$\mathcal{H}_\infty$-Norm Computation for Large and Sparse Descriptor Systems

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In this paper we discuss the problem of computing the $\mathcal{H}_\infty$-norms of transfer functions associated to large-scale descriptor systems. We exploit the relationship between the $\mathcal{H}_\infty$-norm and the structured complex stability radius of the corresponding system pencil. To compute the structured stability radius we consider so-called structured pseudospectra. Namely, we have to find the pseudospectrum that touches the imaginary axis. Therefore, we set up an iteration over the real part of the rightmost pseudoeigenvalue. For that we use a new fast iterative scheme which is based on certain rank-1 perturbations of the system pencil. Finally, we illustrate the performance of our algorithm by using a real-world example.

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1 Introduction and Preliminaries

In this paper we consider linear time-invariant descriptor systems of the form

$$E\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t), \tag{1}$$

where E, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, $x(t) \in \mathbb{R}^n$ is the descriptor vector, $u(t) \in \mathbb{R}^m$ is the input vector, and $y(t) \in \mathbb{R}^p$ is the output vector. Sometimes we will denote such a system by $\Sigma = (\lambda E - A, B, C)$. Throughout this paper we assume that $\lambda E - A$ is a *regular* matrix pencil, i.e., $\det(\lambda E - A) \not\equiv 0$. Furthermore, we assume that all involved matrices are large and sparse with m, $p \ll n$.

By taking the Laplace transforms of both equations in (1) and assuming $Ex(0) = 0$ we obtain the transfer function

$$G(s) = C(sE - A)^{-1}B. \tag{2}$$

We call $G(s)$ (asymptotically) stable if all its *finite* poles are located in the open left half-plane, i.e., $\Pi_{\mathrm{f}}(E, A, B, C) \subset \mathbb{C}^- := \{s \in \mathbb{C} : \mathrm{Re}\,(s) < 0\}$, where $\Pi_{\mathrm{f}}(E, A, B, C)$ denotes the set of all finite poles of $G(s)$. Furthermore, we call $G(s)$ *proper* if $\lim_{\omega\to\infty} \|G(\mathrm{i}\omega)\|_2 < \infty$, otherwise we call it *improper*. By $\mathcal{RH}_\infty^{p\times m}$ we denote the rational Banach space of all stable and proper functions of the form (2). For this space we define the $\mathcal{H}_\infty$*-norm*, given by

$$\|G\|_{\mathcal{H}_\infty} := \sup_{s\in\mathbb{C}^+} \sigma_{\max}(G(s)) = \sup_{\omega\in\mathbb{R}} \sigma_{\max}(G(\mathrm{i}\omega)),$$

with $\mathbb{C}^+ := \{s \in \mathbb{C} : \mathrm{Re}\,(s) > 0\}$ and the maximum singular value $\sigma_{\max}(\cdot)$. Our aim is to compute this norm value under the given assumptions. More details, all proofs and a complete study of numerical results are found in [1].

2 Theoretical Framework

In this section we discuss the theoretical basis of our algorithm. First of all we give the relationship between the $\mathcal{H}_\infty$-norm and the structured complex stability radius of a transfer function $G \in \mathcal{RH}_\infty^{p\times m}(\mathrm{i}\omega)$. We define the numbers

$$r_{\mathbb{C}}^{\mathrm{f}}(E, A, B, C) := \inf\left\{\|\Delta\|_2 : \Pi_{\mathrm{f}}(E, A + B\Delta C, B, C) \cap \mathrm{i}\mathbb{R} \neq \emptyset \text{ with } \Delta \in \mathbb{C}^{m\times p}\right\},$$
$$r_{\mathbb{C}}^{\infty}(E, A, B, C) := \inf\left\{\|\Delta\|_2 : C\left(sE - (A + B\Delta C)\right)^{-1}B \text{ with } \Delta \in \mathbb{C}^{m\times p} \text{ is improper or not well-defined}\right\}.$$

Then the *structured complex stability radius of a transfer function* $G(s)$ is given by

$$r_{\mathbb{C}}(E, A, B, C) := \min\left\{r_{\mathbb{C}}^{\mathrm{f}}(E, A, B, C), r_{\mathbb{C}}^{\infty}(E, A, B, C)\right\}.$$

