Measure-valued and weak solutions to the nonlinear peridynamic model in nonlocal elastodynamics

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Version August 1, 2014

Abstract The peridynamic equation of motion is considered for a large class of nonlinear pairwise force functions modeling isotropic microelastic material. Existence of Young-measure-valued solutions to the peridynamic equation of motion is proven if the underlying basic function space is L^p. Moreover, if the underlying basic function space is $W^{\sigma,p}$ $(0 < \sigma < 1)$, existence of weak solutions is shown. The choice of the function space depends on the singularity of the pairwise force function. The results do not rely upon any convexity assumption whatsoever.

Keywords Peridynamics · nonlocal elasticity · nonlinear elasticity · microelastic material · generalised solution · Young measure · existence · Galerkin approximation

Mathematics Subject Classification (2010) MSC 35Q74 · 74B20 · 74H20 · 35L70 · 35R09

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1 Introduction

In 2000, Silling [34] introduced a new elasticity theory to describe the description of mechanical problems in which discontinuities such as cracks are spontaneously formed or in which long-range interaction takes place. This peridynamic theory is based on the nonlocal pairwise interaction of material points in a finite horizon and the replacement of displacement gradients by divided differences.

Over the last decade, peridynamic theory has been studied and developed intensively. The work published ranges from studying concrete applications and test examples via the analysis of numerical methods to the relation to classical local elasticity theory and the question of well-posedness. In the following, we will concentrate on the latter; for a detailed overview on the whole range of peridynamics, we refer to [15]. First results on the well-posedness of a general linear peridynamic model are provided in [18, 19], further results can be found in [12, 13]. In [20], first results on the one-dimensional nonlinear peridynamic model are obtained. In the recent work [16], we show well-posedness of the multidimensional nonlinear peridynamic model for pairwise force functions satisfying a Lipschitz-type continuity condition with respect to the difference in the deformation. Both local-in-time and global-in-time existence of classical solutions are proven. In [6, 23], by using the direct method of the calculus of variations, existence of minimisers to nonlinear peridynamic energy functionals associated to the steady state problem are obtained. We emphasise that in [6], a weak convexity assumption on the integrand of the energy functional is needed if the underlying function space is L^p (this assumption is dropped when allowing strongly singular integrands, i.e., in the $W^{\sigma,p}$ setting). Moreover, in [23], the energy functional itself is assumed to be strictly convex. However, in classical elasticity, convexity of the stored energy function is known to be an unphysical assumption, see [8, Chapter 4.8] and the references cited therein. It conflicts with both the behaviour of the stored energy when the volume of the deformed body is compressed to zero as well as the axiom of material frame-indifference. Even though the application of these arguments to peridynamics still have to be investigated, the results presented in the following do not rely upon any convexity assumption whatsoever. In particular, we neither require polyconvexity, quasiconvexity (see, e.g, [5, 8, 30]) nor Andrews–Ball type assumptions (see [3, 17]), which are often used to compensate the lack of convexity in nonlinear elasticity.

In this paper, we consider the nonlinear peridynamic problem interpreted as a second order evolution equation. The peridynamic operator relies upon nonlinear pairwise force functions modeling isotropic microelastic material. If the pairwise force function has a (at most) weak singularity, then the underlying basic function space is L^p, i.e., the deformation takes values in L^p. Existence of Young-measure-valued solutions to the peridynamic problem is shown by applying a Galerkin approximation and investigating the limit of approximate solutions. Young measures in (visco)elastodynamics have been studied, e.g., in [9, 31, 32]. However, talking about classical nonlinear elasticity theory, mainly gradient Young measures have been considered. Apart from elasticity, Young-measure-valued solutions are obtained, e.g., in [10] and Young measures for nonlocal functionals are studied in [28]. The use of Young measures in the theory of partial differential equations goes back to [37], see also [38]. We also refer to the monographs [26, 29].

Moreover, in the case of a strongly singular pairwise force function, the underlying basic function space is $W^{\sigma,p}$ for $\sigma \in (0,1)$, i.e., the deformation takes values in $W^{\sigma,p}$. We obtain weak solutions of the nonlinear peridynamic problem. Again the proof relies upon a Galerkin discretisation. The main difference to the L^p setting is that by the structure of the peridynamic operator together with the improved regularity of the function space setting, compactness methods can be used to proceed to the limit in the nonlinearity. Note that Proposition 4.2 below is essential for this result, and we believe that the technique we use there is applicable also to other nonlocal problems. Since the idea of peridynamics is to have a suitable model for nonsmooth deformations, the L^p setting seems to be more relevant in practice. Throughout this paper, we focus on $p \geq 2$ although we believe that the case $p < 2$ can be dealt with similarly. Note that in this work, contrary to the results of [16], no (local) Lipschitz continuity of the pairwise force function is assumed.

The paper is organised as follows: In Section 2, we shortly describe the peridynamic model and the equation we consider here. In Section 3, we study the peridynamic model with basic function space L^p. Properties of the peridynamic operator are investigated in a first subsection. Subsequently, in the second subsection, we prove existence of Young-measure-valued solutions to the nonlinear peridynamic problem by studying the limit of Galerkin-approximate solutions. In Section 4, the peridynamic model with strong singularity and basis function space $W^{\sigma,p}$, $\sigma \in (0,1)$, is considered. Again properties of the peridynamic operator are investigated. In contrast to the setting before, after applying a Galerkin approximation, the limit of approximate solutions can be determined in a more regular sense, which results in the existence of a weak solution.

1.1 Notation

Vectors in $\mathbb{R}^d$, $\mathbb{R}^d$-valued functions and their corresponding function spaces are denoted by boldfaced letters. By the dot $\cdot$ the Euclidean inner product and by $|\cdot|$ the Euclidean norm on $\mathbb{R}^d$ are expressed. We rely upon the usual notation for spaces of (absolutely) continuous and Lebesgue integrable functions, for Sobolev and Sobolev–Slobodetskii spaces, and for spaces of Bochner integrable functions. In particular, $(\cdot,\cdot)$ denotes the inner product in L^2 and $\|\cdot\|_{\sigma,p}$ denotes the norm of the Sobolev–Slobodetskii space $W^{\sigma,p}$, whereas $|\cdot|_{\sigma,p}$ denotes its semi–norm. Moreover, by $\mathscr{C}_w([0,T];X)$, we denote the space of weakly continuous functions mapping $[0,T]$ into a Banach space X. The duality pairing is denoted by $\langle\cdot,\cdot\rangle$. The Lebesgue exponent conjugated to $p \in [2,\infty)$ is denoted by $p' = p/(p-1)$. By $c > 0$, we denote a generic positive constant.

2 Peridynamic equation of motion for isotropic microelastic material

In the following short description of the peridynamic model, we follow Silling [34] except that, right from the beginning, we rely the formulation upon the deformation instead of the displacement.

Note that in this formulation, linearised peridynamics is restricted to a Poisson ratio $\nu = 1/4$. Meanwhile, this has been generalised by the state-based peridynamic model (see [35, 36]).

Let $\boldsymbol{y} : \overline{\Omega} \times [0,T] \to \mathbb{R}^d$ ($d \in \{2,3\}$) be the deformation of a body with volume Ω in the reference configuration considered up to time $T > 0$. Throughout this paper, we assume that $\Omega \subset \mathbb{R}^d$ is a bounded domain and has locally Lipschitz continuous boundary $\partial\Omega$. The peridynamic equation of motion can then be written as

$$\rho(\boldsymbol{x})\partial_{tt}\boldsymbol{y}(\boldsymbol{x},t) = \int_{\Omega} \boldsymbol{f}(\hat{\boldsymbol{x}}-\boldsymbol{x},\boldsymbol{y}(\hat{\boldsymbol{x}},t)-\boldsymbol{y}(\boldsymbol{x},t))\,\mathrm{d}\hat{\boldsymbol{x}} + \boldsymbol{b}(\boldsymbol{x},t) \quad \text{for } (\boldsymbol{x},t) \in \Omega \times (0,T),$$

where ρ denotes the mass density, $\boldsymbol{f}$ the pairwise force function, and $\boldsymbol{b}$ the density of external forces. Without loss of generality, we assume $\rho \equiv 1$ in the sequel.

The equation of motion is supplemented by the initial conditions

$$\boldsymbol{y}(\cdot,0) = \boldsymbol{y}_0\,, \quad \partial_t \boldsymbol{y}(\cdot,0) = \boldsymbol{v}_0 \quad \text{in } \Omega\,.$$

Boundary conditions in the classical sense cannot, in general, be imposed because of the lack of spatial derivatives. Nevertheless, volume constraints, e.g., in a strip along the boundary $\partial\Omega$ can be imposed (see [1]).

For an isotropic microelastic material, the pairwise force function is always given by

$$\boldsymbol{f}(\boldsymbol{\xi},\boldsymbol{\zeta}) = a(|\boldsymbol{\xi}|,|\boldsymbol{\zeta}|)\boldsymbol{\zeta}\,, \quad \boldsymbol{\xi},\boldsymbol{\zeta} \in \mathbb{R}^d,$$

where $a : \mathbb{R}^+ \times \mathbb{R}_0^+ \to \mathbb{R}$ is a suitable scalar function. This special form of the pairwise force function is in accordance with the conservation of angular momentum, which implies that the pairwise force function must have the direction $\boldsymbol{\zeta}$ of the new bond, see [34]. Moreover, it takes into account Newton's third law *actio et reactio* as well as the concept of material frame-indifference [27]. Note that $\xi \mapsto a(\xi,\zeta)$ may have a singularity at $\xi = 0$.

For a function $\boldsymbol{y} = \boldsymbol{y}(\boldsymbol{x})$, we then define the peridynamic integral operator

$$\begin{aligned}(K\boldsymbol{y})(\boldsymbol{x}) &= \int_\Omega \boldsymbol{f}(\hat{\boldsymbol{x}}-\boldsymbol{x},\boldsymbol{y}(\hat{\boldsymbol{x}})-\boldsymbol{y}(\boldsymbol{x}))\,\mathrm{d}\hat{\boldsymbol{x}}\\ &= \int_\Omega a(|\hat{\boldsymbol{x}}-\boldsymbol{x}|,|\boldsymbol{y}(\hat{\boldsymbol{x}})-\boldsymbol{y}(\boldsymbol{x})|)\big(\boldsymbol{y}(\hat{\boldsymbol{x}})-\boldsymbol{y}(\boldsymbol{x})\big)\,\mathrm{d}\hat{\boldsymbol{x}}\,. \end{aligned} \tag{2.1}$$

An essential observation is the following nonlocal integration-by-parts rule: for two (sufficiently smooth) functions $\boldsymbol{y} = \boldsymbol{y}(\boldsymbol{x})$ and $\boldsymbol{z} = \boldsymbol{z}(\boldsymbol{x})$ there holds

$$\int_\Omega (K\boldsymbol{y})(\boldsymbol{x})\cdot \boldsymbol{z}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = -\frac{1}{2}\iint\limits_{\Omega\times\Omega} a(|\hat{\boldsymbol{x}}-\boldsymbol{x}|,|\boldsymbol{y}(\hat{\boldsymbol{x}})-\boldsymbol{y}(\boldsymbol{x})|)\big(\boldsymbol{y}(\hat{\boldsymbol{x}})-\boldsymbol{y}(\boldsymbol{x})\big)\cdot\big(\boldsymbol{z}(\hat{\boldsymbol{x}})-\boldsymbol{z}(\boldsymbol{x})\big)\,\mathrm{d}(\hat{\boldsymbol{x}},\boldsymbol{x})\,. \tag{2.2}$$

This immediately follows from elementary manipulations and is proved in the sequel. In case of a strong singularity of a at $\xi = 0$, the nonlinear form on the right-hand side of (2.2) will be the starting point of our analysis.

The concept of the peridynamic horizon is to restrict the nonlocal interaction to material points lying in a ball of radius $\delta > 0$. We, therefore, assume $a(\xi,\zeta) = 0$ for all $\zeta \in \mathbb{R}_0^+$ if $\xi \geq \delta$, i.e., the integration in (2.2) takes place only on $\Omega \times \Omega$ with $|\hat{\boldsymbol{x}}-\boldsymbol{x}| < \delta$. To be precise, the function a indeed depends on the dimension d and the peridynamic horizon δ. However, for the existence results we derive in the sequel, the concept of the peridynamic horizon does not play a role, and one may take δ as large as the interaction takes place over the whole domain, i.e., $\delta \geq \operatorname{diam}\Omega$.

In this paper, $\delta > 0$ is fixed. For linear peridynamics, it has been shown (see [19]) that for smooth deformations, the peridynamic operator can be developed in terms of higher order differential operators and converges towards a second order differential operator (to be precise, towards the Navier–Lamé operator of linear elasticity) for $\delta \to 0$. A similar technique is used in [4] for higher order gradient methods. Due to these observations, peridynamics can be seen as an approximation of a second order differential operator by an integral operator of zeroth order (weak singularity and L^p setting) or by a fractional differential operator of lower than second order (strong singularity and $W^{\sigma,p}$ setting).

A standard example for a pairwise force function is the proportional microelastic material or bond stretch model with

$$a(\xi,\zeta) = \begin{cases} c_{d,\delta}\,\dfrac{\zeta-\xi}{\xi\zeta} & \text{if } 0<\xi<\delta,\, \zeta>0\,, \\ 0 & \text{otherwise.} \end{cases} \tag{2.3}$$

The constant $c_{d,\delta}$ is then proportional to the bulk modulus and $\delta^{-(d+1)}$, which follows from a comparison of the energy density for an isotropic expansion that arises in the classical and in the peridynamic theory (see, e.g., [18]). The name "bond stretch model" comes from the fact that, for a deformation $\boldsymbol{y} = \boldsymbol{y}(\boldsymbol{x})$ and two positions $\boldsymbol{x}, \hat{\boldsymbol{x}}$, the quantity

$$\frac{|\boldsymbol{y}(\hat{\boldsymbol{x}}) - \boldsymbol{y}(\boldsymbol{x})| - |\hat{\boldsymbol{x}} - \boldsymbol{x}|}{|\hat{\boldsymbol{x}} - \boldsymbol{x}|}$$

is called the bond stretch, i.e., the relative change in the length of a bond between two material points $\boldsymbol{x}$ and $\hat{\boldsymbol{x}}$. In [19], it has been suggested to modify this example such that the interaction tends smoothly towards zero when reaching the peridynamic horizon. This modified model then is more suited from the numerical point of view. Unfortunately, the above example will not fit into our framework so far because of the discontinuity of $\boldsymbol{f}$ in $\boldsymbol{\zeta}$ at $\boldsymbol{\zeta} = \boldsymbol{0}$. See also below where further examples from [34] are discussed.

The pairwise micropotential is (up to an additive function only depending on $|\boldsymbol{\xi}|$, which we choose to be zero) given by

$$w(\boldsymbol{\xi},\boldsymbol{\zeta}) = \int_0^{|\boldsymbol{\zeta}|} a(|\boldsymbol{\xi}|, s)s\,\mathrm{d}s\,, \quad \boldsymbol{\xi},\boldsymbol{\zeta} \in \mathbb{R}^d\,(\boldsymbol{\xi} \neq \boldsymbol{0})\,.$$

Note that $\nabla_{\boldsymbol{\zeta}} w(\boldsymbol{\xi},\boldsymbol{\zeta}) = \boldsymbol{f}(\boldsymbol{\xi},\boldsymbol{\zeta})$.

