Inner-Outer Factorization via Lur'e Equations

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In this paper we construct inner-outer factorizations of transfer functions governed by linear time-invariant differential-algebraic systems. This construction is based on the solution of certain Lur'e equations. In contrast to previous work we do not assume any condition apart from behavioral stabilizability of the underlying system realization.

1 Introduction

We consider differential-algebraic systems

$$\tfrac{\mathrm{d}}{\mathrm{d}t}Ex(t) = Ax(t) + Bu(t), \tag{1a}$$
$$y(t) = Cx(t) + Du(t), \tag{1b}$$

where $E,\, A \in \mathbb{K}^{n\times n}$ (for $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$) are such that the pencil $sE - A \in \mathbb{K}[s]^{n\times n}$ is *regular*, i. e., $\det(sE - A)$ is not the zero polynomial, and $B \in \mathbb{K}^{n\times m}$, $C \in \mathbb{K}^{p\times n}$, $D \in \mathbb{K}^{p\times m}$. The functions $x : \mathbb{R} \to \mathbb{K}^n$, $u : \mathbb{R} \to \mathbb{K}^m$, and $y : \mathbb{R} \to \mathbb{K}^p$ are called *(generalized) state*, *input*, and *output* of the system, respectively. Then the *transfer function* of (1) is

$$G(s) = C(sE - A)^{-1}B + D \in \mathbb{K}(s)^{p\times m}, \tag{2}$$

where $\mathbb{K}(s)$ denotes the field of rational functions with coefficients in $\mathbb{K}$. Conversely, we call (1) a *realization of* $G(s) \in \mathbb{K}(s)^{p\times m}$ if (2) holds true.

In this paper we discuss the construction of *inner-outer factorizations* of $G(s)$, that is

$$G(s) = G_{\mathrm{i}}(s)G_{\mathrm{o}}(s),$$

where the rational matrix $G_{\mathrm{i}}(s) \in \mathbb{K}(s)^{p\times q}$ is inner and $G_{\mathrm{o}}(s) \in \mathbb{K}(s)^{q\times m}$ is outer. Here we call a rational function $G(s) \in \mathbb{K}(s)^{p\times m}$

(i) *outer* if $p = \operatorname{rank}_{\mathbb{K}(s)} G(s)$ and $G(s)$ has no zeros in $\mathbb{C}_+ := \{s \in \mathbb{C} : \mathrm{Re}\,(s) > 0\}$;

(ii) *inner* if $G(s)$ has no poles in $\mathbb{C}_+$ and $G^*(-\overline{s})G(s) = I_m$.

The above factorization plays an important role, e. g., in $\mathcal{H}_\infty$ control [1].

2 Mathematical Preliminaries

We denote by $\Sigma_{n,m,p}(\mathbb{K})$ the set of systems (1) with $E,\, A \in \mathbb{K}^{n\times n}$ such that the pencil $sE - A \in \mathbb{K}[s]^{n\times n}$ is regular and $B \in \mathbb{K}^{n\times m}$, $C \in \mathbb{K}^{p\times n}$, $D \in \mathbb{K}^{p\times m}$, and we write $[E, A, B, C, D] \in \Sigma_{n,m,p}(\mathbb{K})$. The set of control systems (1a) with E, A and B as above is denoted by $\Sigma_{n,m}(\mathbb{K})$, and we write $[E, A, B] \in \Sigma_{n,m}(\mathbb{K})$.

The *behavior* of $[E, A, B] \in \Sigma_{n,m}(\mathbb{K})$ is the set of all solutions of (1a), that is

$$\mathfrak{B}_{[E,A,B]} := \left\{ (x, u) \in \mathcal{L}^2_{\mathrm{loc}}(\mathbb{R}, \mathbb{K}^n) \times \mathcal{L}^2_{\mathrm{loc}}(\mathbb{R}, \mathbb{K}^m) : \tfrac{\mathrm{d}}{\mathrm{d}t}Ex = Ax + Bu \right\},$$

where $\frac{\mathrm{d}}{\mathrm{d}t}$ denotes the distributional derivative. The behavior of $[E, A, B, C, D] \in \Sigma_{n,m,p}(\mathbb{K})$ is defined by

$$\mathfrak{B}_{[E,A,B,C,D]} := \left\{ (x, u, y) \in \mathfrak{B}_{[E,A,B]} \times \mathcal{L}^2_{\mathrm{loc}}(\mathbb{R}, \mathbb{K}^p) : y = Cx + Du \right\}.$$

Next we consider the notion of behavioral stabilizability [2] which we will directly introduce here in terms of an algebraic characterization [3, Cor. 4.3].

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Definition 2.1 The system $[E, A, B] \in \Sigma_{n,m,p}(\mathbb{K})$ is called *behaviorally stabilizable* if

$$\operatorname{rank} \begin{bmatrix} \lambda E - A & B \end{bmatrix} = n \quad \forall\, \lambda \in \overline{\mathbb{C}_+}.$$

Next we introduce the system space $[E, A, B] \in \Sigma_{n,m}(\mathbb{K})$.

Definition 2.2 Let $[E, A, B] \in \Sigma_{n,m}(\mathbb{K})$ be given. Then the *system space* of $[E, A, B]$ is the smallest subspace $\mathcal{V}^{\mathrm{sys}}_{[E,A,B]} \subseteq \mathbb{K}^{n+m}$ such that for all $(x, u) \in \mathfrak{B}_{[E,A,B]}$ we have

$$\begin{pmatrix} x(t) \\ u(t) \end{pmatrix} \in \mathcal{V}^{\mathrm{sys}}_{[E,A,B]} \quad \text{for almost all } t \in \mathbb{R}.$$

For a geometric characterization of $\mathcal{V}^{\mathrm{sys}}_{[E,A,B]}$ we refer to [4].