Lemma 2.1 *It holds that*

$$r_{\mathbb{C}}^{\infty}(E, A, B, C) = \begin{cases} 1/\lim\limits_{\omega\to\infty} \sigma_{\max}(G(\mathrm{i}\omega)) & \text{if } G \not\equiv 0, \\ \infty & \text{if } G \equiv 0. \end{cases}$$

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Proposition 2.2 *It holds*

$$r_{\mathbb{C}}(E,A,B,C) = \begin{cases} \frac{1}{\|G\|_{\mathcal{H}_\infty}} & \text{if } G \not\equiv 0, \\ \infty & \text{if } G \equiv 0. \end{cases} \tag{3}$$

For the remainder of the article we need the following definitions. The *structured ε-pseudospectrum of the transfer function* $G(s)$ with respect to B and C is defined by

$$\Pi_\varepsilon(E,A,B,C) = \left\{ s \in \mathbb{C} : s \in \Pi_{\mathrm{f}}(E, A + B\Delta C, B, C) \text{ for some } \Delta \in \mathbb{C}^{m\times p} \text{ with } \|\Delta\|_2 < \varepsilon \right\}.$$

Furthermore, we need the *structured ε-pseudospectral abscissa*, given by

$$\alpha_\varepsilon(E,A,B,C) := \max\left\{ \operatorname{Re} s : s \in \Pi_\varepsilon(E,A,B,C) \right\}.$$

From these definitions it is clear that $\alpha_{r_{\mathbb{C}}^{\mathrm{f}}}(E,A,B,C) = 0$. The main idea of our algorithm is to find the (unique) root of the function $\alpha(\varepsilon) := \alpha_\varepsilon(E,A,B,C)$. To get an efficient algorithm we need to cheaply evaluate $\alpha(\varepsilon)$ for different values of ε.

2.1 Computation of the Structured ε-Pseudospectral Abscissa

In this subsection we derive a fast algorithm for computing $\alpha(\varepsilon)$.

Lemma 2.3 *Let $s \in \mathbb{C}\backslash\Pi_f(E,A,B,C)$ be given and $\varepsilon > 0$. Then the following statements are equivalent:*

(a) $s \in \Pi_\varepsilon(E,A,B,C)$.

(b) $\sigma_{\max}(G(s)) > \varepsilon^{-1}$.

(c) There exist vectors $u \in \mathbb{C}^m$ and $v \in \mathbb{C}^p$ with $\|u\|_2 < 1$ and $\|v\|_2 < 1$ such that $s \in \Pi_f\left(E, A + \varepsilon B u v^H C, B, C\right)$.

In other words, also the rightmost structured pseudopole is arbitrarily close to the boundary curve $\partial\Pi_\varepsilon(E,A,B,C) := \left\{ s \in \mathbb{C} : \sigma_{\max}\left(G(s)\right) = \varepsilon^{-1} \right\}$. Thus, our strategy consists of computing a sequence of suitable structured rank-1 perturbed pencils $\lambda E - (A + \varepsilon B u v^H C)$ such that one of the perturbed eigenvalues converges to the rightmost structured pseudopole of $G(s)$, similarly as in [2]. We need the following result for the first order perturbation theory of matrix pencils.

Lemma 2.4 *[3] Let $\lambda E - A \in \mathbb{C}[\lambda]^{n\times n}$ be a given matrix pencil with right and left eigenvectors $x,\ y \in \mathbb{C}^n$ corresponding to a simple finite eigenvalue $\lambda = \frac{y^H A x}{y^H E x}$. Let $\lambda E - (A + tBuv^HC)$ be a perturbed matrix pencil with eigenvalue $\tilde{\lambda}$. Then*

$$\tilde{\lambda} = \lambda + t\frac{y^H B u v^H C x}{y^H E x} + \mathcal{O}\left(t^2\right).$$