By Φ, we denote the total macroelastic energy functional, which is given by

$$\Phi(\boldsymbol{y}) = \frac{1}{2}\iint\limits_{\Omega\times\Omega} w(\hat{\boldsymbol{x}} - \boldsymbol{x}, \boldsymbol{y}(\hat{\boldsymbol{x}}) - \boldsymbol{y}(\boldsymbol{x}))\,\mathrm{d}(\hat{\boldsymbol{x}},\boldsymbol{x})\,. \tag{2.4}$$

There holds the relation $\Phi'(\boldsymbol{y}) = -K\boldsymbol{y}$, in the sense of the Gâteaux derivative. This shows indeed that K is a potential operator with potential $-\Phi$. We shall, however, not assume that Φ is convex. We also remark that the concepts of generalised convexity (see, e.g, [5, 8, 30]) do not apply here since the energy functional does not depend on the deformation gradient rather than on differences in the deformation. Even though a characterisation of weak lower semi-continuity for nonlocal derivative-free functionals is given in [14], we emphasise that the generalised convexity condition given there is not assumed here (in contrast to [6] for the L^p setting).

We close this section with a final remark on linear peridynamics.

Remark 2.1 In linear peridynamics, the behaviour under small deformations is investigated. Hence, the variable of consideration is, opposed to nonlinear elasticity, the displacement (instead of the deformation). Similarly, instead of considering the difference in the deformation field, one looks at the difference in the displacement field, $\boldsymbol{\eta} := \boldsymbol{\zeta} - \boldsymbol{\xi}$. The pairwise force function $\tilde{\boldsymbol{f}}(\boldsymbol{\xi},\boldsymbol{\eta}) = a(|\boldsymbol{\xi}|, |\boldsymbol{\eta}+\boldsymbol{\xi}|)(\boldsymbol{\eta}+\boldsymbol{\xi})$ is linearised in $\boldsymbol{\eta} \approx \boldsymbol{0}$, i.e., for small deformations there holds

$$\tilde{\boldsymbol{f}}(\boldsymbol{\xi},\boldsymbol{\eta}) \approx \tilde{\boldsymbol{f}}(\boldsymbol{\xi},\boldsymbol{0}) + \nabla_{\boldsymbol{\eta}}\tilde{\boldsymbol{f}}(\boldsymbol{\xi},\boldsymbol{0})\boldsymbol{\eta}\,.$$

The zeroth order term can be taken into the right-hand side $\boldsymbol{b}$ and is therefore not relevant. Due to the structure of the gradient, one arrives at

$$\tilde{\boldsymbol{f}}(\boldsymbol{\xi},\boldsymbol{\eta}) = \lambda(|\boldsymbol{\xi}|)(\boldsymbol{\xi}\otimes\boldsymbol{\xi})\boldsymbol{\eta} = \lambda(|\boldsymbol{\xi}|)(\boldsymbol{\xi}\cdot\boldsymbol{\eta})\boldsymbol{\xi}\,, \quad \tilde{w}(\boldsymbol{\xi},\boldsymbol{\eta}) = \frac{1}{2}\lambda(|\boldsymbol{\xi}|)\,(\boldsymbol{\xi}\cdot\boldsymbol{\eta})^2\,,$$

where $\otimes$ denotes the dyadic product. Here, λ is a scalar function given by the formula $\lambda(\xi) = \partial^{0,1}a(\xi,\xi)/\xi$, where $\partial^{0,1}$ denotes the derivative with respect to the second argument. Note that this type of linearised pairwise force function appears for any pairwise force function describing isotropic microelastic materials (see [34]). Due to the matrix-vector structure, the linearisation does not fit into our framework. Indeed, for the linear problem, $\tilde{\boldsymbol{f}}$ is no longer in the direction of $\boldsymbol{\zeta}$ but of $\boldsymbol{\xi}$ and is a function of the projection $\boldsymbol{\xi}\cdot\boldsymbol{\eta}$. Thus, in that sense, the nonlinear model is not a generalisation of the linear one. Moreover, a coercivity condition of the micropotential such as

$$\tilde{w}(\boldsymbol{\xi},\boldsymbol{\eta}) \geq c\,|\boldsymbol{\eta}|^2 - \alpha(|\boldsymbol{\xi}|) \quad \text{for all } \boldsymbol{\xi},\boldsymbol{\eta}\in\mathbb{R}^d\,,\ |\boldsymbol{\xi}| < \delta\,,$$

for an integrable function α cannot be satisfied for the linearised model. This is seen by considering a sequence of vectors $\{\boldsymbol{\eta}_n\}$ each orthogonal to a given (and fixed) vector $\boldsymbol{\xi}$ with length $|\boldsymbol{\eta}_n| = n\,|\boldsymbol{\xi}|$.

3 Basic function space is L^p

If the function a possesses no singularity or if the singularity of a at $\xi = 0$ is only a weak singularity then we can work in a functional analytic setting based on Lebesgue spaces. More precisely, we make the following assumptions, where $\delta > 0$ is the given peridynamic horizon and $p \in [2,\infty)$ is a fixed number.

Assumptions.

(a1) For all $\zeta \in \mathbb{R}_0^+$, the function $\xi \mapsto a(\xi,\zeta)$, $\mathbb{R}^+ \to \mathbb{R}$, is Lebesgue-measurable.
(a2) For almost all $\xi \in \mathbb{R}^+$, the function $\boldsymbol{\zeta} \to a(\xi,|\boldsymbol{\zeta}|)\boldsymbol{\zeta}$, $\mathbb{R}^d \to \mathbb{R}^d$, is continuous.
(a3) For all $\zeta \in \mathbb{R}_0^+$, there holds $a(\xi,\zeta) = 0$ if $\xi \geq \delta$.
(a4) There exist constants $c_0, c_1, c_2 \geq 0$ and $\gamma_1, \gamma_2 \in [0,d)$ such that for all $\zeta \in \mathbb{R}_0^+$ and almost all $\xi \in (0,\delta)$

$$|a(\xi,\zeta)|\,\zeta \leq c_0 + c_1\xi^{-\gamma_1} + c_2\xi^{-\gamma_2}\zeta^{p-1}\,.$$

(a5) There exist $\mu > 0$ and a nonnegative function $\alpha \in L^1(0,\delta)$ such that for all $\zeta \in \mathbb{R}_0^+$ and almost all $\xi \in (0,\delta)$

$$\int_0^\zeta a(\xi,s)\,s\,\mathrm{d}s \geq \mu\zeta^p - \alpha(\xi)\xi^{-d+1}\,.$$

Due to Assumptions (a1) and (a4), $\xi \mapsto a(\xi,\zeta)$ is integrable on $(0,\delta)$. Note that the constants c_0, c_1, c_2 in Assumption (a4) will depend on the dimension d and the peridynamic horizon δ. This has to be taken into account when considering the limit behaviour for vanishing nonlocality, i.e., if $\delta \to 0$, which is not in the scope of this paper. Moreover, μ may depend on d and δ. Note that due to $\zeta^p\xi^{-\gamma_2} \geq \zeta^p\delta^{-\gamma_2}$, Assumptions (a4) and (a5) do not conflict each other.

Example 3.1 The original bond stretch model given by (2.3) fulfills all the assumptions except the continuity requirement (a2). In particular, with Young's inequality, we find that (a4) is satisfied with $c_1 = 0$, $c_0 = c_2 = c_{d,\delta}$, $\gamma_2 = 1$ and $p = 2$. With respect to Assumption (a5), we observe that

$$\int_0^\zeta \frac{s-\xi}{\xi}\,\mathrm{d}s = \frac{\zeta^2}{2\xi} - \zeta \ge \frac{\zeta^2}{2\delta} - \zeta \ge \frac{\zeta^2}{4\delta} - \delta$$

since $\xi < \delta$ and $\zeta \le \zeta^2/(4\delta) + \delta$ by Young's inequality.

Example 3.2 In [34, p. 185], the example $a(\xi,\zeta) = (\zeta - \xi)^2$ $(\xi < \delta)$ is given. This example fulfills all the assumptions above. In particular, (a4) and (a5) are satisfied with $c_1 = \gamma_2 = 0$, $p = 4$ and α being constant (which is depending on δ).

Example 3.3 In [34, p. 186], the example $w(\boldsymbol{\xi},\boldsymbol{\zeta}) = b(|\boldsymbol{\xi}|)(|\boldsymbol{\zeta}|^2 - |\boldsymbol{\xi}|^2)^2$ $(|\boldsymbol{\xi}| < \delta)$ is given, where $b = b(|\boldsymbol{\xi}|)$ is the shielding function, which is assumed to be nonnegative. This leads to $a(\xi,\zeta) = 4b(\xi)(\zeta^2 - \xi^2)$ $(\xi < \delta)$. With Young's inequality, we find $c > 0$ such that

$$|a(\xi,\zeta)|\,\zeta \le c\,b(\xi)\,\delta^3 + c\,b(\xi)\,\zeta^3$$

for all $\xi \in \mathbb{R}^+$ and $\zeta \in \mathbb{R}_0^+$, as well as

$$w(\boldsymbol{\xi},\boldsymbol{\zeta}) \ge \frac{1}{2}\,b(|\boldsymbol{\xi}|)\,|\boldsymbol{\zeta}|^4 - b(|\boldsymbol{\xi}|)\,|\boldsymbol{\xi}|^4\,.$$

For a suitable nonnegative shielding function, which is bounded from above and from below by positive constants, all the assumptions above are fulfilled.

Example 3.4 By analogy with the p-Laplacian, the example $a(\xi,\zeta) = \zeta^{p-2}$ for $p \in [2,\infty)$ fulfills the assumptions with $c_0 = c_1 = \gamma_2 = 0$, $c_2 = 1$, $\mu = 1/p$ and $\alpha \equiv 0$.

3.1 Properties of the peridynamic operator

The following proposition shows that the peridynamic operator (2.1) is well-defined and satisfies a certain growth condition.

Proposition 3.1 *The peridynamic operator K maps $\boldsymbol{L}^p(\Omega)$ into its dual and satisfies the estimate*

$$\|K\boldsymbol{y}\|_{0,p'} \le c\left(1 + \|\boldsymbol{y}\|_{0,p}^{p-1}\right).$$

Furthermore, $K : \boldsymbol{L}^p(\Omega) \to \boldsymbol{L}^{p'}(\Omega)$ is demicontinuous and the integration-by-parts rule (2.2) *holds for all $\boldsymbol{y},\boldsymbol{z} \in \boldsymbol{L}^p(\Omega)$.*

Proof Let $\boldsymbol{y} \in \boldsymbol{L}^p(\Omega)$. The Assumptions (a1) and (a2) ensure measurability of the integrand in the definition (2.1) of $K\boldsymbol{y}$. Moreover, with Assumptions (a3) and (a4), there holds for almost all $\boldsymbol{x} \in \Omega$

$$\begin{aligned}|(K\boldsymbol{y})(\boldsymbol{x})| &\le c_0\,|H(\boldsymbol{x})| + c_1 \int_{H(\boldsymbol{x})} |\hat{\boldsymbol{x}} - \boldsymbol{x}|^{-\gamma_1}\,\mathrm{d}\hat{\boldsymbol{x}} + c_2 \int_{H(\boldsymbol{x})} |\hat{\boldsymbol{x}} - \boldsymbol{x}|^{-\gamma_2}\,|\boldsymbol{y}(\hat{\boldsymbol{x}}) - \boldsymbol{y}(\boldsymbol{x})|^{p-1}\,\mathrm{d}\hat{\boldsymbol{x}}\\ &=: c_0\,|H(\boldsymbol{x})| + c_1 I_1(\boldsymbol{x}) + c_2 I_2(\boldsymbol{x})\,,\end{aligned}$$

where

$$H(\boldsymbol{x}) := \{\hat{\boldsymbol{x}} \in \Omega : |\hat{\boldsymbol{x}} - \boldsymbol{x}| < \delta\}\,.$$

Note that the volume of $H(\boldsymbol{x})$ can be estimated by the volume of the ball with centre zero and radius δ. The first integral $I_1(\boldsymbol{x})$ on the right-hand side is finite since $\gamma_1 < d$ and can be estimated by

$$I_1(\boldsymbol{x}) \le \int_{B(\boldsymbol{0};\delta)} |\boldsymbol{\xi}|^{-\gamma_1} \, \mathrm{d}\boldsymbol{\xi} \, .$$

For the second integral, we use $|\boldsymbol{y}(\hat{\boldsymbol{x}}) - \boldsymbol{y}(\boldsymbol{x})|^{p-1} \le c(|\boldsymbol{y}(\hat{\boldsymbol{x}})|^{p-1} + |\boldsymbol{y}(\boldsymbol{x})|^{p-1})$ and $\gamma_2 < d$ to obtain

$$I_2(\boldsymbol{x}) \le c\,|\boldsymbol{y}(\boldsymbol{x})|^{p-1} + c \int_{H(\boldsymbol{x})} |\hat{\boldsymbol{x}} - \boldsymbol{x}|^{-\gamma_2} \, |\boldsymbol{y}(\hat{\boldsymbol{x}})|^{p-1} \, \mathrm{d}\hat{\boldsymbol{x}} .$$

To get existence of the second term, we define the measurable function

$$(\hat{\boldsymbol{x}},\boldsymbol{x}) \mapsto |\hat{\boldsymbol{x}} - \boldsymbol{x}|^{-\gamma_2} \, |\boldsymbol{y}(\hat{\boldsymbol{x}})|^{p-1} \, \chi_{(0,\delta)}(|\hat{\boldsymbol{x}} - \boldsymbol{x}|) =: F(\hat{\boldsymbol{x}},\boldsymbol{x}) \, .$$

Here, $\chi_{(0,\delta)}$ denotes the characteristic function of the interval $(0,\delta)$. Since $\gamma_2 < d$ and $\boldsymbol{y} \in \boldsymbol{L}^p(\Omega)$, we observe that

$$\int_\Omega |F(\hat{\boldsymbol{x}},\boldsymbol{x})| \, \mathrm{d}\boldsymbol{x} < \infty \quad \text{as well as} \quad \int_\Omega \int_\Omega |F(\hat{\boldsymbol{x}},\boldsymbol{x})| \, \mathrm{d}\boldsymbol{x} \mathrm{d}\hat{\boldsymbol{x}} < \infty \, .$$

Therefore, by Tonelli's theorem [7, Theorem 4.4], we get $F \in L^1(\Omega \times \Omega)$. This allows us to apply Fubini's theorem [7, Theorem 4.5] to obtain that $\hat{\boldsymbol{x}} \mapsto F(\hat{\boldsymbol{x}},\boldsymbol{x})$ as well as $\boldsymbol{x} \mapsto \int_\Omega F(\hat{\boldsymbol{x}},\boldsymbol{x}) \, \mathrm{d}\hat{\boldsymbol{x}}$ are integrable. Consequently, $K\boldsymbol{y}$ exists and is integrable.

