[ = C_1 (sE_{11} - A_{11})^{-1} B_1 + D - C_2 A_{22}^{-1} B_2 \]

\[ = G_{sp}(s) \quad = M_0 \]

Which matrix should be perturbed?

- Keep \( \lambda E_{11} - A_{11} \) to preserve poles of the system \( \implies \) stability preservation,
- keep \( A_{22}, B_2, C_2, D \) to preserve behaviour for large frequencies,
Choice of the Perturbation

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- Keep \( \lambda E_{11} - A_{11} \) to preserve poles of the system \( \Rightarrow \) stability preservation,
- keep \( A_{22}, B_2, C_2, D \) to preserve behaviour for large frequencies,
- free parameters \( B_1, C_1 \).
Choice of the System Norm

We choose the $\mathcal{H}_2$-norm to measure the error, i.e.,

$$\|\mathcal{E}_{sp}\|_{\mathcal{H}_2} := \|\tilde{G}_{sp} - G_{sp}\|_{\mathcal{H}_2} = \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} \|\mathcal{E}_{sp}(i\omega)\|_F^2 \, d\omega\right)^{1/2}.$$
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In the sequel: only perturbations of $B_1$!
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In the sequel: only perturbations of $B_1$!

**Theorem** [Stykel ’06]

Let $G_o$ be the unique, positive definite solution of the generalized Lyapunov equation

$$E_{11}^T G_o A_{11} + A_{11}^T G_o E_{11} = -C_1^T C_1,$$

and $G_o = R^T R$ a Cholesky factorization. Then

$$\|\mathcal{E}_{sp}\|_{\mathcal{H}_2} = \|R\Delta\|_F \quad \text{with} \quad \Delta := \tilde{B}_1 - B_1.$$
Choice of the System Norm

Remark

In the case that the transfer function is given as

\[ G_p(s) = [C_1 \quad C_2] \left( s \begin{bmatrix} E_{11} & E_{12} \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \right)^{-1} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} + D, \]

it is not necessary to decouple \( G_p \) into \( G_{sp} \) and \( M_0 \), since all required matrices are already available!
Computation of the Optimal Perturbation

Let \( \lambda \mathcal{N} - \mathcal{M} \) be the \( \text{sH}/H \) matrix pencil associated to a system in scattering representation (similar for admittance/impedance representation)

\[
\lambda \mathcal{N} - \mathcal{M} = \lambda \begin{bmatrix}
\mathcal{E}_{11} & \mathcal{E}_{12} & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & \mathcal{E}_{11}^T \\
0 & 0 & \mathcal{E}_{12}^T \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{bmatrix}
- \begin{bmatrix}
\mathcal{A}_{11} & \mathcal{A}_{12} & \mathcal{B}_1 \\
0 & \mathcal{A}_{22} & \mathcal{B}_2 \\
\mathcal{C}_1 & \mathcal{C}_2 & \mathcal{D} \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{bmatrix} \end{equation}

\[
\mathcal{J} \hat{\mathcal{M}} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix},
\mathcal{J} = \begin{bmatrix}
0 & I_n & \mathcal{N} \\
0 & I_n + 2m & 0 \\
\end{bmatrix}.
\]
Computation of the Optimal Perturbation

The perturbed pencil is $\lambda \mathcal{N} - \left( \mathcal{M} + \hat{\mathcal{M}} \right)$ with

$$
\hat{\mathcal{M}} = \begin{bmatrix}
0 & 0 & \Delta & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\Delta^T & 0 & 0
\end{bmatrix}.
$$
Computation of the Optimal Perturbation

\[ J \hat{M} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\Delta & 0 & 0 \\
0 & 0 & -\Delta & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}, \quad J = \begin{bmatrix}
0 & I_{n+m} \\
-I_{n+m} & 0
\end{bmatrix} \]
Computation of the Optimal Perturbation

\[
\mathcal{J}\hat{M} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\Delta^T & 0 & 0 \\
0 & 0 & -\Delta & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}, \quad \mathcal{J} = \begin{bmatrix}
0 & I_{n+m} \\
-l_{n+m} & 0 \\
\end{bmatrix}
\]

First Order Perturbation of Eigenvalues

Let \(i\omega_j\) be a simple, finite, purely imaginary eigenvalue of \(\lambda\mathcal{N} - \mathcal{M}\) with eigenvector \(v_j\). Then

\[
\tilde{\omega}_j - \omega_j = \frac{v_j^H \mathcal{J}\hat{M} v_j}{i v_j^H \mathcal{J}\mathcal{N} v_j} + O \left( \|\hat{M}\|_2^2 \right), \quad j = 1, \ldots, k.
\]
Computation of the Optimal Perturbation

\[ \mathcal{J} \hat{M} = \begin{bmatrix}
  0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & -\Delta^T \\
  0 & 0 & -\Delta & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}, \quad \mathcal{J} = \begin{bmatrix}
  0 & I_{n+m} \\
  -I_{n+m} & 0 \\
\end{bmatrix} \]