Now, we describe how such rank-1 perturbations can be constructed in an optimal way. Therefore, let $\lambda E - A$ with a simple eigenvalue λ and corresponding right and left eigenvectors x, y with $y^H E x > 0$ be given. Furthermore, let $u \in \mathbb{C}^m$ and $v \in \mathbb{C}^p$ with $\|u\|_2 = \|v\|_2 = 1$ be given vectors. Then, by Lemma 2.4 it holds

$$\operatorname{Re}\left(\left.\frac{\mathrm{d}\tilde{\lambda}(t)}{\mathrm{d}t}\right|_{t=0} \right) = \frac{\operatorname{Re}\left(y^H B u v^H C x\right)}{y^H E x} \le \frac{\left\|y^H B\right\|_2 \|Cx\|_2}{y^H E x}. \tag{4}$$

Equality in (4) holds for $u = \frac{B^T y}{\|B^T y\|_2}$, $v = \frac{Cx}{\|Cx\|_2}$. Hence, local maximal growth in $\operatorname{Re}\left(\tilde{\lambda}(t)\right)$ as t increases from 0 is achieved for this choice of u and v. In this way we generate the initial perturbation. Next we consider subsequent perturbations. Let therefore $\lambda E - \hat{A} := \lambda E - (A + \varepsilon B\hat{u}\hat{v}^H C)$ with a simple eigenvalue $\hat{\lambda}$ and associated right and left eigenvectors $\hat{x}$, $\hat{y}$ with $\hat{y}^H E \hat{x} > 0$ be the perturbed matrix pencil. Let in addition vectors $u \in \mathbb{C}^m$, $v \in \mathbb{C}^p$ with $\|u\|_2 = \|v\|_2 = 1$ be given. We consider the family of perturbations of the matrix pencil $\lambda E - \hat{A}$ of the form

$$\lambda E - \left(\hat{A} + tB\left(uv^H - \hat{u}\hat{v}^H\right)C\right),$$

which are structured ε-norm rank-1 perturbations of $\lambda E - A$ for $t = 0$ and $t = \varepsilon$. For the perturbed eigenvalue, for simplicity called again $\tilde{\lambda}$, we obtain

$$\operatorname{Re}\left(\left.\frac{\mathrm{d}\tilde{\lambda}(t)}{\mathrm{d}t}\right|_{t=0} \right) = \frac{\operatorname{Re}\left(\hat{y}^H B\left(uv^H - \hat{u}\hat{v}^H\right)C\hat{x}\right)}{\hat{y}^H E \hat{x}} \le \frac{\left\|\hat{y}^H B\right\|_2 \|C\hat{x}\|_2 - \operatorname{Re}\left(\hat{y}^H B\hat{u}\hat{v}^H C\hat{x}\right)}{\hat{y}^H E\hat{x}}. \tag{5}$$

Similarly as above, equality in (5) holds for $u = \frac{B^T\hat{y}}{\|B^T\hat{y}\|_2}$, $v = \frac{C\hat{x}}{\|C\hat{x}\|_2}$. So, the basic algorithm consists of successively choosing an eigenvalue and constructing the perturbations described above by using the corresponding eigenvectors.

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2.2 Choice of the Eigenvalues

An important question is how to actually choose the eigenvalues that should be perturbed. Intuitively, in each step one would choose the rightmost eigenvalue of the perturbed pencil to construct the next perturbation. However, that might not be a good choice. Note, that the perturbability of an eigenvalue λ with right and left eigenvectors x and y highly depends on $\left\|B^T y\right\|_2$ and $\|Cx\|_2$. If these values are small, no large perturbation is possible. We note, that these values are strongly related to the usual concepts for controllability and observability. Large values of $\left\|B^T y\right\|_2$ indicate a good controllability at λ. Similarly, large values of $\|Cx\|_2$ indicate a good observability at λ.

