Solution of Descriptor Lur'e Equations via Even Matrix Pencils

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Descriptor Lur'e equations are an important tool for the solution of linear-quadratic optimal control problems for differential-algebraic systems. In this article we discuss how one can construct all solutions of these equations by the deflating subspaces of associated even matrix pencils.

1 Introduction

Assume that we have given a descriptor system

$$E\dot{x}(t) = Ax(t) + Bu(t), \tag{1}$$

where $E, A \in \mathbb{K}^{n,n}$, $B \in \mathbb{K}^{n,m}$ are such that the pencil $sE - A \in \mathbb{K}[s]^{n,n}$ is *regular*, i.e., there exists a $\lambda \in \mathbb{C}$ such that $\det(\lambda E - A) \neq 0$. For $Q = Q^* \in \mathbb{K}^{n,n}$, $S \in \mathbb{K}^{n,m}$, $R = R^* \in \mathbb{K}^{m,m}$ we consider *descriptor Lur'e equations* of the form

$$\begin{aligned} A^*X + X^*A + Q &= K^*K + H^*\Sigma H, \\ X^*B + S &= K^*L + H^*\Sigma J, \quad E^*X = X^*E, \\ R &= L^*L + J^*\Sigma J, \end{aligned} \tag{2}$$

which play an important role for optimal control problems of systems of the form (1), see [1]. Let $r = \operatorname{rank} E$ and define the *system space* of (1) by

$$\mathcal{V} := \left\{ \begin{pmatrix} x \\ u \end{pmatrix} \in \mathbb{K}^{n+m} : Ax + Bu \in \operatorname{im} E \right\} \subseteq \mathbb{K}^{n+m}. \tag{3}$$

Then a sextuple (X, K, L, H, J, Σ) consisting of some

a) matrices $X \in \mathbb{K}^{n,n}$, $K \in \mathbb{K}^{p,n}$, $L \in \mathbb{K}^{p,m}$ for some $p \in \mathbb{N}_0$ with $\ker \begin{bmatrix} K & L \end{bmatrix} = \mathcal{V}^{\perp}$;

b) matrices $H \in \mathbb{K}^{n-r,n}$, $J \in \mathbb{K}^{n-r,m}$ with $\ker \begin{bmatrix} H & J \end{bmatrix} = \mathcal{V}$;

c) a signature matrix $\Sigma \in \mathbb{R}^{n-r,n-r}$ (that is, $\Sigma = \operatorname{diag}(-I_{p_1}, I_{p_2})$ for some $p_1, p_2 \in \mathbb{N}_0$)

is called *solution of the descriptor Lur'e equation* (2), if it fulfills (2) and $\operatorname{rank}_{\mathbb{K}(s)} \begin{bmatrix} -sE + A & B \\ K & L \end{bmatrix} = n + p$. In this article we study the relationship of (2) to even *matrix pencils* of the form

$$s\mathcal{E} - \mathcal{A} = \begin{bmatrix} 0 & -sE + A & B \\ sE^* + A^* & Q & S \\ B^* & S^* & R \end{bmatrix}, \tag{4}$$

namely we will show how to construct solutions of (2) via deflating subspaces of $s\mathcal{E} - \mathcal{A}$.

2 Main Result

Definition 2.1 a) A matrix Y is called a *basis matrix* for a subspace $\mathcal{Y} \subset \mathbb{K}^N$ if it has full column rank and $\operatorname{im} Y = \mathcal{Y}$.

b) A subspace $\mathcal{Y} \subset \mathbb{C}^N$ is called *deflating subspace* for the pencil $s\mathcal{E} - \mathcal{A} \in \mathbb{C}[s]^{N,N}$ if, for a basis matrix $Y \in \mathbb{C}^{N,k}$ of $\mathcal{Y}$, there exists some $l \in \mathbb{N}_0$, a matrix $Z \in \mathbb{C}^{N,l}$ and a pencil $s\widetilde{\mathcal{E}} - \widetilde{\mathcal{A}} \in \mathbb{C}[s]^{l,k}$ with $\operatorname{rank}_{\mathbb{C}(s)} \big(s\widetilde{\mathcal{E}} - \widetilde{\mathcal{A}}\big) = l$ such that

$$(s\mathcal{E} - \mathcal{A})Y = Z\big(s\widetilde{\mathcal{E}} - \widetilde{\mathcal{A}}\big).$$