We proceed to show that the estimate asserted holds. The same arguments as above result in the integrability of

$$(\hat{\boldsymbol{x}},\boldsymbol{x}) \mapsto |\hat{\boldsymbol{x}} - \boldsymbol{x}|^{-\gamma_2} \, |\boldsymbol{y}(\hat{\boldsymbol{x}}) - \boldsymbol{y}(\boldsymbol{x})|^p \, \chi_{(0,\delta)}(|\hat{\boldsymbol{x}} - \boldsymbol{x}|) \quad \text{on } \Omega \times \Omega \, .$$

Hence, we obtain with $\gamma_2 = \gamma_2/p + \gamma_2/p'$ and Hölder's inequality the estimate

$$I_2(\boldsymbol{x}) \le \left(\int_{H(\boldsymbol{x})} |\hat{\boldsymbol{x}} - \boldsymbol{x}|^{-\gamma_2} \, \mathrm{d}\hat{\boldsymbol{x}} \right)^{1/p} \left(\int_{H(\boldsymbol{x})} |\hat{\boldsymbol{x}} - \boldsymbol{x}|^{-\gamma_2} \, |\boldsymbol{y}(\hat{\boldsymbol{x}}) - \boldsymbol{y}(\boldsymbol{x})|^p \, \mathrm{d}\hat{\boldsymbol{x}} \right)^{1/p'} ,$$

where the first integral on the right-hand side is again finite since $\gamma_2 < d$. Therefore,

$$\int_\Omega |(K\boldsymbol{y})(\boldsymbol{x})|^{p'} \, \mathrm{d}x \le c + c \iint\limits_{\substack{\Omega \times \Omega \\ |\hat{\boldsymbol{x}} - \boldsymbol{x}| < \delta}} |\hat{\boldsymbol{x}} - \boldsymbol{x}|^{-\gamma_2} \, |\boldsymbol{y}(\hat{\boldsymbol{x}}) - \boldsymbol{y}(\boldsymbol{x})|^p \, \mathrm{d}(\hat{\boldsymbol{x}},\boldsymbol{x}) \, .$$

The assertion now follows using $|\boldsymbol{y}(\hat{\boldsymbol{x}}) - \boldsymbol{y}(\boldsymbol{x})|^p \le c\big(|\boldsymbol{y}(\hat{\boldsymbol{x}})|^p + |\boldsymbol{y}(\boldsymbol{x})|^p\big)$, symmetry and Fubini's theorem since

$$\begin{aligned} \iint\limits_{\substack{\Omega \times \Omega \\ |\hat{\boldsymbol{x}} - \boldsymbol{x}| < \delta}} |\hat{\boldsymbol{x}} - \boldsymbol{x}|^{-\gamma_2} \, |\boldsymbol{y}(\hat{\boldsymbol{x}}) - \boldsymbol{y}(\boldsymbol{x})|^p \, \mathrm{d}(\hat{\boldsymbol{x}},\boldsymbol{x}) &\le c \iint\limits_{\substack{\Omega \times \Omega \\ |\hat{\boldsymbol{x}} - \boldsymbol{x}| < \delta}} |\hat{\boldsymbol{x}} - \boldsymbol{x}|^{-\gamma_2} \, |\boldsymbol{y}(\boldsymbol{x})|^p \, \mathrm{d}(\hat{\boldsymbol{x}},\boldsymbol{x}) \\ &= c \int_\Omega \int_{H(\boldsymbol{x})} |\hat{\boldsymbol{x}} - \boldsymbol{x}|^{-\gamma_2} \, \mathrm{d}\hat{\boldsymbol{x}} \, |\boldsymbol{y}(\boldsymbol{x})|^p \, \mathrm{d}\boldsymbol{x} \end{aligned}$$

and since the inner integral on the right-hand side is again finite because of $\gamma_2 < d$. Putting all together gives the estimate asserted.

Next we show the demicontinuity. Suppose $\{\boldsymbol{y}_n\}$ is a sequence in $\boldsymbol{L}^p(\Omega)$ converging strongly to $\boldsymbol{y}$. Then there exists a subsequence (not relabeled) converging almost everywhere to $\boldsymbol{y}$ that is dominated by a positive function $h \in L^p(\Omega)$,

$$\boldsymbol{y}_n(\boldsymbol{x}) \to \boldsymbol{y}(\boldsymbol{x}) \quad \text{and} \quad |\boldsymbol{y}_n(\boldsymbol{x})| \leq h(\boldsymbol{x}) \quad \text{for almost every } \boldsymbol{x} \in \Omega\,,$$

see, e.g., [7, Theorem 4.9]. For $\boldsymbol{z} \in \boldsymbol{L}^p(\Omega)$ there holds

$$\begin{aligned}\langle K\boldsymbol{y}_n, \boldsymbol{z}\rangle &= \int_\Omega (K\boldsymbol{y}_n)(\boldsymbol{x}) \cdot \boldsymbol{z}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \\ &= \iint\limits_{\Omega\times\Omega} a(|\hat{\boldsymbol{x}}-\boldsymbol{x}|\,, |\boldsymbol{y}_n(\hat{\boldsymbol{x}}) - \boldsymbol{y}_n(\boldsymbol{x})|)\big(\boldsymbol{y}_n(\hat{\boldsymbol{x}}) - \boldsymbol{y}_n(\boldsymbol{x})\big) \cdot \boldsymbol{z}(\boldsymbol{x})\,\mathrm{d}\,(\hat{\boldsymbol{x}},\boldsymbol{x})\,.\end{aligned}$$

Due to Assumption (a3), the integrand of the right-hand side converges almost everywhere on $\Omega \times \Omega$. Moreover, by similar calculations as above, one can construct a dominating $L^1(\Omega\times\Omega)$ function for the integrand by replacing $|\boldsymbol{y}_n(\boldsymbol{x})|$ with $h(\boldsymbol{x})$ appropriately. Lebesgue's theorem on dominated convergence allows us to deduce that

$$\langle K\boldsymbol{y}_n, \boldsymbol{z}\rangle \to \iint\limits_{\Omega\times\Omega} a(|\hat{\boldsymbol{x}}-\boldsymbol{x}|\,, |\boldsymbol{y}(\hat{\boldsymbol{x}}) - \boldsymbol{y}(\boldsymbol{x})|)\big(\boldsymbol{y}(\hat{\boldsymbol{x}}) - \boldsymbol{y}(\boldsymbol{x})\big) \cdot \boldsymbol{z}(\boldsymbol{x})\,\mathrm{d}\,(\hat{\boldsymbol{x}},\boldsymbol{x}) = \langle K\boldsymbol{y}, \boldsymbol{z}\rangle\,.$$

By a standard contradiction argument, the whole sequence converges, which proves the demicontinuity.

It remains to show the integration-by-parts formula. Using the symmetry of a in both of its arguments, by relabeling variables there holds

$$\langle K\boldsymbol{y}, \boldsymbol{z}\rangle = \int_\Omega \int_\Omega a(|\hat{\boldsymbol{x}}-\boldsymbol{x}|\,, |\boldsymbol{y}(\hat{\boldsymbol{x}}) - \boldsymbol{y}(\boldsymbol{x})|)\big(\boldsymbol{y}(\boldsymbol{x}) - \boldsymbol{y}(\hat{\boldsymbol{x}})\big) \cdot \boldsymbol{z}(\hat{\boldsymbol{x}})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\hat{\boldsymbol{x}}\,.$$

Applying Fubini's theorem to change the order of integration yields

$$\langle K\boldsymbol{y}, \boldsymbol{z}\rangle = -\int_\Omega \int_\Omega a(|\hat{\boldsymbol{x}}-\boldsymbol{x}|\,, |\boldsymbol{y}(\hat{\boldsymbol{x}}) - \boldsymbol{y}(\boldsymbol{x})|)\big(\boldsymbol{y}(\hat{\boldsymbol{x}}) - \boldsymbol{y}(\boldsymbol{x})\big) \cdot \boldsymbol{z}(\hat{\boldsymbol{x}})\,\mathrm{d}\hat{\boldsymbol{x}}\,\mathrm{d}\boldsymbol{x}\,.$$

Hence, we have

$$\begin{aligned}\langle K\boldsymbol{y}, \boldsymbol{z}\rangle &= \frac{1}{2}\langle K\boldsymbol{y}, \boldsymbol{z}\rangle + \frac{1}{2}\langle K\boldsymbol{y}, \boldsymbol{z}\rangle \\ &= -\frac{1}{2}\int_\Omega \int_\Omega a(|\hat{\boldsymbol{x}}-\boldsymbol{x}|\,, |\boldsymbol{y}(\hat{\boldsymbol{x}}) - \boldsymbol{y}(\boldsymbol{x})|)\big(\boldsymbol{y}(\hat{\boldsymbol{x}}) - \boldsymbol{y}(\boldsymbol{x})\big) \cdot \big(\boldsymbol{z}(\hat{\boldsymbol{x}}) - \boldsymbol{z}(\boldsymbol{x})\big)\,\mathrm{d}\hat{\boldsymbol{x}}\,\mathrm{d}\boldsymbol{x}\,.\end{aligned}$$

A final application of Fubini's theorem proves the assertion. □

Remark 3.1 As usual, for a function $\boldsymbol{y} : [0,T] \to \boldsymbol{L}^p(\Omega)$, we define $K\boldsymbol{y} : [0,T] \to \boldsymbol{L}^{p'}(\Omega)$ via $[K\boldsymbol{y}](t) := K\boldsymbol{y}(t)$. If $\boldsymbol{y} \in L^p(0,T;\boldsymbol{L}^p(\Omega))$ then $K\boldsymbol{y} \in L^{p'}(0,T;\boldsymbol{L}^{p'}(\Omega))$. This follows from Pettis' theorem (see [11, Theorem 2, p. 42]) together with the demicontinuity of $K : \boldsymbol{L}^p(\Omega) \to \boldsymbol{L}^{p'}(\Omega)$, which ensures Bochner measurability of $K\boldsymbol{y}$, and the growth estimate in the proposition above.

We next show that the peridynamic operator is indeed a potential operator and that the potential satisfies a certain growth condition.

Proposition 3.2 *The total macroelastic energy functional Φ given by* (2.4) *is well-defined on $\boldsymbol{L}^p(\Omega)$ and satisfies the estimate*

$$|\Phi(\boldsymbol{y})| \le c\,\|\boldsymbol{y}\|_{0,p}\left(1+\|\boldsymbol{y}\|_{0,p}^{p-1}\right).$$

Furthermore, the negative peridynamic operator $-K : \boldsymbol{L}^p(\Omega) \to \boldsymbol{L}^{p'}(\Omega)$ is the Gâteaux derivative of $\Phi : \boldsymbol{L}^p(\Omega) \to \mathbb{R}$.

Proof First of all, we observe that the pairwise micropotential is well-defined and that, with Assumptions (a3) and (a4), for almost all $\boldsymbol{\xi} \in \mathbb{R}^d$ with $|\boldsymbol{\xi}| < \delta$ and all $\boldsymbol{\zeta} \in \mathbb{R}^d$

$$|w(\boldsymbol{\xi},\boldsymbol{\zeta})| \le c_0|\boldsymbol{\zeta}| + c_1|\boldsymbol{\xi}|^{-\gamma_1}|\boldsymbol{\zeta}| + c_2\frac{1}{p}|\boldsymbol{\xi}|^{-\gamma_2}|\boldsymbol{\zeta}|^p. \tag{3.1}$$

Let $\boldsymbol{y},\boldsymbol{z} \in \boldsymbol{L}^p(\Omega)$. Measurability of the integrand in the definition of Φ follows from Assumptions (a1) and (a2). With (3.1), Φ can be estimated similarly as in the proof of Proposition 3.1 (note that we need Tonelli's and Fubini's theorem again), and we find

$$|\Phi(\boldsymbol{y})| \le c\left(\|\boldsymbol{y}\|_{0,1} + \|\boldsymbol{y}\|_{0,p}^p\right),$$

which implies the estimate asserted.

By definition of the Gâteaux derivative and using in particular Assumption (a2) as well as Lebesgue's theorem on dominated convergence together with Assumptions (a3) and (a4), we have

$$
\begin{aligned}
&\langle \Phi'(\boldsymbol{y}),\boldsymbol{z}\rangle \\
&= \lim_{\theta\to 0}\frac{1}{\theta}\big(\Phi(\boldsymbol{y}+\theta\boldsymbol{z}) - \Phi(\boldsymbol{y})\big) \\
&= \frac{1}{2}\lim_{\theta\to 0}\frac{1}{\theta}\iint\limits_{\Omega\times\Omega}\big(w(\hat{\boldsymbol{x}}-\boldsymbol{x},\boldsymbol{y}(\hat{\boldsymbol{x}})-\boldsymbol{y}(\boldsymbol{x})+\theta(\boldsymbol{z}(\hat{\boldsymbol{x}})-\boldsymbol{z}(\boldsymbol{x}))) - w(\hat{\boldsymbol{x}}-\boldsymbol{x},\boldsymbol{y}(\hat{\boldsymbol{x}})-\boldsymbol{y}(\boldsymbol{x}))\big)\,\mathrm{d}(\hat{\boldsymbol{x}},\boldsymbol{x}) \\
&= \frac{1}{2}\lim_{\theta\to 0}\frac{1}{\theta}\iint\limits_{\Omega\times\Omega}\int_{|\boldsymbol{y}(\hat{\boldsymbol{x}})-\boldsymbol{y}(\boldsymbol{x})|}^{|\boldsymbol{y}(\hat{\boldsymbol{x}})-\boldsymbol{y}(\boldsymbol{x})+\theta(\boldsymbol{z}(\hat{\boldsymbol{x}})-\boldsymbol{z}(\boldsymbol{x}))|} a(|\hat{\boldsymbol{x}}-\boldsymbol{x}|,s)\,s\,\mathrm{d}s\,\mathrm{d}(\hat{\boldsymbol{x}},\boldsymbol{x}) \\
&= \frac{1}{2}\lim_{\theta\to 0}\iint\limits_{\Omega\times\Omega}\int_0^1 a\big(|\hat{\boldsymbol{x}}-\boldsymbol{x}|, |\boldsymbol{y}(\hat{\boldsymbol{x}})-\boldsymbol{y}(\boldsymbol{x})| + \tilde{\theta}\big(|\boldsymbol{y}(\hat{\boldsymbol{x}})-\boldsymbol{y}(\boldsymbol{x})+\theta(\boldsymbol{z}(\hat{\boldsymbol{x}})-\boldsymbol{z}(\boldsymbol{x}))| - |\boldsymbol{y}(\hat{\boldsymbol{x}})-\boldsymbol{y}(\boldsymbol{x})|\big)\big) \\
&\qquad\qquad \times\Big(|\boldsymbol{y}(\hat{\boldsymbol{x}})-\boldsymbol{y}(\boldsymbol{x})| + \tilde{\theta}\big(|\boldsymbol{y}(\hat{\boldsymbol{x}})-\boldsymbol{y}(\boldsymbol{x})+\theta(\boldsymbol{z}(\hat{\boldsymbol{x}})-\boldsymbol{z}(\boldsymbol{x}))| - |\boldsymbol{y}(\hat{\boldsymbol{x}})-\boldsymbol{y}(\boldsymbol{x})|\big)\Big) \\
&\qquad\qquad\qquad \times\frac{|\boldsymbol{y}(\hat{\boldsymbol{x}})-\boldsymbol{y}(\boldsymbol{x})+\theta(\boldsymbol{z}(\hat{\boldsymbol{x}})-\boldsymbol{z}(\boldsymbol{x}))| - |\boldsymbol{y}(\hat{\boldsymbol{x}})-\boldsymbol{y}(\boldsymbol{x})|}{\theta}\,\mathrm{d}\tilde{\theta}\,\mathrm{d}(\hat{\boldsymbol{x}},\boldsymbol{x}) \\
&= \frac{1}{2}\iint\limits_{\Omega\times\Omega} a(|\hat{\boldsymbol{x}}-\boldsymbol{x}|,|\boldsymbol{y}(\hat{\boldsymbol{x}})-\boldsymbol{y}(\boldsymbol{x})|)\big(\boldsymbol{y}(\hat{\boldsymbol{x}})-\boldsymbol{y}(\boldsymbol{x})\big)\cdot\big(\boldsymbol{z}(\hat{\boldsymbol{x}})-\boldsymbol{z}(\boldsymbol{x})\big)\,\mathrm{d}(\hat{\boldsymbol{x}},\boldsymbol{x}) \\
&= -\langle K\boldsymbol{y},\boldsymbol{z}\rangle.
\end{aligned}
$$

□

Finally, we show that the potential corresponding to the negative peridynamic operator is bounded from below.

Proposition 3.3 *The total macroelastic energy functional $\Phi : \boldsymbol{L}^p(\Omega) \to \mathbb{R}$ is bounded from below and satisfies the following estimates: There exist $\lambda > 0$, $\kappa_0, \kappa_1 \geq 0$ such that for all $\boldsymbol{y} \in \boldsymbol{L}^p(\Omega)$*

$$\Phi(\boldsymbol{y}) \geq -\kappa_0 \,, \tag{3.2a}$$

$$\Phi(\boldsymbol{y}) \geq \lambda \, \|\boldsymbol{y}\|_{0,p}^p - \kappa_1 \left(1 + \|\boldsymbol{y}\|_{0,1}^p\right) . \tag{3.2b}$$

In order to prove the second inequality, we need the following lemma.