First Order Perturbation of Eigenvalues

Let \( i \omega_j \) be a simple, finite, purely imaginary eigenvalue of \( \lambda N - M \) with eigenvector \( v_j \). Then

\[ \tilde{\omega}_j - \omega_j = \frac{v_j^H \mathcal{J} \hat{M} v_j}{i v_j^H \mathcal{J} N v_j} + O(\|\hat{M}\|_2^2), \quad j = 1, \ldots, k. \]

Computation of \( v_j \) \( \implies \) see talk at GAMM Annual Meeting in Graz!
Computation of the Optimal Perturbation

\[ \tilde{\omega}_j - \omega_j = \frac{v_j^H \mathcal{J} \hat{M} v_j}{i v_j^H \mathcal{J} \mathcal{N} v_j} \]
Computation of the Optimal Perturbation

\[ \tilde{\omega}_j - \omega_j = \frac{v_j^H J \hat{M} v_j}{i v_j^H J N v_j} \]

\[ = - \frac{v_{j3}^H \Delta^T v_{j4} + v_{j4}^H \Delta v_{j3}}{i v_j^H J N v_j} \]
Computation of the Optimal Perturbation

\[ \tilde{\omega}_j - \omega_j = \frac{v_j^H J \hat{M} v_j}{i v_j^H J N v_j} \]

\[ = -\frac{v_{j3}^H \Delta^T v_{j4} + v_{j4}^H \Delta v_{j3}}{i v_j^H J N v_j} \]

\[ = -\frac{2 \text{Re} \left( v_{j4}^H \Delta v_{j3} \right)}{i v_j^H J N v_j} \]
Compute the Optimal Perturbation

\[ \tilde{\omega}_j - \omega_j = \frac{v_j^H \hat{J} \hat{M} v_j}{i v_j^H \hat{J} \mathcal{N} v_j} \]

\[ = - \frac{v_{j3}^H \Delta^T v_{j4} + v_{j4}^H \Delta v_{j3}}{i v_j^H \mathcal{N} v_j} \]

\[ = - \frac{2 \text{Re} (v_{j4}^H \Delta v_{j3})}{i v_j^H \mathcal{N} v_j} \]

\[ = - \frac{2 \text{Re} (v_{j3}^T \otimes v_{j4}^H) \text{vec}(\Delta)}{i v_j^H \mathcal{N} v_j} \]
Computation of the Optimal Perturbation

\[ \tilde{\omega}_j - \omega_j = -\frac{2 \text{Re} \left( v^T_j \otimes v^H_j \right) \text{vec}(\Delta)}{i v^H_j J N v_j} \]

Doing this for all imaginary eigenvalues, we obtain

\[ Z \text{vec}(\Delta) = \tilde{\omega} - \omega. \]
Computation of the Optimal Perturbation

\[
\tilde{\omega}_j - \omega_j = -\frac{2 \Re (v_{j3}^T \otimes v_{j4}^H) \text{vec}(\Delta)}{i v_j^H J \mathcal{N} v_j}
\]

Doing this for all imaginary eigenvalues, we obtain

\[
Z \text{vec}(\Delta) = \tilde{\omega} - \omega.
\]

Minimize \(\|\mathcal{E}_{sp}\|_{\mathcal{H}_2}\) by solving the minimization problem

\[
\min_{\Delta \in \mathbb{R}^{m \times n_f}} \|R\Delta\|_F \quad \text{subject to} \quad Z \text{vec}(\Delta) = \tilde{\omega} - \omega.
\]
Computation of the Optimal Perturbation

$$\tilde{\omega}_j - \omega_j = -\frac{2 \text{Re} (v_{j3}^T \otimes v_{j4}^H) \text{vec}(\Delta)}{i v_j^H J N v_j}$$

Doing this for all imaginary eigenvalues, we obtain

$$Z \text{vec}(\Delta) = \tilde{\omega} - \omega.$$

Minimize $\|E_{sp}\|_{\mathcal{H}_2}$ by solving the minimization problem

$$\min_{\Delta \in \mathbb{R}^{m \times nf}} \|R \Delta\|_F \quad \text{subject to} \quad Z \text{vec}(\Delta) = \tilde{\omega} - \omega.$$  