Consequently, for our algorithm we look for eigenvalues that have both sufficiently large real part and a high controllability and observability. An algorithm which unites both concepts and can compute the desired eigenvalues is the *(subspace accelerated MIMO) dominant pole algorithm (SAMDP)*, introduced in [4,5]. Assume that $\lambda E - A$ has only simple eigenvalues λ_k with left and right eigenvectors y_k and x_k, normalized such that $y_k^H E x_k = 1$. Then

$$G(s) = \sum_{k=1}^{n} \frac{R_k}{s - \lambda_k} + R_\infty \quad \text{with the } \textit{residues} \quad R_k = C x_k y_k^H B, \quad R_\infty = \lim_{\omega \to \infty} G(\mathrm{i}\omega).$$

Then, $\|R_k\|_2 = \|Cx_k\|_2 \left\|B^H y_k\right\|_2$ is a measure for simultaneous controllability and observability of λ_k. We observe that if λ_j is close to the imaginary axis and $\|R_j\|_2$ is large, then for $\omega \approx \mathrm{Im}(\lambda_j)$ it holds

$$G(\mathrm{i}\omega) \approx \frac{R_j}{-\mathrm{Re}(\lambda_j)} + \sum_{\substack{k=1 \\ k \neq j}}^{n} \frac{R_k}{\mathrm{i}\omega - \lambda_k} + R_\infty,$$

and therefore $\|G(\mathrm{i}\omega)\|_2$ is large, too. Therefore we call an eigenvalue $\lambda_j \in \Lambda_{\mathrm{f}}(E, A)$ *dominant pole* of $G(s)$, if

$$\frac{\|R_k\|_2}{|\mathrm{Re}(\lambda_k)|} < \frac{\|R_j\|_2}{|\mathrm{Re}(\lambda_j)|}, \quad k = 1, \ldots, n, \quad k \neq j. \tag{6}$$

The most dominant poles can be determined by SAMDP and are essentially what we are looking for. However, we also deal with positive structured pseudospectral abscissae. By using the definition (6), the eigenvalues tend to loose dominance as soon as they have crossed the imaginary axis into the right half-plane. This is of course an undesired behavior since this could lead to convergence problems when the rightmost pseudoeigenvalue is "far" in the right half-plane. Therefore we use an alternative dominance measure. We call an eigenvalue $\lambda_j \in \Lambda_{\mathrm{f}}(E, A)$ *exponentially dominant pole* of $G(s)$, if

$$\|R_k\|_2 \exp(\beta\, \mathrm{Re}(\lambda_k)) < \|R_j\|_2 \exp(\beta\, \mathrm{Re}(\lambda_j)), \quad k = 1, \ldots, n, \quad k \neq j.$$

The parameter β is a weighting factor which defines the trade-off between the influence of the residue and real part of the eigenvalues. In our numerical experiments it turned out that $\beta = 100$ is a good choice for many examples (high weight on the real part). Since SAMDP delivers the poles which generate the peaks in the frequency response of a system and due to the relation (3), we can also determine very good initial estimates for $r_{\mathbb{C}}^{\mathrm{f}}(E, A, B, C)$.

2.3 Algorithmic Details

In this subsection we present some pseudocode of the derived method. Algorithm 1 summarizes the procedure for the computation of the structured ε-pseudospectral abscissa. In our implementation of the algorithm we initially use estimates of the eigenvectors x and y that are used to construct the optimal perturbation to accelerate the computations. This is possible since we can take the eigenvectors returned by the previous $\alpha(\varepsilon)$-evaluations.

We mention the drawback that the algorithm not necessarily converges to the globally rightmost value on the boundary of the ε-pseudospectrum $\partial\Pi_\varepsilon(E, A, B, C)$. Mostly it does but in some rare situations the algorithm converges only to a local maximizer. This especially happens in the first iteration of the root-finding algorithm when no good estimates of the optimal eigenvectors are available. Therefore, sometimes one has to try several dominant poles to find the global maximizer in the beginning. To find the root of $\alpha(\varepsilon)$ we can use any root-finding algorithm that does not require derivative information.

3 Numerical Results

In this section we briefly present the results for one particular example (`M20PI_n`) taken from Joost Rommes' website[1]. The example has $n = 1182$ states and $m = p = 3$ inputs and outputs. We implemented the method for different root-finding schemes. Table 1 summarizes the results for the secant method. It lists all iterates of Algorithm 1 for each iteration of the root finder. The corresponding transfer function with the computed $\mathcal{H}_\infty$-norm is depicted in Figure 1.