Lemma 3.1 *Let $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^d$ for $d \in \mathbb{N}$. Then for $p \in [2, \infty)$ there holds the estimate*

$$|\boldsymbol{a}-\boldsymbol{b}|^p \geq \frac{1}{2}\big(|\boldsymbol{a}|^p + |\boldsymbol{b}|^p\big) - \frac{p}{2}\left(|\boldsymbol{a}|^{p-1}|\boldsymbol{b}| + |\boldsymbol{b}|^{p-1}|\boldsymbol{a}|\right).$$

A proof can be found in the Appendix.

Proof (of Proposition 3.3) With the definition of Φ, the definition of the pairwise micropotential w, and Assumptions (a3) and (a5), we immediately find for all $\boldsymbol{y} \in \boldsymbol{L}^p(\Omega)$

$$\begin{aligned}
\Phi(\boldsymbol{y}) &\geq -\frac{1}{2} \iint\limits_{\substack{\Omega\times\Omega \\ |\hat{\boldsymbol{x}}-\boldsymbol{x}|<\delta}} \alpha(|\hat{\boldsymbol{x}}-\boldsymbol{x}|)|\hat{\boldsymbol{x}}-\boldsymbol{x}|^{-d+1} \,\mathrm{d}(\hat{\boldsymbol{x}},\boldsymbol{x}) \\
&\geq -\frac{1}{2}\int_\Omega \int_{B(\boldsymbol{x};\delta)} \alpha(|\hat{\boldsymbol{x}}-\boldsymbol{x}|)|\hat{\boldsymbol{x}}-\boldsymbol{x}|^{-d+1} \,\mathrm{d}\hat{\boldsymbol{x}}\,\mathrm{d}\boldsymbol{x} =: -\kappa_0 \,,
\end{aligned}$$

where $B(\boldsymbol{x};\delta) \subset \mathbb{R}^d$ denotes the Euclidean ball of radius δ with origin $\boldsymbol{x}$. Since $\alpha \in L^1(0,\delta)$, the inner integral is finite. This proves the first assertion.

For the second estimate, we obtain for all $\boldsymbol{y} \in \boldsymbol{L}^p(\Omega)$

$$\Phi(\boldsymbol{y}) \geq \frac{\mu}{2} \iint\limits_{\substack{\Omega\times\Omega \\ |\hat{\boldsymbol{x}}-\boldsymbol{x}|<\delta}} |\boldsymbol{y}(\hat{\boldsymbol{x}}) - \boldsymbol{y}(\boldsymbol{x})|^p \,\mathrm{d}(\hat{\boldsymbol{x}},\boldsymbol{x}) - \kappa_0.$$

Note that the double integral on the right-hand side can be estimated from above by $\|\boldsymbol{y}\|_{0,p}^p$ but it is not possible to prove that the double integral can be estimated from below by $\|\boldsymbol{y}\|_{0,p}^p$ (which would be possible if one would prescribe the values of the deformation on a set of nonzero measure, see [1]). However, applying Lemma 3.1 to the integrand yields

$$\begin{aligned}
\iint\limits_{\substack{\Omega\times\Omega \\ |\hat{\boldsymbol{x}}-\boldsymbol{x}|<\delta}} |\boldsymbol{y}(\hat{\boldsymbol{x}}) - \boldsymbol{y}(\boldsymbol{x})|^p \,\mathrm{d}(\hat{\boldsymbol{x}},\boldsymbol{x}) &\geq \frac{1}{2} \iint\limits_{\substack{\Omega\times\Omega \\ |\hat{\boldsymbol{x}}-\boldsymbol{x}|<\delta}} \big(|\boldsymbol{y}(\hat{\boldsymbol{x}})|^p + |\boldsymbol{y}(\boldsymbol{x})|^p\big) \,\mathrm{d}(\hat{\boldsymbol{x}},\boldsymbol{x}) \\
&\quad - \frac{p}{2} \iint\limits_{\substack{\Omega\times\Omega \\ |\hat{\boldsymbol{x}}-\boldsymbol{x}|<\delta}} \left(|\boldsymbol{y}(\hat{\boldsymbol{x}})|^{p-1}|\boldsymbol{y}(\boldsymbol{x})| + |\boldsymbol{y}(\boldsymbol{x})|^{p-1}|\boldsymbol{y}(\hat{\boldsymbol{x}})|\right) \mathrm{d}(\hat{\boldsymbol{x}},\boldsymbol{x}) \\
&\geq \Lambda \, \|\boldsymbol{y}\|_{0,p}^p - p\,\|\boldsymbol{y}\|_{0,p-1}^{p-1}\,\|\boldsymbol{y}\|_{0,1} \,,
\end{aligned}$$

where $\Lambda := \inf_{\boldsymbol{x}\in\Omega}|H(\boldsymbol{x})|$ is positive since Ω is a Lipschitz domain and thus satisfies the cone property. By Hölder's and Young's inequality the assertion follows. □

3.2 Existence of measure-valued solutions

In what follows, we identify the $\mathbb{R}^d$-valued deformation $\boldsymbol{y} = \boldsymbol{y}(\boldsymbol{x},t)$ with the $\boldsymbol{L}^p(\Omega)$-valued abstract function $\boldsymbol{y} = \boldsymbol{y}(t)$ via $[\boldsymbol{y}(t)](\boldsymbol{x}) := \boldsymbol{y}(\boldsymbol{x},t)$. In this way, the peridynamic model can be described in terms of the abstract evolution equation

$$\boldsymbol{y}'' - K\boldsymbol{y} = \boldsymbol{b} \quad \text{in } (0,T)\,, \tag{3.3a}$$

supplemented by initial conditions

$$\boldsymbol{y}(0) = \boldsymbol{y}_0 \quad \text{and} \quad \boldsymbol{y}'(0) = \boldsymbol{v}_0\,. \tag{3.3b}$$

Here, $'$ denotes the weak time derivative. In this section, we prove the existence of Young-measure-valued solution.

Definition 3.1 Let an external force density $\boldsymbol{b} \in L^1(0,T;\boldsymbol{L}^2(\Omega))$ and initial data $\boldsymbol{y}_0 \in \boldsymbol{L}^p(\Omega)$, $\boldsymbol{v}_0 \in \boldsymbol{L}^2(\Omega)$ be given. A function $\boldsymbol{y} : [0,T] \to \boldsymbol{L}^p(\Omega)$ with

$$\begin{aligned} \boldsymbol{y} &\in L^\infty(0,T;\boldsymbol{L}^p(\Omega))\,, \\ \boldsymbol{y}' &\in L^\infty(0,T;\boldsymbol{L}^2(\Omega))\,, \\ \boldsymbol{y}'' &\in L^1(0,T;\boldsymbol{L}^{p'}(\Omega)) \end{aligned}$$

is called generalised solution to the peridynamic problem under Assumptions (a1)–(a5) if there exists a Young measure $\nu = \{\nu_{\boldsymbol{x},t}\}_{(\boldsymbol{x},t)\in\Omega\times(0,T)}$ such that

$$\boldsymbol{y}'' - \overline{\boldsymbol{K}} = \boldsymbol{b} \quad \text{in } L^1(0,T;\boldsymbol{L}^{p'}(\Omega))\,,$$

where $\overline{\boldsymbol{K}} \in L^{p'}(0,T;\boldsymbol{L}^{p'}(\Omega))$ with

$$\boldsymbol{y}(\boldsymbol{x},t) = \int_{\mathbb{R}^d} \boldsymbol{\lambda}\,\mathrm{d}\nu_{\boldsymbol{x},t}(\boldsymbol{\lambda}) \quad \text{a.e. in } \Omega\times(0,T)\,, \tag{3.4a}$$

$$[\overline{\boldsymbol{K}}(t)](\boldsymbol{x}) = \int_\Omega \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} a(|\hat{\boldsymbol{x}}-\boldsymbol{x}|,|\hat{\boldsymbol{\lambda}}-\boldsymbol{\lambda}|)\big(\hat{\boldsymbol{\lambda}}-\boldsymbol{\lambda}\big)\,\mathrm{d}\nu_{\hat{\boldsymbol{x}},t}(\hat{\boldsymbol{\lambda}})\,\mathrm{d}\nu_{\boldsymbol{x},t}(\boldsymbol{\lambda})\,\mathrm{d}\hat{\boldsymbol{x}}\,, \tag{3.4b}$$

and $\boldsymbol{y}(0) = \boldsymbol{y}_0$ in $\boldsymbol{L}^p(\Omega)$ as well as $\boldsymbol{y}'(0) = \boldsymbol{v}_0$ in $\boldsymbol{L}^2(\Omega)$.

Remark 3.2 Observe that, since $\boldsymbol{y} \in L^\infty(0,T;\boldsymbol{L}^p(\Omega))$ and $\boldsymbol{y}' \in L^\infty(0,T;\boldsymbol{L}^2(\Omega))$, we have $\boldsymbol{y} \in W^{1,\infty}(0,T;\boldsymbol{L}^2(\Omega)) \subseteq \mathscr{AC}([0,T];\boldsymbol{L}^2(\Omega))$. Thus, by [25, Chapitre 3, Lemme 8.1], it follows $\boldsymbol{y} \in \mathscr{C}_w([0,T];\boldsymbol{L}^p(\Omega))$. Repeating this argument, since $\boldsymbol{y}' \in L^\infty(0,T;\boldsymbol{L}^2(\Omega))$ and $\boldsymbol{y}'' \in L^1(0,T;\boldsymbol{L}^{p'}(\Omega))$, we have $\boldsymbol{y}' \in W^{1,1}(0,T;\boldsymbol{L}^{p'}(\Omega)) \subseteq \mathscr{AC}([0,T];\boldsymbol{L}^{p'}(\Omega))$. Thus, again by [25, Chapitre 3, Lemme 8.1], it follows $\boldsymbol{y}' \in \mathscr{C}_w([0,T];\boldsymbol{L}^2(\Omega))$.

Theorem 3.1 *If $\boldsymbol{b} \in L^1(0,T;\boldsymbol{L}^2(\Omega))$, $\boldsymbol{y}_0 \in \boldsymbol{L}^p(\Omega)$ and $\boldsymbol{v}_0 \in \boldsymbol{L}^2(\Omega)$ then there exists a generalised solution in the sense of Definition 3.1 to the peridynamic problem.*

Proof We divide the proof into various steps which arise naturally when applying a Galerkin approximation.

Step 1: Galerkin approximation. Note that we have the Gelfand

$$\boldsymbol{L}^p(\Omega) \overset{d}{\hookrightarrow} \boldsymbol{L}^2(\Omega) \overset{d}{\hookrightarrow} \boldsymbol{L}^{p'}(\Omega), \tag{3.5}$$

where $\overset{d}{\hookrightarrow}$ denotes a dense and continuous embedding. Let $\{\boldsymbol{V}_\ell\}_{\ell\in\mathbb{N}}$ be a Galerkin scheme for the separable Banach space $\boldsymbol{L}^p(\Omega)$, i.e.,

$$\boldsymbol{V}_\ell = \operatorname{span}\{\boldsymbol{\varphi}_1,\ldots,\boldsymbol{\varphi}_\ell\}, \quad \overline{\bigcup_{\ell\in\mathbb{N}} \boldsymbol{V}_\ell} = \boldsymbol{L}^p(\Omega), \tag{3.6}$$

with a Galerkin basis $\{\boldsymbol{\varphi}_j\}_{j=1}^{\ell}$. Then, problem (3.3) can be approximated to find a function $\boldsymbol{y}_\ell : [0,T] \to \boldsymbol{V}_\ell$ such that

$$\langle \boldsymbol{y}_\ell''(t), \boldsymbol{z}_\ell\rangle - \langle K\boldsymbol{y}_\ell(t), \boldsymbol{z}_\ell\rangle = (\boldsymbol{b}(t), \boldsymbol{z}_\ell) \quad \text{for all } \boldsymbol{z}_\ell \in \boldsymbol{V}_\ell, \text{ a.e. in } (0,T), \tag{3.7a}$$

$$\boldsymbol{y}_\ell(0) = \boldsymbol{y}_{0,\ell}, \tag{3.7b}$$

$$\boldsymbol{y}_\ell'(0) = \boldsymbol{v}_{0,\ell}. \tag{3.7c}$$

In (3.7b) and (3.7c), the initial values are elements of $\boldsymbol{V}_\ell$ such that $\boldsymbol{y}_{0,\ell}$ converges strongly to $\boldsymbol{y}_0$ in $\boldsymbol{L}^p(\Omega)$ and $\boldsymbol{v}_{0,\ell}$ converges strongly to $\boldsymbol{v}_0$ in $\boldsymbol{L}^2(\Omega)$ as $\ell \to \infty$.

Step 2: Existence of an approximate solution. We claim that for fixed $\ell \in \mathbb{N}$ there exists a solution $\boldsymbol{y}_\ell \in \mathscr{C}^1([0,T];\boldsymbol{V}_\ell)$ with $\boldsymbol{y}_\ell'' \in L^1(0,T;\boldsymbol{V}_\ell)$ to the approximate problem (3.7). To prove this, we apply a theorem on maximal solutions in the sense of Carathéodory.

In order to show the claim, we reformulate (3.7) equivalently via the identification

$$\boldsymbol{y}_\ell(t) = \sum_{j=1}^{\ell} y_\ell^j(t)\boldsymbol{\varphi}_j, \quad \boldsymbol{y}_{0,\ell} = \sum_{j=1}^{\ell} y_{0,\ell}^j \boldsymbol{\varphi}_j \quad \boldsymbol{v}_{0,\ell} = \sum_{j=1}^{\ell} v_{0,\ell}^j \boldsymbol{\varphi}_j$$

for functions $y_\ell^j : [0,T] \to \mathbb{R}$ and numbers $y_{0,\ell}^j, v_{0,\ell}^j \in \mathbb{R}$. Defining the vector-valued function $\boldsymbol{Y}_\ell : [0,T] \to \mathbb{R}^\ell$ via $(\boldsymbol{Y}_\ell)_k = y_\ell^k$ as well as the vectors $(\boldsymbol{Y}_{0,\ell})_k = y_{0,\ell}^k$ and $(\boldsymbol{V}_{0,\ell})_k = v_{0,\ell}^k$, the initial value problem (3.7) equivalently reads

$$M\boldsymbol{Y}_\ell''(t) = \boldsymbol{g}\big(t, \boldsymbol{Y}_\ell(t)\big), \tag{3.8a}$$

$$\boldsymbol{Y}_\ell(0) = \boldsymbol{Y}_{0,\ell}, \tag{3.8b}$$

$$\boldsymbol{Y}_\ell'(0) = \boldsymbol{V}_{0,\ell}, \tag{3.8c}$$

as an initial value problem in $\mathbb{R}^\ell$. In this setting, we equip $\mathbb{R}^\ell$ with the norm $\|\boldsymbol{Z}\|_{\mathbb{R}^\ell} := \|\boldsymbol{z}\|_{0,p}$ where $\boldsymbol{z} = \sum_{j=1}^{\ell} z^j \boldsymbol{\varphi}_j \in \boldsymbol{V}_\ell$ and z^j are the components of $\boldsymbol{Z}$. In (3.8), M is an invertible matrix with $M_{j,k} = (\boldsymbol{\varphi}_j, \boldsymbol{\varphi}_k)$ and the k-th component of the right-hand side $\boldsymbol{g} : [0,T] \times \mathbb{R}^\ell \to \mathbb{R}^\ell$ reads

$$g_k(t,\boldsymbol{Z}) := (\boldsymbol{b}(t), \boldsymbol{\varphi}_k) + \langle K\boldsymbol{z}, \boldsymbol{\varphi}_k\rangle.$$

Note that, in view of Proposition 3.1, $\boldsymbol{g}$ is a Carathéodory function (remember that K is demicontinuous) and there holds

$$|g_k(t,\boldsymbol{Z})| \le c\,\|\boldsymbol{\varphi}_k\|_{0,p}\left(1 + \|\boldsymbol{b}(t)\|_{0,2} + \|\boldsymbol{Z}\|_{\mathbb{R}^\ell}^{p-1}\right).$$

Thus, for each compact subset $\mathscr{K} \subset \mathbb{R}^\ell$, there exists a function $m_{\mathscr{K}} \in L^1(0,T)$ such that

$$|g_k(t,\boldsymbol{Z})| \le m_{\mathscr{K}}(t) \quad \text{for almost every } t \in (0,T) \text{ and all } \boldsymbol{Z} \in \mathscr{K}.$$

Now (3.8) can be formulated as a first order system in $\mathbb{R}^{2\ell}$. An application of [22, Theorem 5.2] provides the existence of a maximal solution $\boldsymbol{y}_\ell \in W^{2,1}(I;\boldsymbol{V}_\ell)$ on an interval I. If the solution does not blow up, then $I = (0,T)$ (similar to the arguments in [2, Corollary 7.7]). Due to the a propri estimates proved in the next step, we can exclude blow-ups. This proves the claim.