Change of basis $\Delta_R := R \Delta$ leads to

$$\min_{\Delta_R \in \mathbb{R}^{m \times nf}} \|\Delta_R\|_F \quad \text{subject to} \quad Z_R \text{vec}(\Delta_R) = \tilde{\omega} - \omega.$$
Computation of the Optimal Perturbation

\[ \tilde{\omega}_j - \omega_j = -\frac{2 \text{Re} (v^T_j \otimes v^H_j) \text{vec}(\Delta)}{iv_j^H JN v_j} \]

Doing this for all imaginary eigenvalues, we obtain

\[ Z \text{vec}(\Delta) = \tilde{\omega} - \omega. \]

Minimize \( \|E_{sp}\|_{\mathcal{H}_2} \) by solving the minimization problem

\[ \min_{\Delta \in \mathbb{R}^{m \times nf}} \|R\Delta\|_F \quad \text{subject to} \quad Z \text{vec}(\Delta) = \tilde{\omega} - \omega. \]

Change of basis \( \Delta_R := R\Delta \) leads to

\[ \min_{\Delta_R \in \mathbb{R}^{m \times nf}} \|\text{vec} (\Delta_R)\|_2 \quad \text{subject to} \quad Z_R \text{vec}(\Delta_R) = \tilde{\omega} - \omega. \]
Computation of the Optimal Perturbation

\[
\tilde{\omega}_j - \omega_j = -\frac{2 \Re (\nu^T_j \otimes \nu^H_j) \text{vec} (\Delta)}{i \nu^H_j J N \nu_j}
\]

Doing this for all imaginary eigenvalues, we obtain

\[Z \text{vec} (\Delta) = \tilde{\omega} - \omega.\]

Minimize \(\|\mathcal{E}_{sp}\|_{\mathcal{H}_2}\) by solving the minimization problem

\[
\min_{\Delta \in \mathbb{R}^{m \times n_f}} \|R \Delta\|_F \quad \text{subject to} \quad Z \text{vec} (\Delta) = \tilde{\omega} - \omega.
\]

Change of basis \(\Delta_R := R \Delta\) leads to

\[
\min_{\Delta_R \in \mathbb{R}^{m \times n_f}} \|\text{vec} (\Delta_R)\|_2 \quad \text{subject to} \quad Z_R \text{vec} (\Delta_R) = \tilde{\omega} - \omega.
\]

Solution: \(\text{vec} (\Delta_R) = Z_R^\dagger (\tilde{\omega} - \omega), \quad \Delta = R^{-1} \Delta_R.\)
1 Preliminaries

2 Computation of the System Decomposition

3 Passivity Enforcement of the Proper Part

4 Numerical Example

5 Conclusion and Outlook
Example Description

- artificial example system in scattering representation with $n = 12$, $m = 2$,
- passivity violation in $[0.300808, 0.322311] \cup [0.411681, 0.651213]$,
- $\| G \|_{H_2} = 1.3686$, $\| G \|_{H_\infty} = 1.0553$. 

![Graph showing the frequency response of $\sigma(G(i\omega))$](attachment://graph.png)
Some Results

Facts:
- $\alpha = 0.5$
- $\varepsilon_r = 0.01750$
- iter.: 2

![Graph showing the relationship between $\omega$ and $\sigma(G(\omega))$]
Some Results

Facts:

- $\alpha = 0.4$
- $\varepsilon_r = 0.01429$
- iter.: 2
Some Results

Facts:
- $\alpha = 0.3$
- $\varepsilon_r = 0.01074$
- iter.: 2
Some Results

Facts:

- $\alpha = 0.2$
- $\varepsilon_r = 0.00805$
- iter.: 21

![Graph showing the comparison between original and passified systems](image)
Some Results

Facts:

- $\alpha = 0.1$
- $\varepsilon_r = 0.00694$
- iter.: 30
Some Results

Facts:

- $\alpha = 0.05$
- $\varepsilon_r = 0.00665$
- iter.: 72
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Conclusion and Outlook

What we have done

- **Characterization of passivity** for systems in scattering and admittance/impedance representation in terms of the transfer functions,
- computation of relevant subsystems and Markov parameters,
- **passivity enforcement of the proper part** using perturbations of $sH/H$ matrix pencils.
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What we could not present here

- Asymptotic passivity enforcement for passivity violations at $\omega = \infty$,
- enforcing positive semidefiniteness of $M_1$.  
  \[\text{[Wang et al. ’10]}\]
Conclusion and Outlook

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- **Characterization of passivity** for systems in scattering and admittance/impedance representation in terms of the transfer functions,
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- Asymptotic passivity enforcement for passivity violations at $\omega = \infty$,
- enforcing positive semidefiniteness of $M_1$. [Wang et al. ’10]

Further research directions

- Large-scale systems,
- consider other matrices than $B_1$ for perturbation.
Thank you!

References