[1] `http://sites.google.com/site/rommes/software/`

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Algorithm 1 Computation of the structured pseudospectral abscissa

Input: System $\Sigma = (\lambda E - A, B, C)$, perturbation level ε, tolerance on relative change τ.
Output: $\alpha_\varepsilon(E, A, B, C)$.

1: Compute the exponentially dominant pole λ_0 of $(\lambda E - A, B, C)$ with left and right eigenvectors y_0 and x_0.
2: Compute the perturbation $\hat{A} = A + \varepsilon \frac{BB^T y_0 x_0^H C^T C}{\|B^T y_0\|_2 \|C x_0\|_2}$.
3: **for** $j = 1, 2, \ldots$ **do**
4: Compute the exponentially dominant pole λ_j of $(\lambda E - \hat{A}, B, C)$ with left and right eigenvectors y_j and x_j.
5: **if** $|\mathrm{Re}\,(\lambda_j) - \mathrm{Re}\,(\lambda_{j-1})| < \tau\,|\mathrm{Re}\,(\lambda_j)|$ **then**
6: Set $k = j$.
7: Break.
8: **end if**
9: Compute the perturbation $\hat{A} = A + \varepsilon \frac{BB^T y_j x_j^H C^T C}{\|B^T y_j\|_2 \|C x_j\|_2}$.
10: **end for**
11: $\alpha_\varepsilon(E, A, B, C) = \mathrm{Re}\,(\lambda_k)$.

Table 1 Convergence history for the `M20PI_n` example using the secant method

	k			
	1	2	3	4
$\mathrm{Re}(\lambda_{\mathrm{dom}})$	-6.7945e-02	-6.0215e+00	-3.7397e-04	3.6361e-11
	2.3140e-03	-6.0212e+00	-3.4533e-05	3.9060e-11
	3.0285e-03	—	-3.2591e-05	3.8927e-11
	3.0355e-03	—	-3.2572e-05	—
	3.0356e-03	—	—	—
ε_k	2.582502e-01	2.066001e-01	2.582241e-01	2.582244e-01

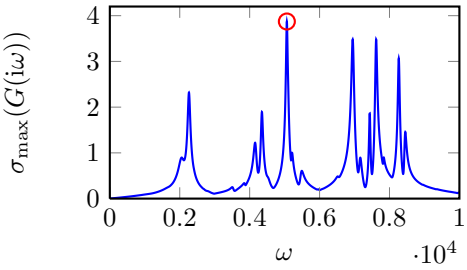

Fig. 1 Transfer function plot for the `M20PI_n` example with computed $\mathcal{H}_\infty$-norm (red circle)

In Table 2, the results for different root-finding algorithms are listed where n_{outer} denotes the number of $\alpha(\varepsilon)$-evaluations in the root finder and n_{inner} is the total number of iterations of Algorithm 1. The application of the secant method is the fastest. In particular we compare these results with those obtained by some standard software packages such as MATLAB 2010b and SLICOT[2]. In both packages, so-called level-set methods are implemented [6]. Besides the much higher computational complexity, it turns out that MATLAB was not able to compute the correct result for this example.

4 Conclusions

In this paper we have introduced a new iterative scheme for computing the $\mathcal{H}_\infty$-norm of a transfer function. This routine uses the relationship between the $\mathcal{H}_\infty$-norm the structured complex stability radius of a matrix or a pencil. Based on the method introduced in [2], the algorithm computes a sequence of structured pseudospectral abscissae. This is done by computing an optimal rank-1 perturbation of the system such that one of the eigenvalues of the perturbed matrix or pencil converges to the rightmost structured pseudopole of the transfer function.

Table 2 Comparison of different root-finding methods and available software packages

method	$\|G\|_{\mathcal{H}_\infty}$	n_{outer}	n_{inner}	time in s
bisection	3.872601	19	41	17.34
secant method	3.872601	4	15	6.58
false position method	3.872601	6	22	9.49
Illinois method	3.872601	6	20	8.98
Pegasus method	3.872601	5	18	7.93
Müller's method	3.872601	6	20	8.90
Ridders' method	3.872601	9	34	14.39
MATLAB `norm`	**9.924038**	—	—	307.04
SLICOT `AB13HD`	3.872601	—	—	281.36

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[2] `http://www.slicot.org/`

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