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Theorem 2.2 *Let the system $[E, A, B]$, the even matrix pencil $s\mathcal{E} - \mathcal{A}$ as in* (4) *and Popov function*

$$\Phi(s) = \begin{bmatrix} (-\overline{s}E - A)^{-1}B \\ I_m \end{bmatrix}^* \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{bmatrix} (sE - A)^{-1}B \\ I_m \end{bmatrix}$$

be given. Consider the following statements:

1) The descriptor Lur'e equation (2) *is solvable.*

2) It holds $\Phi(\mathrm{i}\omega) \geq 0$ for all $\omega \in \mathbb{R}$ with $\mathrm{i}\omega \notin \sigma(E, A)$, and there exist $Y_1,\ Y_2 \in \mathbb{K}^{n,n+m}$, $Y_3 \in \mathbb{K}^{m,n+m}$, $Z_1,\ Z_2 \in \mathbb{K}^{n,n+p}$, $Z_3 \in \mathbb{K}^{m,n+p}$ such that for

$$Y = \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix}, \qquad Z = \begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \end{bmatrix}, \tag{5}$$

the following holds true:

i) the space $\operatorname{im} Y$ is $n + m$-dimensional and $\mathcal{E}$-neutral;

ii) $\mathcal{V} \subset \operatorname{im} \begin{bmatrix} Y_2 \\ Y_3 \end{bmatrix}$ with $\mathcal{V}$ as in (3)*;*

iii) $\operatorname{rank} EY_2 = r$;

iv) there exist $\tilde{\mathcal{E}}, \tilde{\mathcal{A}} \in \mathbb{K}^{n+p,n+m}$, such that

$$(s\mathcal{E} - \mathcal{A})Y = Z\big(s\tilde{\mathcal{E}} - \tilde{\mathcal{A}}\big). \tag{6}$$

Then the following implications hold:

a) Statement 1) implies 2).

b) If $[E, A, B]$ is impulse controllable, then 2) also implies 1).

In the case where 2) holds true, a solution of the descriptor Lur'e equations exists with $\operatorname{rank}_{\mathbb{K}(s)} \Phi(s) = p$ and there exists a subspace $\operatorname{im} Y$ with $\operatorname{rank} Y_2 = n$ and a matrix $Y_x^- \in \mathbb{K}^{n+m,n}$ with $Y_x Y_x^- = I_n$ such that

$$X = Y_\mu Y_x^-. \tag{7}$$

It remains to check under which conditions on $s\tilde{\mathcal{E}} - \tilde{\mathcal{A}}$ and Y, the deflating subspace $\operatorname{im} Y$ indeed defines a solution of the descriptor Lur'e equation, in particular we check whether iii) is fulfilled. This is summarized in the next theorem.

Theorem 2.3 *Assume that $[E, A, B]$ is impulse controllable and let the descriptor Lur'e equation* (2) *with associated even matrix pencil $s\mathcal{E} - \mathcal{A}$ as in* (4) *be solvable. Let an $n+m$-dimensional $\mathcal{E}$-neutral space $\operatorname{im} Y$ with Y as in* (5) *be given such that* (6) *holds for some $Z \in \mathbb{K}^{2n+m,n+p}$, $\tilde{\mathcal{E}}, \tilde{\mathcal{A}} \in \mathbb{K}^{n+p,n+m}$ and $\mathcal{V} \subset \operatorname{im} \begin{bmatrix} Y_2 \\ Y_3 \end{bmatrix}$ with $\mathcal{V}$ as in* (3)*. If for all generalized eigenvalues λ of the pencil $s\tilde{\mathcal{E}} - \tilde{\mathcal{A}}$, the number $-\overline{\lambda}$ is not an uncontrollable mode of $[E, A, B]$, then $\operatorname{rank} EY_2 = r$.*

3 Summary and Outlook

In this paper we have discussed relations between the solutions of a descriptor Lur'e equation and the deflating subspaces of an associated even matrix pencil. In particular, we have given equivalent conditions for the solvability of the descriptor Lur'e equation and the existence of a deflating subspace. Moreover, we have given a sufficient condition on the subspace that allows to construct a solution from it. In [2], we also give more details on how to construct the solution by a transformation of the even pencil to even Kronecker canonical form [3] and investigate the relation to the optimal control problem. Moreover, the numerical solution of descriptor Lur'e equations is currently under investigation.

References

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