3 Construction of Inner-Outer Factorizations

We construct inner-outer factorizations of arbitrary rational matrices. The basis for such a construction will be the Lur'e equation

$$\begin{bmatrix} A^*XE + E^*XA + C^*C & E^*XB + C^*D \\ B^*XE + D^*C & D^*D \end{bmatrix} =_{\mathcal{V}^{\mathrm{sys}}_{[E,A,B]}} \begin{bmatrix} K^* \\ L^* \end{bmatrix} \begin{bmatrix} K & L \end{bmatrix}, \quad X = X^*, \tag{3}$$

where we use

$$M =_{\mathcal{V}} N \quad \Longleftrightarrow \quad v^*(M - N)v = 0 \quad \forall\, v \in \mathcal{V}$$

as a notational convention for Hermitian matrices $M,\ N \in \mathbb{K}^{n \times n}$ and a subspace $\mathcal{V} \subseteq \mathbb{K}^n$.

We are particularly interested in *stabilizing solutions*, i. e., triples $(X, K, L) \in \mathbb{K}^{n \times n} \times \mathbb{K}^{q \times n} \times \mathbb{K}^{q \times m}$ that satisfy (3) and

$$\operatorname{rank} \begin{bmatrix} -\lambda E + A & B \\ K & L \end{bmatrix} = n + q \quad \forall\, \lambda \in \mathbb{C}_+.$$

First we present the general idea for our approach: The *Popov function* corresponding to the Lur'e equation (3) is

$$\Phi(s) = \begin{bmatrix} (-\overline{s}E - A)^{-1}B \\ I_m \end{bmatrix}^* \begin{bmatrix} C^*C & C^*D \\ D^*C & D^*D \end{bmatrix} \begin{bmatrix} (sE - A)^{-1}B \\ I_m \end{bmatrix} = G(-\overline{s})^* G(s). \tag{4}$$

For a stabilizing solution (X, K, L) of (3) we obtain from [4, Rem. 5.7] that $\Phi(s) = W(-\overline{s})^* W(s)$ for the outer function $W(s) = K(sE - A)^{-1}B + L$. Assume that an inner-outer factorization $G(s) = G_{\mathrm{i}}(s)G_{\mathrm{o}}(s)$ with $G_{\mathrm{i}}(s) \in \mathbb{K}(s)^{p \times q}$ and $G_{\mathrm{o}}(s) \in \mathbb{K}(s)^{q \times m}$ exists. Then (4) and the property $G_{\mathrm{i}}(-\overline{s})^* G_{\mathrm{i}}(s) = I_q$ imply that

$$G(-\overline{s})^* G(s) = G_{\mathrm{o}}(-\overline{s})^* G_{\mathrm{i}}(-\overline{s})^* G_{\mathrm{i}}(s) G_{\mathrm{o}}(s) = G_{\mathrm{o}}(-\overline{s})^* G_{\mathrm{o}}(s).$$

This justifies the ansatz $G_{\mathrm{o}}(s) = W(s) = K(sE - A)^{-1}B + L$. The inner factor will be constructed by $G_{\mathrm{i}}(s) = G(s)G_{\mathrm{o}}(s)^-$, where $G_{\mathrm{o}}(s)^-$ denotes a right inverse of $G_{\mathrm{o}}(s)$. Thereby, we will construct a right inverse of $G_{\mathrm{o}}(s)$ by $Z(G_{\mathrm{o}}(s)Z)^{-1}$, where $Z \in \mathbb{R}^{m \times q}$ is a matrix such that $G_{\mathrm{o}}(s)Z$ is invertible.

The following theorem [5] shows that the idea outlined above can indeed be used to construct inner-outer factorizations.

Theorem 3.1 *Let $[E, A, B, C, D] \in \Sigma_{n,m,p}(\mathbb{K})$ behaviorally stabilizable with transfer function $G(s) \in \mathbb{K}(s)^{p \times m}$ and let $q = \operatorname{rank}_{\mathbb{K}(s)} G(s)$. Then there exist a matrix $Z \in \mathbb{K}^{m \times q}$ with $\operatorname{rank}_{\mathbb{K}(s)} G(s)Z = q$ and a stabilizing solution (X, K, L) of the Lur'e equation* (3)*. Further, an inner-outer factorization is given by $G(s) = G_{\mathrm{i}}(s)G_{\mathrm{o}}(s)$, where $G_{\mathrm{i}}(s) \in \mathbb{K}(s)^{p \times q}$ is the transfer function of*

$$[E_{\mathrm{i}}, A_{\mathrm{i}}, B_{\mathrm{i}}, C_{\mathrm{i}}, D_{\mathrm{i}}] := \left[\begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A & BZ \\ K & LZ \end{bmatrix}, \begin{bmatrix} 0 \\ -I_q \end{bmatrix}, \begin{bmatrix} C & DZ \end{bmatrix}, 0_{p \times q} \right] \in \Sigma_{n+q,q,p}(\mathbb{K}),$$

and $G_o(s) \in \mathbb{K}(s)^{q \times m}$ is the transfer function of

$$[E_{\mathrm{o}}, A_{\mathrm{o}}, B_{\mathrm{o}}, C_{\mathrm{o}}, D_{\mathrm{o}}] := [E, A, B, K, L] \in \Sigma_{n,m,q}(\mathbb{K}).$$

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