Step 3: A priori estimates for the approximate solution. Considering (3.7a) with $\boldsymbol{z}_\ell = \boldsymbol{y}_\ell'(t)$ yields

$$\langle \boldsymbol{y}_\ell''(t), \boldsymbol{y}_\ell'(t)\rangle + \langle \boldsymbol{\Phi}'(\boldsymbol{y}_\ell(t)), \boldsymbol{y}_\ell'(t)\rangle \le \|\boldsymbol{b}(t)\|_{0,2} \left\|\boldsymbol{y}_\ell'(t)\right\|_{0,2}.$$

Thus we arrive at

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\boldsymbol{y}_\ell'(t)\right\|_{0,2}^2 + \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\Phi}(\boldsymbol{y}_\ell(t)) \le \|\boldsymbol{b}(t)\|_{0,2}\left\|\boldsymbol{y}_\ell'(t)\right\|_{0,2},$$

since $\boldsymbol{y}_\ell \in W^{2,1}(0,T;\boldsymbol{V}_\ell)$ and since $\boldsymbol{\Phi}' = -K$ is demicontinuous (see also [21, Lemma 4.1, p. 90]). Defining the positive (see Proposition 3.3) function $w(t) := 1 + \left\|\boldsymbol{y}_\ell'(t)\right\|_{0,2}^2 + 2\kappa_0 + 2\boldsymbol{\Phi}(\boldsymbol{y}_\ell(t))$, this yields

$$\frac{1}{2}w'(t) \le \|\boldsymbol{b}(t)\|_{0,2}\sqrt{w(t)}.$$

Dividing by $\sqrt{w(t)}$ and integrating from 0 to t results in the boundedness of $w(t)$ and, for almost all $t \in (0,T)$

$$\left\|\boldsymbol{y}_\ell'(t)\right\|_{0,2}^2 + \boldsymbol{\Phi}(\boldsymbol{y}_\ell(t)) \le c\left(1 + \left\|\boldsymbol{v}_{0,\ell}\right\|_{0,2}^2 + \boldsymbol{\Phi}(\boldsymbol{y}_{0,\ell}) + \|\boldsymbol{b}\|_{L^1(0,T;\boldsymbol{L}^2(\Omega))}^2\right).$$

As convergent sequences, the sequences $\{\boldsymbol{v}_{0,\ell}\} \subset \boldsymbol{L}^2(\Omega)$ and $\{\boldsymbol{y}_{0,\ell}\} \subset \boldsymbol{L}^p(\Omega)$ are bounded. With the boundedness of $\boldsymbol{\Phi}$ (see Proposition 3.2), we find that the right-hand side of the foregoing estimate is bounded. Finally, since $\boldsymbol{\Phi}(\boldsymbol{y}_\ell(t))$ is bounded from below by a constant, see (3.2a), we gain boundedness of $\{\boldsymbol{y}_\ell'\}$ in $L^\infty(0,T;\boldsymbol{L}^2(\Omega))$. Therefore, the sequence $\{\boldsymbol{y}_\ell\}$ is bounded in $L^\infty(0,T;\boldsymbol{L}^1(\Omega))$, and an application of (3.2b) yields the boundedness of $\{\boldsymbol{y}_\ell\}$ also in $L^\infty(0,T;\boldsymbol{L}^p(\Omega))$.

Step 4: Passage to the limit. The above a priori estimates provide the existence of a subsequence (not relabeled) and an element $\boldsymbol{y}$ such that $\{\boldsymbol{y}_\ell\}$ converges weakly* in $L^\infty(0,T;\boldsymbol{L}^p(\Omega))$ to $\boldsymbol{y}$. Moreover, we gain the existence of a subsequence of derivatives $\{\boldsymbol{y}_\ell'\}$ (not relabeled) converging weakly* in $L^\infty(0,T;\boldsymbol{L}^2(\Omega))$ to some element, which by the definition of the weak derivative is shown to be $\boldsymbol{y}'$, such that

$$\boldsymbol{y}_\ell \overset{*}{\rightharpoonup} \boldsymbol{y} \quad \text{in } L^\infty(0,T;\boldsymbol{L}^p(\Omega)),$$
$$\boldsymbol{y}_\ell' \overset{*}{\rightharpoonup} \boldsymbol{y}' \quad \text{in } L^\infty(0,T;\boldsymbol{L}^2(\Omega)).$$

In the following, we would like to pass to the limit in (3.7). For ϕ in $\mathscr{C}_c^\infty(0,T)$, there holds

$$-\int_0^T (\boldsymbol{y}_\ell'(t), \boldsymbol{z}_m)\,\phi'(t)\,\mathrm{d}t - \int_0^T \langle K\boldsymbol{y}_\ell(t), \boldsymbol{z}_m\rangle\,\phi(t)\,\mathrm{d}t = \int_0^T (\boldsymbol{b}(t), \boldsymbol{z}_m)\,\phi(t)\,\mathrm{d}t \tag{3.9}$$

for all $\boldsymbol{z}_m \in \boldsymbol{V}_m$ with $m \in \mathbb{N}$ and $m \le \ell$. Since the sequence of time derivatives $\{\boldsymbol{y}_\ell'\}$ converges weakly* in $L^\infty(0,T;\boldsymbol{L}^2(\Omega))$, the first term of (3.9) converges to $-\int_0^T \langle \boldsymbol{y}'(t), \boldsymbol{z}_m\rangle\,\phi'(t)\,\mathrm{d}t$.

It remains to investigate the peridynamic operator. First, from the observations above, we deduce that the sequence $\{\boldsymbol{y}_\ell\}$ is bounded in $\boldsymbol{L}^p(\Omega \times (0,T))$. By [29, Theorem 6.2],

$\{\boldsymbol{y}_\ell\}$ generates a Young measure $\nu = \{\nu_{\boldsymbol{x},t}\}_{(\boldsymbol{x},t)\in\Omega\times(0,T)}$ such that (3.4a) holds. With [6, Lemma 3.3], which goes back to [28, Proposition 2.3], the sequence $\{\tilde{\boldsymbol{y}}_\ell\}$ defined through

$$\tilde{\boldsymbol{y}}_\ell(\hat{\boldsymbol{x}},\boldsymbol{x},t) := \boldsymbol{y}_\ell(\hat{\boldsymbol{x}},t) - \boldsymbol{y}_\ell(\boldsymbol{x},t)\,, \quad (\hat{\boldsymbol{x}},\boldsymbol{x},t) \in \Omega\times\Omega\times(0,T)\,,$$

generates a Young measure $\tilde{\nu} = \{\tilde{\nu}_{\hat{\boldsymbol{x}},\boldsymbol{x},t}\}_{(\hat{\boldsymbol{x}},\boldsymbol{x},t)\in\Omega\times\Omega\times(0,T)}$ with $\tilde{\nu}_{\hat{\boldsymbol{x}},\boldsymbol{x},t} = \nu_{\hat{\boldsymbol{x}},t} \ominus \nu_{\boldsymbol{x},t}$. In [6, Definition 3.2], the operator $\ominus$ is called *convolution difference of probability measures* (see the reference for a precise definition). However, for any continuous compactly supported function g there holds

$$\int_{\mathbb{R}^d} g(\tilde{\boldsymbol{\lambda}})\,\mathrm{d}(\nu_1\ominus\nu_2)(\tilde{\boldsymbol{\lambda}}) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} g(\boldsymbol{\lambda}_1-\boldsymbol{\lambda}_2)\,\mathrm{d}\nu_1(\boldsymbol{\lambda}_1)\,\mathrm{d}\nu_2(\boldsymbol{\lambda}_2)\,.$$

Second, we introduce the Nemytskii operator $\boldsymbol{F} = \boldsymbol{F}(\boldsymbol{y})$ for $\boldsymbol{y}$ as a function of $\Omega\times(0,T)$ into $\mathbb{R}^d$ by

$$(\boldsymbol{F}\boldsymbol{y})(\hat{\boldsymbol{x}},\boldsymbol{x},t) := \boldsymbol{f}(\hat{\boldsymbol{x}}-\boldsymbol{x},\boldsymbol{y}(\hat{\boldsymbol{x}},t)-\boldsymbol{y}(\boldsymbol{x},t)) = a(|\hat{\boldsymbol{x}}-\boldsymbol{x}|,|\boldsymbol{y}(\hat{\boldsymbol{x}},t)-\boldsymbol{y}(\boldsymbol{x},t)|)\big(\boldsymbol{y}(\hat{\boldsymbol{x}},t)-\boldsymbol{y}(\boldsymbol{x},t)\big)\,.$$

Note that due to Assumptions (a1) and (a2), $\boldsymbol{f}$ is a Carathéodory function. Now, since the sequence $\{\boldsymbol{F}(\boldsymbol{y}_\ell)\}$ is bounded in $\boldsymbol{L}^{p'}(\Omega\times\Omega\times(0,T))$ (this follows from the calculations of Proposition 3.1 and the boundedness of $\{\boldsymbol{y}_\ell\}$ in $L^\infty(0,T;\boldsymbol{L}^p(\Omega))$), a subsequence exists which converges weakly to a certain element $\overline{\boldsymbol{f}} \in \boldsymbol{L}^{p'}(\Omega\times\Omega\times(0,T))$. By the fundamental property of Young measures (see, e.g. [29, Theorem 6.2]), we get

$$\overline{\boldsymbol{f}}(\hat{\boldsymbol{x}},\boldsymbol{x},t) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \boldsymbol{f}(\hat{\boldsymbol{x}}-\boldsymbol{x},\hat{\boldsymbol{\lambda}}-\boldsymbol{\lambda})\,\mathrm{d}\nu_{\hat{\boldsymbol{x}},t}(\hat{\boldsymbol{\lambda}})\,\mathrm{d}\nu_{\boldsymbol{x},t}(\boldsymbol{\lambda}) \quad \text{a.e. on } \Omega\times\Omega\times(0,T)\,.$$

Thus, since $\overline{\boldsymbol{f}} \in \boldsymbol{L}^{p'}(\Omega\times\Omega\times(0,T))$, the abstract function $\overline{\boldsymbol{K}}$ defined through

$$[\overline{\boldsymbol{K}}(t)](\boldsymbol{x}) := \int_\Omega \overline{\boldsymbol{f}}(\hat{\boldsymbol{x}},\boldsymbol{x},t)\,\mathrm{d}\hat{\boldsymbol{x}}\,,$$

see also (3.4b), is an element of $L^{p'}(0,T;\boldsymbol{L}^{p'}(\Omega))$. In conclusion, as $\ell\to\infty$, there holds

$$\int_0^T \langle K\boldsymbol{y}_\ell(t),\boldsymbol{z}_m\rangle\phi(t)\,\mathrm{d}t \to \int_0^T \langle\overline{\boldsymbol{K}}(t),\boldsymbol{z}_m\rangle\phi(t)\,\mathrm{d}t\,.$$

By the limited completeness of the Galerkin scheme (3.6) and since $\mathscr{C}_c^\infty(0,T)\otimes\boldsymbol{L}^p(\Omega)$ is dense with respect to the weak* convergence in $L^\infty(0,T;\boldsymbol{L}^p(\Omega))$, $\boldsymbol{y}\in L^\infty(0,T;\boldsymbol{L}^p(\Omega))$ with $\boldsymbol{y}'\in L^\infty(0,T;\boldsymbol{L}^2(\Omega))$ solves the equation (3.3a). This also shows that the second weak derivative $\boldsymbol{y}''$ exists and is in $L^1(0,T;\boldsymbol{L}^2(\Omega))$.

It remains to show that $\boldsymbol{y}$ satisfies the initial conditions. We observe that due to the convergence of $\boldsymbol{y}_\ell$ above, there holds $\boldsymbol{y}_\ell \rightharpoonup \boldsymbol{y}$ in $W^{1,2}(0,T;\boldsymbol{L}^2(\Omega)) \hookrightarrow \mathscr{C}([0,T];\boldsymbol{L}^2(\Omega))$. Since the trace operator $\Gamma_0 : W^{1,2}(0,T;\boldsymbol{L}^2(\Omega)) \to \boldsymbol{L}^2(\Omega)$, $\Gamma_0\boldsymbol{z} = \boldsymbol{z}(0)$, is linear and bounded and thus weakly-weakly continuous, it follows $\boldsymbol{y}_\ell(0) \rightharpoonup \boldsymbol{y}(0)$ in $\boldsymbol{L}^2(\Omega)$. However, since $\boldsymbol{y}_\ell(0) = \boldsymbol{y}_{0,\ell} \to \boldsymbol{y}_0$ in $\boldsymbol{L}^p(\Omega)$, we deduce $\boldsymbol{y}(0) = \boldsymbol{y}_0$.

Finally, we obtain for all $\boldsymbol{z}_m \in \boldsymbol{V}_m$ with $m \le \ell$ on the one hand

$$\begin{aligned} -(\boldsymbol{y}'(0),\boldsymbol{z}_m) &= \int_0^T \left[(\boldsymbol{y}'(t),\boldsymbol{z}_m)\,\frac{T-t}{T}\right]'\mathrm{d}t \\ &= -\frac{1}{T}\int_0^T (\boldsymbol{y}'(t),\boldsymbol{z}_m)\,\mathrm{d}t + \int_0^T \langle\overline{\boldsymbol{K}}(t),\boldsymbol{z}_m\rangle\,\frac{T-t}{t}\,\mathrm{d}t + \int_0^T (\boldsymbol{b}(t),\boldsymbol{z}_m)\,\frac{T-t}{t}\,\mathrm{d}t\,, \end{aligned}$$

and on the other hand

$$-(\boldsymbol{v}_{0,\ell},\boldsymbol{z}_m) = -\frac{1}{T}\int_0^T (\boldsymbol{y}_\ell'(t),\boldsymbol{z}_m)\,\mathrm{d}t + \int_0^T \langle K\boldsymbol{y}_\ell(t),\boldsymbol{z}_m\rangle \frac{T-t}{t}\,\mathrm{d}t + \int_0^T (\boldsymbol{b}(t),\boldsymbol{z}_m)\,\frac{T-t}{t}\,\mathrm{d}t\,.$$

Passing to the limit in the second equation yields $\boldsymbol{y}'(0) = \boldsymbol{v}_0$.

□

Remark 3.3 Note that if $\boldsymbol{y}_\ell$ can be shown to converge strongly to $\boldsymbol{y}$ in $\boldsymbol{L}^p(\Omega\times(0,T))$, then the Young measure reduces to the concentrated measure $\nu_{\boldsymbol{x},t} = \delta_{\boldsymbol{y}(\boldsymbol{x},t)}$. This follows from the Young-measure property [29, Proposition 6.12]. In that case, one can pass to the limit directly and does not need to use Young measures. Nevertheless, strong convergence cannot be expected in the case of weak singularity since the embeddings in (3.5) are not compact.

4 Basic function space is $W^{\sigma,p}$ $(0<\sigma<1)$

In what follows, we would like to allow a strong singularity of a at $\xi = 0$. The functional analytic framework can then be based on Sobolev–Slobodetskii spaces. We then, however, are left with the question of how to define the peridynamic operator. One suitable way is to start with the nonlinear form given by the right-hand side of (2.2). This is also suited in view of the weak formulation and solutions in the sense of distributions. The choice of a smaller basic function space with more regular functions results in an improved existence theorem.

We make the following assumptions, where $\delta > 0$ is the given peridynamic horizon and $p\in[2,\infty)$ and $\sigma\in(0,1)$ are fixed numbers. Compared to the L^p setting in Section 3, Assumptions (a4) and (a5) are replaced by adapted versions, while (a1)–(a3) remain unchanged.

Assumptions.

(a1) For all $\zeta\in\mathbb{R}_0^+$, the function $\xi\mapsto a(\xi,\zeta)$, $\mathbb{R}^+\to\mathbb{R}$, is Lebesgue-measurable.
(a2) For almost all $\xi\in\mathbb{R}^+$, the function $\boldsymbol{\zeta}\to a(\xi,|\boldsymbol{\zeta}|)\boldsymbol{\zeta}$, $\mathbb{R}^d\to\mathbb{R}^d$, is continuous.
(a3) For all $\zeta\in\mathbb{R}_0^+$, there holds $a(\xi,\zeta)=0$ if $\xi\geq\delta$.
(a4’) There exist constants $c_0,c_1,c_2\geq 0$, $\gamma_1\in[0,d+\sigma)$ and $\gamma_2\in[0,d+\sigma p]$ such that for all $\zeta\in\mathbb{R}_0^+$ and almost all $\xi\in(0,\delta)$

$$|a(\xi,\zeta)|\,\zeta \leq c_0 + c_1\xi^{-\gamma_1} + c_2\xi^{-\gamma_2}\zeta^{p-1}\,.$$

(a5’) There exist $\mu>0$ and a nonnegative function $\alpha\in L^1(0,\delta)$ such that for all $\zeta\in\mathbb{R}_0^+$ and almost all $\xi\in(0,\delta)$

$$\int_0^\zeta a(\xi,s)\,s\,\mathrm{d}s \geq \mu\xi^{-(d+\sigma p)}\zeta^p - \alpha(\xi)\xi^{-d+1}\,.$$

Example 4.1 By analogy with the fractional p-Laplacian, the example $a(\xi,\zeta)=\xi^{-(d+\sigma p)}\zeta^{p-2}$ for $p\in[2,\infty)$ fulfills the assumptions with $c_0=c_1=0$, $c_2=1$, $\gamma_2=d+\sigma p$, $\mu=1/p$ and $\alpha\equiv 0$.

4.1 Properties of the peridynamic operator

Proposition 4.1 *The form $k : \boldsymbol{W}^{\sigma,p}(\Omega) \times \boldsymbol{W}^{\sigma,p}(\Omega) \to \mathbb{R}$ with*

$$k(\boldsymbol{y},\boldsymbol{z}) := -\frac{1}{2} \iint\limits_{\Omega\times\Omega} a(|\hat{\boldsymbol{x}}-\boldsymbol{x}|, |\boldsymbol{y}(\hat{\boldsymbol{x}})-\boldsymbol{y}(\boldsymbol{x})|)\big(\boldsymbol{y}(\hat{\boldsymbol{x}})-\boldsymbol{y}(\boldsymbol{x})\big)\cdot\big(\boldsymbol{z}(\hat{\boldsymbol{x}})-\boldsymbol{z}(\boldsymbol{x})\big)\,\mathrm{d}(\hat{\boldsymbol{x}},\boldsymbol{x}) \tag{4.1}$$

is well-defined and satisfies for all $\boldsymbol{y},\boldsymbol{z} \in \boldsymbol{W}^{\sigma,p}(\Omega)$ the estimate

$$|k(\boldsymbol{y},\boldsymbol{z})| \le c\left(1+|\boldsymbol{y}|_{\sigma,p}^{p-1}\right)|\boldsymbol{z}|_{\sigma,p}\,. \tag{4.2}$$

Furthermore, the form $k : \boldsymbol{W}^{\sigma,p}(\Omega)\times\boldsymbol{W}^{\sigma,p}(\Omega) \to \mathbb{R}$ is continuous in its first and second argument.

Remark 4.1 As usual, we can associate with the form k the operator

$$K : \boldsymbol{W}^{\sigma,p}(\Omega) \to (\boldsymbol{W}^{\sigma,p}(\Omega))^*\,, \quad \langle K\boldsymbol{y},\boldsymbol{z}\rangle := k(\boldsymbol{y},\boldsymbol{z}) \quad \text{for } \boldsymbol{y},\boldsymbol{z}\in\boldsymbol{W}^{\sigma,p}(\Omega)\,. \tag{4.3}$$

This operator, which we still call the peridynamic operator, then satisfies the estimate

$$\|K\boldsymbol{y}\|_{(\boldsymbol{W}^{\sigma,p}(\Omega))^*} \le c\left(1+|\boldsymbol{y}|_{\sigma,p}^{p-1}\right) \tag{4.4}$$

and is demicontinuous.

Suppose that $\boldsymbol{y}\in\boldsymbol{W}^{\sigma p',p-1}(\Omega)$ with $\sigma p > d$ and $\sigma p' < 1$ and that either $c_1 = 0$ or $\gamma_1 < d$. Then $\boldsymbol{W}^{\sigma p',p-1}(\Omega) \hookrightarrow \boldsymbol{W}^{\sigma,p}(\Omega) \hookrightarrow \mathscr{C}(\overline{\Omega})$ and Fubini's theorem shows that the mapping

$$\hat{\boldsymbol{x}} \mapsto a(|\hat{\boldsymbol{x}}-\boldsymbol{x}|, |\boldsymbol{y}(\hat{\boldsymbol{x}})-\boldsymbol{y}(\boldsymbol{x})|)\big(\boldsymbol{y}(\hat{\boldsymbol{x}})-\boldsymbol{y}(\boldsymbol{x})\big)$$

is integrable over $H(\boldsymbol{x})$ for almost all $\boldsymbol{x}\in\Omega$, and we can define the peridynamic operator as in (2.1). Moreover, Fubini's theorem also shows that $K\boldsymbol{y}\in\boldsymbol{L}^1(\Omega)$ for K given by (2.1). With (2.2), we find that

$$\langle \boldsymbol{z}, K\boldsymbol{y}\rangle = k(\boldsymbol{y},\boldsymbol{z}) = \langle K\boldsymbol{y},\boldsymbol{z}\rangle\,,$$

where on the left-hand side we have the operator given by (2.1) and the duality pairing between $\boldsymbol{L}^\infty(\Omega)$ and $\boldsymbol{L}^1(\Omega)$ but on the right-hand side we have the operator $K : \boldsymbol{W}^{\sigma,p}(\Omega) \to (\boldsymbol{W}^{\sigma,p}(\Omega))^*$ given by (4.3) and the duality pairing between $(\boldsymbol{W}^{\sigma,p}(\Omega))^*$ and $\boldsymbol{W}^{\sigma,p}(\Omega)$. This finally proves that under the additional assumptions above, the peridynamic operator (4.3) again possesses the representation (2.1).

Remark 4.2 For a function $\boldsymbol{y} : [0,T] \to \boldsymbol{W}^{\sigma,p}(\Omega)$, we define $K\boldsymbol{y} : [0,T] \to (\boldsymbol{W}^{\sigma,p}(\Omega))^*$ via $[K\boldsymbol{y}](t) := K\boldsymbol{y}(t)$. If $\boldsymbol{y}\in L^p(0,T;\boldsymbol{W}^{\sigma,p}(\Omega))$ then $K\boldsymbol{y}\in L^{p'}(0,T;(\boldsymbol{W}^{\sigma,p}(\Omega))^*)$. This follows again from Pettis' theorem (see [11, Theorem 2, p. 42]) together with the demicontinuity of $K : \boldsymbol{W}^{\sigma,p}(\Omega) \to (\boldsymbol{W}^{\sigma,p}(\Omega))^*$ and (4.4).

Proof (of Proposition 4.1) Let $\boldsymbol{y},\boldsymbol{z}\in\boldsymbol{W}^{\sigma,p}(\Omega)$. Assumptions (a1) and (a2) ensure Lebesgue-measurability of the integrand in the definition of $k(\boldsymbol{y},\boldsymbol{z})$. Moreover with Assumptions (a3) and (a4'), there holds

$$\begin{aligned}
|k(\boldsymbol{y},\boldsymbol{z})| \le{}& c_0 \iint\limits_{\substack{\Omega\times\Omega\\ |\hat{\boldsymbol{x}}-\boldsymbol{x}|<\delta}} |\boldsymbol{z}(\hat{\boldsymbol{x}})-\boldsymbol{z}(\boldsymbol{x})|\,\mathrm{d}(\hat{\boldsymbol{x}},\boldsymbol{x}) + c_1 \iint\limits_{\substack{\Omega\times\Omega\\ |\hat{\boldsymbol{x}}-\boldsymbol{x}|<\delta}} |\hat{\boldsymbol{x}}-\boldsymbol{x}|^{-\gamma_1}|\boldsymbol{z}(\hat{\boldsymbol{x}})-\boldsymbol{z}(\boldsymbol{x})|\,\mathrm{d}(\hat{\boldsymbol{x}},\boldsymbol{x}) \\
&+ c_2 \iint\limits_{\substack{\Omega\times\Omega\\ |\hat{\boldsymbol{x}}-\boldsymbol{x}|<\delta}} |\hat{\boldsymbol{x}}-\boldsymbol{x}|^{-\gamma_2}|\boldsymbol{y}(\hat{\boldsymbol{x}})-\boldsymbol{y}(\boldsymbol{x})|^{p-1}|\boldsymbol{z}(\hat{\boldsymbol{x}})-\boldsymbol{z}(\boldsymbol{x})|\,\mathrm{d}(\hat{\boldsymbol{x}},\boldsymbol{x}) \\
=:{}& c_0 I_0 + c_1 I_1 + c_2 I_2\,.
\end{aligned}$$

We first observe that $I_0 \le c\,|\boldsymbol{z}|_{\sigma,p}$. For the integral I_1, we find with Hölder's inequality

$$I_1 = \iint\limits_{\substack{\Omega\times\Omega\\ |\hat{\boldsymbol{x}}-\boldsymbol{x}|<\delta}} |\hat{\boldsymbol{x}}-\boldsymbol{x}|^{-\gamma_1+(d+\sigma p)/p}\,\frac{|\boldsymbol{z}(\hat{\boldsymbol{x}})-\boldsymbol{z}(\boldsymbol{x})|}{|\hat{\boldsymbol{x}}-\boldsymbol{x}|^{(d+\sigma p)/p}}\,\mathrm{d}\,(\hat{\boldsymbol{x}},\boldsymbol{x})$$

$$\le \Bigg(\iint\limits_{\substack{\Omega\times\Omega\\ |\hat{\boldsymbol{x}}-\boldsymbol{x}|<\delta}} |\hat{\boldsymbol{x}}-\boldsymbol{x}|^{-\gamma_1 p'+(d+\sigma p)p'/p}\,\mathrm{d}\,(\hat{\boldsymbol{x}},\boldsymbol{x})\Bigg)^{1/p'} |\boldsymbol{z}|_{\sigma,p}\,. \tag{4.5}$$

Since $-\gamma_1 p' + (d+\sigma p)p'/p = (-\gamma_1+d+\sigma)p' - d > -d$, the integral on the right-hand side in (4.5) is finite. For I_2, we observe because of $-\gamma_2+d+\sigma p \ge 0$ with Hölder's inequality

$$I_2 = \iint\limits_{\substack{\Omega\times\Omega\\ |\hat{\boldsymbol{x}}-\boldsymbol{x}|<\delta}} |\hat{\boldsymbol{x}}-\boldsymbol{x}|^{-\gamma_2+d+\sigma p}\,\frac{|\boldsymbol{y}(\hat{\boldsymbol{x}})-\boldsymbol{y}(\boldsymbol{x})|^{p-1}}{|\hat{\boldsymbol{x}}-\boldsymbol{x}|^{(d+\sigma p)/p'}}\,\frac{|\boldsymbol{z}(\hat{\boldsymbol{x}})-\boldsymbol{z}(\boldsymbol{x})|}{|\hat{\boldsymbol{x}}-\boldsymbol{x}|^{(d+\sigma p)/p}}\,\mathrm{d}\,(\hat{\boldsymbol{x}},\boldsymbol{x}) \le c\,|\boldsymbol{y}|_{\sigma,p}^{p-1}\,|\boldsymbol{z}|_{\sigma,p}\,. \tag{4.6}$$

This also proves the estimate asserted.

It remains to show the continuity of $k(\cdot,\boldsymbol{z}) : \boldsymbol{W}^{\sigma,p}(\Omega) \to \mathbb{R}$ for arbitrary but fixed $\boldsymbol{z}$ in $\boldsymbol{W}^{\sigma,p}(\Omega)$. Suppose $\{\boldsymbol{y}_n\}$ is a sequence in $\boldsymbol{W}^{\sigma,p}(\Omega)$ converging strongly to $\boldsymbol{y}$. Then there exists a subsequence of $\{\boldsymbol{y}_n\}$ (not relabeled) converging almost everywhere to $\boldsymbol{y}$ on Ω,

$$\boldsymbol{y}_n(\boldsymbol{x}) \to \boldsymbol{y}(\boldsymbol{x}) \quad \text{a.e. on } \Omega\,.$$

Due to the continuity requirement on a (see Assumption (a2)), the integrand of $k(\boldsymbol{y}_n,\boldsymbol{z})$ converges almost everywhere on $\Omega\times\Omega$ to

$$a(|\hat{\boldsymbol{x}}-\boldsymbol{x}|\,,|\boldsymbol{y}(\hat{\boldsymbol{x}})-\boldsymbol{y}(\boldsymbol{x})|)\big(\boldsymbol{y}(\hat{\boldsymbol{x}})-\boldsymbol{y}(\boldsymbol{x})\big)\cdot\big(\boldsymbol{z}(\hat{\boldsymbol{x}})-\boldsymbol{z}(\boldsymbol{x})\big)\,.$$

Moreover, defining

$$\tilde{\boldsymbol{y}}_n(\hat{\boldsymbol{x}},\boldsymbol{x}) := \frac{|\boldsymbol{y}_n(\hat{\boldsymbol{x}})-\boldsymbol{y}_n(\boldsymbol{x})|^p}{|\hat{\boldsymbol{x}}-\boldsymbol{x}|^{d+\sigma p}}\,, \quad \tilde{\boldsymbol{y}}(\hat{\boldsymbol{x}},\boldsymbol{x}) := \frac{|\boldsymbol{y}(\hat{\boldsymbol{x}})-\boldsymbol{y}(\boldsymbol{x})|^p}{|\hat{\boldsymbol{x}}-\boldsymbol{x}|^{d+\sigma p}}\,,$$

the sequence $\{\tilde{\boldsymbol{y}}_n\}$ converges to $\tilde{\boldsymbol{y}}$ in $L^1(\Omega\times\Omega)$. Therefore, there exists a positive function $h \in L^1(\Omega\times\Omega)$ dominating the integrand of $|\boldsymbol{y}_n|_{\sigma,p}$, i.e.,

$$\tilde{\boldsymbol{y}}_n(\hat{\boldsymbol{x}},\boldsymbol{x}) \le h(\hat{\boldsymbol{x}},\boldsymbol{x}) \quad \text{a.e. on } \Omega\times\Omega\,.$$

With h one can construct a dominating $L^1(\Omega\times\Omega)$ function for the integrand $k(\boldsymbol{y}_n,\boldsymbol{z})$ by repeating the calculations above but applying Young's inequality instead of Hölder's inequality in (4.6). Lebesgue's theorem on dominated convergence allows us to deduce that $k(\boldsymbol{y}_n,\boldsymbol{z})$ converges to $k(\boldsymbol{y},\boldsymbol{z})$. By a standard contradiction argument, the whole sequence converges, which proves the assertion.

Finally, the form $k : \boldsymbol{W}^{\sigma,p}(\Omega)\times\boldsymbol{W}^{\sigma,p}(\Omega) \to \mathbb{R}$ is linear and bounded and thus continuous in its second argument. □

It turns out that we also need a different type of continuity result. If we assume a bit more regularity on $\boldsymbol{z}$, then it suffices to assume strong convergence in a weaker space to gain convergence of the peridynamic form k. This observation is crucial to the upcoming existence result.

Proposition 4.2 *Let* $0 < \eta < \min\left\{\frac{1-\sigma}{p-1}, \sigma\right\}$ *be fixed. Then the form given by* (4.1) *is also well-defined on* $\boldsymbol{W}^{\sigma-\eta,p}(\Omega) \times \boldsymbol{W}^{\sigma+\eta(p-1),p}(\Omega)$ *and satisfies for all* $\boldsymbol{y} \in \boldsymbol{W}^{\sigma-\eta,p}(\Omega)$ *and* $\boldsymbol{z} \in \boldsymbol{W}^{\sigma+\eta(p-1),p}(\Omega)$ *the estimate*

$$|k(\boldsymbol{y},\boldsymbol{z})| \leq c\left(1+|\boldsymbol{y}|_{\sigma-\eta,p}^{p-1}\right)|\boldsymbol{z}|_{\sigma+\eta(p-1),p}\,.$$

Furthermore, $k : \boldsymbol{W}^{\sigma-\eta,p}(\Omega) \times \boldsymbol{W}^{\sigma+\eta(p-1),p}(\Omega) \to \mathbb{R}$ *is continuous in its first and second argument.*

Proof First observe that $\sigma-\eta > 0$ and $\sigma+\eta(p-1) < 1$. Now let $\boldsymbol{y} \in \boldsymbol{W}^{\sigma-\eta,p}(\Omega)$ and $\boldsymbol{z} \in \boldsymbol{W}^{\sigma+\eta(p-1),p}(\Omega)$. Following the lines of the proof of Proposition 4.1, it suffices to adapt the estimate on I_2. We deduce

$$I_2 = \iint\limits_{\substack{\Omega\times\Omega\\ |\hat{\boldsymbol{x}}-\boldsymbol{x}|<\delta}} |\hat{\boldsymbol{x}}-\boldsymbol{x}|^{\beta}\,\frac{|\boldsymbol{y}(\hat{\boldsymbol{x}})-\boldsymbol{y}(\boldsymbol{x})|^{p-1}}{|\hat{\boldsymbol{x}}-\boldsymbol{x}|^{(d+(\sigma-\eta)p)/p'}}\,\frac{|\boldsymbol{z}(\hat{\boldsymbol{x}})-\boldsymbol{z}(\boldsymbol{x})|}{|\hat{\boldsymbol{x}}-\boldsymbol{x}|^{(d+(\sigma+\eta(p-1))p)/p}}\,\mathrm{d}(\hat{\boldsymbol{x}},\boldsymbol{x})$$

for

$$\beta := -\gamma_2 + (d+(\sigma-\eta)p)/p' + (d+(\sigma+\eta(p-1))p)/p = -\gamma_2 + d + \sigma p\,. \tag{4.7}$$

Since $\gamma_2 \leq d+\sigma p$, we have $\beta \geq 0$. By Hölder's inequality, it follows

$$I_2 \leq c\,|\boldsymbol{y}|_{\sigma-\eta,p}^{p-1}\,|\boldsymbol{z}|_{\sigma+\eta(p-1),p}\,,$$

which yields well-definition and the estimate asserted.

In order to show continuity of the form k in its first argument, we copy the strategy of Proposition 4.1. Suppose $\{\boldsymbol{y}_n\}$ is a sequence in $\boldsymbol{W}^{\sigma-\eta,p}(\Omega)$ that converges strongly to $\boldsymbol{y} \in \boldsymbol{W}^{\sigma-\eta,p}(\Omega)$ and fix $\boldsymbol{z} \in \boldsymbol{W}^{\sigma+\eta(p-1),p}(\Omega)$. Then there exists a subsequence (not relabeled) and $h \in L^1(\Omega\times\Omega)$ such that

$$\boldsymbol{y}_n(\boldsymbol{x}) \to \boldsymbol{y}(\boldsymbol{x}) \quad \text{a.e. on } \Omega\,, \qquad \frac{|\boldsymbol{y}_n(\hat{\boldsymbol{x}})-\boldsymbol{y}_n(\boldsymbol{x})|^p}{|\hat{\boldsymbol{x}}-\boldsymbol{x}|^{d+(\sigma-\eta)p}} \leq h(\hat{\boldsymbol{x}},\boldsymbol{x}) \quad \text{a.e. on } \Omega\times\Omega\,.$$

First we observe that due to the pointwise convergence and Assumption (a2), the integrand of $k(\boldsymbol{y}_n,\boldsymbol{z})$ converges almost everywhere. The goal is to construct a dominating $L^1(\Omega\times\Omega)$ function in order to pass to the limit with Lebesgue's theorem on dominated convergence. Again by Assumption (a4'), the integrand of the peridynamic form can be estimated by

$$\begin{aligned}
&a(|\hat{\boldsymbol{x}}-\boldsymbol{x}|,|\boldsymbol{y}_n(\hat{\boldsymbol{x}})-\boldsymbol{y}_n(\boldsymbol{x})|)\big(\boldsymbol{y}_n(\hat{\boldsymbol{x}})-\boldsymbol{y}_n(\boldsymbol{x})\big)\cdot\big(\boldsymbol{z}(\hat{\boldsymbol{x}})-\boldsymbol{z}(\boldsymbol{x})\big)\\
&\leq c_0\,|\boldsymbol{z}(\hat{\boldsymbol{x}})-\boldsymbol{z}(\boldsymbol{x})| + c_1\,|\hat{\boldsymbol{x}}-\boldsymbol{x}|^{-\gamma_1}\,|\boldsymbol{z}(\hat{\boldsymbol{x}})-\boldsymbol{z}(\boldsymbol{x})| + c_2\,|\hat{\boldsymbol{x}}-\boldsymbol{x}|^{-\gamma_2}\,|\boldsymbol{y}_n(\hat{\boldsymbol{x}})-\boldsymbol{y}_n(\boldsymbol{x})|^{p-1}\,|\boldsymbol{z}(\hat{\boldsymbol{x}})-\boldsymbol{z}(\boldsymbol{x})|\\
&\qquad\qquad =: c_0 I_0(\hat{\boldsymbol{x}},\boldsymbol{x}) + c_1 I_1(\hat{\boldsymbol{x}},\boldsymbol{x}) + c_2 I_2(\hat{\boldsymbol{x}},\boldsymbol{x})\,.
\end{aligned}$$

The integrands I_0 and I_1 are in $L^1(\Omega\times\Omega)$ (see (4.5)). For the remaining integrand I_2, we observe with Young's inequality (see (4.7))

$$I_2(\hat{\boldsymbol{x}},\boldsymbol{x}) \leq \frac{\delta^\beta}{p'}h(\hat{\boldsymbol{x}},\boldsymbol{x}) + \frac{\delta^\beta}{p}\,\frac{|\boldsymbol{z}(\hat{\boldsymbol{x}})-\boldsymbol{z}(\boldsymbol{x})|^p}{|\hat{\boldsymbol{x}}-\boldsymbol{x}|^{d+(\sigma+\eta(p-1))p}}\,,$$

and the right-hand side is integrable on $\Omega \times \Omega$. Hence with Lebesgue's theorem on dominated convergence, there holds

$$k(\boldsymbol{y}_n, \boldsymbol{z}) \to k(\boldsymbol{y}, \boldsymbol{z}),$$

and by a standard contradiction argument, the whole sequence converges.

Finally, the form $k : \boldsymbol{W}^{\sigma-\eta,p}(\Omega) \times \boldsymbol{W}^{\sigma+\eta(p-1),p}(\Omega) \to \mathbb{R}$ is linear and bounded and thus continuous in its second argument. □

As in the previous section, we can prove that the peridynamic operator is a potential operator and that the potential satisfies a certain growth condition.

Proposition 4.3 *The total macroelastic energy functional Φ is well-defined on $\boldsymbol{W}^{\sigma,p}(\Omega)$ and satisfies the estimate*

$$|\Phi(\boldsymbol{y})| \leq c\, |\boldsymbol{y}|_{\sigma,p} \left(1 + |\boldsymbol{y}|_{\sigma,p}^{p-1}\right).$$

Furthermore, the negative peridynamic operator $-K : \boldsymbol{W}^{\sigma,p}(\Omega) \to (\boldsymbol{W}^{\sigma,p}(\Omega))^*$ is the Gâteaux derivative of $\Phi : \boldsymbol{W}^{\sigma,p}(\Omega) \to \mathbb{R}$.

Proof The first steps are as in the proof of Proposition 3.2. With (3.1), we then find similarly to the arguments in the proof of Proposition 4.1 the asserted estimate. The rest of the proof follows the same lines as that of Proposition 3.2. □

Finally, we show coercivity-type properties of the macroelastic energy functional.

Proposition 4.4 *The total macroelastic energy functional $\Phi : \boldsymbol{W}^{\sigma,p}(\Omega) \to \mathbb{R}$ is bounded from below and satisfies the following estimates: There exist $\lambda > 0$, $\kappa_0, \kappa_1, \kappa_2 \geq 0$ such that for all $\boldsymbol{y} \in \boldsymbol{W}^{\sigma,p}(\Omega)$*

$$\Phi(\boldsymbol{y}) \geq -\kappa_0, \tag{4.8a}$$

$$\Phi(\boldsymbol{y}) \geq \lambda\, \|\boldsymbol{y}\|_{0,p}^p - \kappa_1 \left(1 + \|\boldsymbol{y}\|_{0,1}^p\right), \tag{4.8b}$$

$$\Phi(\boldsymbol{y}) \geq \lambda\, |\boldsymbol{y}|_{\sigma,p}^p - \kappa_2 \left(1 + \|\boldsymbol{y}\|_{0,p}^p\right). \tag{4.8c}$$

Proof The first estimate (4.8a) is proven as in Proposition 3.3. To prove (4.8b), we start with

$$\Phi(\boldsymbol{y}) \geq \frac{\mu}{2} \iint\limits_{\substack{\Omega\times\Omega \\ |\hat{\boldsymbol{x}}-\boldsymbol{x}|<\delta}} \frac{|\boldsymbol{y}(\hat{\boldsymbol{x}}) - \boldsymbol{y}(\boldsymbol{x})|^p}{|\hat{\boldsymbol{x}} - \boldsymbol{x}|^{d+\sigma p}} \,\mathrm{d}(\hat{\boldsymbol{x}}, \boldsymbol{x}) - \kappa_0 \tag{4.9}$$

for all $\boldsymbol{y} \in \boldsymbol{W}^{\sigma,p}(\Omega)$. We observe with $\bar{\boldsymbol{y}}(\boldsymbol{x}) := \frac{1}{|H(\boldsymbol{x})|} \int_{H(\boldsymbol{x})} \boldsymbol{y}(\hat{\boldsymbol{x}}) \,\mathrm{d}\hat{\boldsymbol{x}}$ and $|\bar{\boldsymbol{y}}(\boldsymbol{x})| \leq \frac{1}{\Lambda} \|\boldsymbol{y}\|_{0,1}$ (remember $\Lambda := \inf_{\boldsymbol{x}\in\Omega} |H(\boldsymbol{x})| > 0$) that

$$\begin{aligned}\int_\Omega |\boldsymbol{y}(\boldsymbol{x}) - \bar{\boldsymbol{y}}(\boldsymbol{x})|^p \,\mathrm{d}\boldsymbol{x} &\leq \frac{1}{\Lambda} \int_\Omega \int_{H(\boldsymbol{x})} |\boldsymbol{y}(\hat{\boldsymbol{x}}) - \boldsymbol{y}(\boldsymbol{x})|^p \,\mathrm{d}\hat{\boldsymbol{x}}\,\mathrm{d}\boldsymbol{x} \\ &\leq \frac{\delta^{d+\sigma p}}{\Lambda} \iint\limits_{\substack{\Omega\times\Omega \\ |\hat{\boldsymbol{x}}-\boldsymbol{x}|<\delta}} \frac{|\boldsymbol{y}(\hat{\boldsymbol{x}}) - \boldsymbol{y}(\boldsymbol{x})|^p}{|\hat{\boldsymbol{x}} - \boldsymbol{x}|^{d+\sigma p}} \,\mathrm{d}(\hat{\boldsymbol{x}}, \boldsymbol{x}).\end{aligned}$$

Furthermore, in view of Lemma 3.1, we conclude that

$$\int_\Omega |\boldsymbol{y}(\boldsymbol{x}) - \bar{\boldsymbol{y}}(\boldsymbol{x})|^p \,\mathrm{d}\boldsymbol{x} \geq \frac{1}{2} \|\boldsymbol{y}\|_{0,p}^p - \frac{p}{2\Lambda} \|\boldsymbol{y}\|_{0,p-1}^{p-1} \|\boldsymbol{y}\|_{0,1} - \frac{p}{2\Lambda^{p-1}} \|\boldsymbol{y}\|_{0,1}^p.$$

Combining both inequalities with (4.9), applying Hölder's inequality to $\|\boldsymbol{y}\|_{0,p-1}$ and finally applying Young's inequality yields (4.8b). For the third estimate (4.8c), we start again with (4.9). Applying

$$\iint\limits_{\substack{\Omega\times\Omega \\ |\hat{\boldsymbol{x}}-\boldsymbol{x}|\geq\delta}} \frac{|\boldsymbol{y}(\hat{\boldsymbol{x}})-\boldsymbol{y}(\boldsymbol{x})|^p}{|\hat{\boldsymbol{x}}-\boldsymbol{x}|^{d+\sigma p}}\,\mathrm{d}\,(\hat{\boldsymbol{x}},\boldsymbol{x}) \leq c\,\|\boldsymbol{y}\|_{0,p}^p$$

with another constant c depending on $1/\delta$ proves this assertion.

□

4.2 Existence of weak solutions

We start with a Galerkin approximation and are able to apply compactness methods to handle the limit of the nonlinear term by making use of the regularity of the test function. This is the main difference compared to Theorem 3.1.

Definition 4.1 Let an external force density $\boldsymbol{b}\in L^1(0,T;\boldsymbol{L}^2(\Omega))$ as well as initial data $\boldsymbol{y}_0\in \boldsymbol{W}^{\sigma,p}(\Omega)$, $\sigma\in(0,1)$, and $\boldsymbol{v}_0\in\boldsymbol{L}^2(\Omega)$ be given. A function $\boldsymbol{y}:[0,T]\to\boldsymbol{W}^{\sigma,p}(\Omega)$ with

$$\begin{aligned}\boldsymbol{y}&\in L^\infty(0,T;\boldsymbol{W}^{\sigma,p}(\Omega))\,,\\ \boldsymbol{y}'&\in L^\infty(0,T;\boldsymbol{L}^2(\Omega))\,,\\ \boldsymbol{y}''&\in L^1(0,T;(\boldsymbol{W}^{\sigma,p}(\Omega))^*)\end{aligned}$$

is called weak solution to the peridynamic problem under Assumptions (a1)–(a3), (a4'), (a5') if there holds

$$\boldsymbol{y}''-K\boldsymbol{y}=\boldsymbol{b}\quad\text{in } L^1(0,T;(\boldsymbol{W}^{\sigma,p}(\Omega))^*) \tag{4.10}$$

and $\boldsymbol{y}(0)=\boldsymbol{y}_0$ in $\boldsymbol{W}^{\sigma,p}(\Omega)$ as well as $\boldsymbol{y}'(0)=\boldsymbol{v}_0$ in $\boldsymbol{L}^2(\Omega)$.

Remark 4.3 Since $\boldsymbol{y}\in L^\infty(0,T;\boldsymbol{W}^{\sigma,p}(\Omega))$ as well as $\boldsymbol{y}'\in L^\infty(0,T;\boldsymbol{L}^2(\Omega))$, we conclude that $\boldsymbol{y}\in W^{1,\infty}(0,T;\boldsymbol{L}^2(\Omega))\subseteq\mathscr{AC}([0,T];\boldsymbol{L}^2(\Omega))$. Thus, by [25, Chapitre 3, Lemme 8.1], it follows $\boldsymbol{y}\in\mathscr{C}_w([0,T];\boldsymbol{W}^{\sigma,p}(\Omega))$. Since $\boldsymbol{y}'\in L^\infty(0,T;\boldsymbol{L}^2(\Omega))$ and $\boldsymbol{y}''\in L^1(0,T;(\boldsymbol{W}^{\sigma,p}(\Omega))^*)$, there similarly holds $\boldsymbol{y}'\in W^{1,1}(0,T;(\boldsymbol{W}^{\sigma,p}(\Omega))^*)\subseteq\mathscr{AC}([0,T];(\boldsymbol{W}^{\sigma,p}(\Omega))^*)$. Therefore, again in view of [25, Chapitre 3, Lemme 8.1], it follows $\boldsymbol{y}'\in\mathscr{C}_w([0,T];\boldsymbol{L}^2(\Omega))$.

Theorem 4.1 *If $\boldsymbol{b}\in L^1(0,T;\boldsymbol{L}^2(\Omega))$, $\boldsymbol{y}_0\in\boldsymbol{W}^{\sigma,p}(\Omega)$, $\sigma\in(0,1)$, and $\boldsymbol{v}_0\in\boldsymbol{L}^2(\Omega)$ then there exists a weak solution in the sense of Definition 4.1 to the peridynamic problem.*

Proof The first part of the proof follows almost the same lines as those of the proof of Theorem 3.1.

Step 1: Galerkin approximation. Note that we now have the Gelfand triple

$$\boldsymbol{W}^{\sigma,p}(\Omega)\overset{c,d}{\hookrightarrow}\boldsymbol{L}^2(\Omega)\overset{c,d}{\hookrightarrow}(\boldsymbol{W}^{\sigma,p}(\Omega))^*\,.$$

Here, $\overset{c,d}{\hookrightarrow}$ denotes a dense and compact embedding. Let $\{\boldsymbol{V}_\ell\}_{\ell\in\mathbb{N}}$ be a Galerkin scheme of the separable Banach space $\boldsymbol{W}^{\sigma,p}(\Omega)$, i.e.,

$$\boldsymbol{V}_\ell=\operatorname{span}\{\boldsymbol{\varphi}_1,\dots,\boldsymbol{\varphi}_\ell\}\,,\qquad \overline{\bigcup_{\ell\in\mathbb{N}}\boldsymbol{V}_\ell}=\boldsymbol{W}^{\sigma,p}(\Omega)\,, \tag{4.11}$$

with a Galerkin basis $\{\boldsymbol{\varphi}_j\}_{j=1}^{\ell}$. Without loss of generality, we assume that for each $\ell \in \mathbb{N}$ the space $\boldsymbol{V}_\ell$ is a subset of $\boldsymbol{W}^{\sigma+\eta(p-1),p}(\Omega)$ (note that $\mathscr{C}^\infty(\overline{\Omega})$ is dense in $\boldsymbol{W}^{\sigma,p}(\Omega)$) for $\eta > 0$ chosen later in the proof. Then the peridynamic problem of Definition 4.1 can be approximated to find $\boldsymbol{y}_\ell : [0,T] \to \boldsymbol{V}_\ell$ such that

$$\langle \boldsymbol{y}_\ell''(t), \boldsymbol{z}_\ell\rangle - k(\boldsymbol{y}_\ell(t), \boldsymbol{z}_\ell) = (\boldsymbol{b}(t), \boldsymbol{z}_\ell) \quad \text{for all } \boldsymbol{z}_\ell \in \boldsymbol{V}_\ell, \text{ a.e. in } (0,T), \tag{4.12a}$$

$$\boldsymbol{y}_\ell(0) = \boldsymbol{y}_{0,\ell}, \tag{4.12b}$$

$$\boldsymbol{y}_\ell'(0) = \boldsymbol{v}_{0,\ell}, \tag{4.12c}$$

with $\boldsymbol{y}_{0,\ell}$ converging strongly to $\boldsymbol{y}_0$ in $\boldsymbol{W}^{\sigma,p}(\Omega)$ and $\boldsymbol{v}_{0,\ell}$ converging strongly to $\boldsymbol{v}_0$ in $\boldsymbol{L}^2(\Omega)$ as $\ell \to \infty$.

Step 2: Existence of an approximate solution. We claim that for fixed $\ell \in \mathbb{N}$ there exists a solution $\boldsymbol{y}_\ell \in W^{2,1}(0,T;\boldsymbol{V}_\ell)$ to the approximate problem (4.12). The proof follows the same lines as Step 2 of the proof of Theorem 3.1.

Step 3: A priori estimates for the approximate solution. We test the equation with $\boldsymbol{z} = \boldsymbol{y}_\ell'(t)$, $t \in (0,T)$. This yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{y}_\ell'(t)\|_{0,2}^2 + \frac{\mathrm{d}}{\mathrm{d}t}\Phi(\boldsymbol{y}_\ell(t)) \le \|\boldsymbol{b}(t)\|_{0,2}\left\|\boldsymbol{y}_\ell'(t)\right\|_{0,2}$$

since $\boldsymbol{y}_\ell \in W^{2,1}(0,T;\boldsymbol{V}_\ell)$ and since $\Phi' = -K : \boldsymbol{W}^{\sigma,p}(\Omega) \to (\boldsymbol{W}^{\sigma,p}(\Omega))^*$ is demicontinuous. Similarly to Step 3 of the proof of Theorem 3.1, this yields boundedness of $\{\boldsymbol{y}_\ell'\}$ in $L^\infty(0,T;\boldsymbol{L}^2(\Omega))$ (recall that because of (4.8a) the macroelastic potential is bounded from below) and hence of $\{\boldsymbol{y}_\ell\}$ in $L^\infty(0,T;\boldsymbol{L}^2(\Omega))$ (recall that $\{\boldsymbol{y}_{0,\ell}\}$ is bounded in $\boldsymbol{W}^{\sigma,p}(\Omega)$). Applying (4.8b) yields boundedness of $\{\boldsymbol{y}_\ell\}$ in $L^\infty(0,T;\boldsymbol{L}^p(\Omega))$, which finally results – employing (4.8c) – in the boundedness in $L^\infty(0,T;\boldsymbol{W}^{\sigma,p}(\Omega))$.

Step 4: Passage to the limit. From the a priori estimates above, we gain the existence of an element $\boldsymbol{y}$ and a subsequence of $\{\boldsymbol{y}_\ell\}$ (not relabeled) such that the subsequence converges weakly* to $\boldsymbol{y}$ in $L^\infty(0,T;\boldsymbol{W}^{\sigma,p}(\Omega))$. Moreover, $\{\boldsymbol{y}_\ell'\}$ converges weakly* to some element in $L^\infty(0,T;\boldsymbol{L}^2(\Omega))$, which can be shown to be $\boldsymbol{y}'$. However, due to the structure of k, weak* convergence is not sufficient to pass to the limit in the peridynamic form – strong convergence is needed. Even though we do not get strong convergence in a space with values in $\boldsymbol{W}^{\sigma,p}(\Omega)$, we can make use of Proposition 4.2. Therefore, let $0 < \eta < \min\{\sigma, \frac{1-\sigma}{p-1}\}$ be fixed. Since $\boldsymbol{W}^{\sigma,p}(\Omega)$ is compactly embedded in $\boldsymbol{W}^{\sigma-\eta,p}(\Omega)$ (which is continuously embedded in $\boldsymbol{L}^2(\Omega)$), in view of the Lions–Aubin lemma [33, Lemma 7.7], there holds

$$L^p(0,T;\boldsymbol{W}^{\sigma,p}(\Omega)) \cap W^{1,\infty}(0,T;\boldsymbol{L}^2(\Omega)) \overset{c}{\hookrightarrow} L^p(0,T;\boldsymbol{W}^{\sigma-\eta,p}(\Omega)).$$

We obtain thus a subsequence of $\{\boldsymbol{y}_\ell\}$ converging strongly in $\boldsymbol{L}^p(0,T;\boldsymbol{W}^{\sigma-\eta,p}(\Omega))$ and also almost everywhere in time. Altogether, we can deduce the existence of an element $\boldsymbol{y}$ such that (passing to a subsequence if necessary)

$$\boldsymbol{y}_\ell(t) \to \boldsymbol{y}(t) \quad \text{in } \boldsymbol{W}^{\sigma-\eta,p}(\Omega) \quad \text{a.e. in } (0,T), \tag{4.13a}$$

$$\boldsymbol{y}_\ell \overset{*}{\rightharpoonup} \boldsymbol{y} \quad \text{in } L^\infty(0,T;\boldsymbol{W}^{\sigma,p}(\Omega)), \tag{4.13b}$$

$$\boldsymbol{y}_\ell' \overset{*}{\rightharpoonup} \boldsymbol{y}' \quad \text{in } L^\infty(0,T;\boldsymbol{L}^2(\Omega)). \tag{4.13c}$$

It remains to pass to the limit in (4.12). For $\phi \in \mathscr{C}_c^\infty(0,T)$ and $\boldsymbol{z}_m \in \boldsymbol{V}_m$, there holds for all $m \le \ell$, $m \in \mathbb{N}$,

$$-\int_0^T (\boldsymbol{y}_\ell'(t), \boldsymbol{z}_m)\,\phi'(t)\,\mathrm{d}t - \int_0^T k(\boldsymbol{y}_\ell(t), \boldsymbol{z}_m)\,\phi(t)\,\mathrm{d}t = \int_0^T (\boldsymbol{b}(t), \boldsymbol{z}_m)\,\phi(t)\,\mathrm{d}t\,.$$

Now we pass to the limit as $\ell \to \infty$. The acceleration term converges because of (4.13c). To get convergence of the nonlinear peridynamic operator, we cannot use the continuity of k in its first argument from Proposition 3.1. However, since $\boldsymbol{z}_m$ is smooth (remember that $\boldsymbol{V}_m \subset \boldsymbol{W}^{\sigma+\eta(p-1),p}(\Omega)$), we use Proposition 4.2 together with (4.13a) to see that, as $\ell \to \infty$,

$$k(\boldsymbol{y}_\ell(t), \boldsymbol{z}_m) \to k(\boldsymbol{y}(t), \boldsymbol{z}_m) \quad \text{a.e. in } (0,T)\,.$$

Moreover, (4.2) and the boundedness of $\{\boldsymbol{y}_\ell\}$ in $L^\infty(0,T;\boldsymbol{W}^{\sigma,p}(\Omega))$ show that the sequence $\{k(\boldsymbol{y}_\ell(t), \boldsymbol{z}_m)\}$ is bounded in $L^\infty(0,T)$. Lebesgue's theorem on dominated convergence now yields

$$-\int_0^T (\boldsymbol{y}'(t), \boldsymbol{z}_m)\,\phi'(t)\,\mathrm{d}t - \int_0^T k(\boldsymbol{y}(t), \boldsymbol{z}_m)\,\phi(t)\,\mathrm{d}t = \int_0^T (\boldsymbol{b}(t), \boldsymbol{z}_m)\,\phi(t)\,\mathrm{d}t$$

for all $\boldsymbol{z}_m \in \boldsymbol{V}_m$, $m \in \mathbb{N}$. By the limited completeness of the Galerkin scheme (see (4.11)) and since $\mathscr{C}_c^\infty(0,T) \otimes \boldsymbol{W}^{\sigma,p}(\Omega)$ is dense with respect to the weak* convergence in $L^\infty(0,T;\boldsymbol{W}^{\sigma,p}(\Omega))$, $\boldsymbol{y}$ solves the equation (4.10).

By similar arguments as in the proof of Theorem 3.1, we obtain $\boldsymbol{y}(0) = \boldsymbol{y}_0$ and $\boldsymbol{y}'(0) = \boldsymbol{v}_0$. □

Appendix

Proof (of Lemma 3.1) The case $p = 2$ is obvious. Let $p > 2$. Due to [24], we have the identity

$$\begin{aligned}\Big(|\boldsymbol{b}|^{p-2}\boldsymbol{b} - |\boldsymbol{a}|^{p-2}\boldsymbol{a}\Big)\cdot(\boldsymbol{b}-\boldsymbol{a}) \\ = \frac{|\boldsymbol{b}|^{p-2} + |\boldsymbol{a}|^{p-2}}{2}\,|\boldsymbol{b}-\boldsymbol{a}|^2 + \frac{\Big(|\boldsymbol{b}|^{p-2} - |\boldsymbol{a}|^{p-2}\Big)\Big(|\boldsymbol{b}|^2 - |\boldsymbol{a}|^2\Big)}{2}\end{aligned} \tag{4.14}$$

which follows from simple calculations. Applying Young's inequality to the first term of the right-hand side yields

$$\frac{|\boldsymbol{b}|^{p-2} + |\boldsymbol{a}|^{p-2}}{2}\,|\boldsymbol{b}-\boldsymbol{a}|^2 \le \frac{2}{p}\,|\boldsymbol{b}-\boldsymbol{a}|^p + \frac{p-2}{2p}\,(|\boldsymbol{a}|^p + |\boldsymbol{b}|^p)\,.$$

The second term of the right-hand side of (4.14) can be estimated through

$$\frac{\Big(|\boldsymbol{b}|^{p-2} - |\boldsymbol{a}|^{p-2}\Big)\Big(|\boldsymbol{b}|^2 - |\boldsymbol{a}|^2\Big)}{2} \le \frac{1}{2}\,(|\boldsymbol{a}|^p + |\boldsymbol{b}|^p)\,.$$

Thus, by the Cauchy–Schwarz inequality, we have

$$\begin{aligned}|\boldsymbol{a}|^p + |\boldsymbol{b}|^p - |\boldsymbol{a}|^{p-1}\,|\boldsymbol{b}| - |\boldsymbol{b}|^{p-1}\,|\boldsymbol{a}| &\le \Big(|\boldsymbol{b}|^{p-2}\boldsymbol{b} - |\boldsymbol{a}|^{p-2}\boldsymbol{a}\Big)\cdot(\boldsymbol{b}-\boldsymbol{a}) \\ &\le \frac{2}{p}\,|\boldsymbol{b}-\boldsymbol{a}|^p + \frac{p-1}{p}\,(|\boldsymbol{a}|^p + |\boldsymbol{b}|^p)\,.\end{aligned}$$

Rearranging the terms proves the claim. □

Acknowledgements The authors would like to thank François Murat (Paris) for all the fruitful discussions.

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