Finite speed of propagation for stochastic porous media equations.

Benjamin Gess

1Institut für Mathematik, Humboldt-Universität zu Berlin

5th Spring School on Evolution Equations, Analytical and Numerical Aspects of Evolution Equations, TU Berlin, March 2013

preprint:[arXiv:1210.2415]
Outline

1. The deterministic case

2. The stochastic case
The deterministic case
Recall: The deterministic porous medium equation

\[ \frac{d}{dt} u = \Delta (u^m), \]  

(1)

for non-negative initial conditions \( u_0 \geq 0 \) [Vázquez, 2006].

In the superlinear case \( m > 1 \), (1) has degenerate diffusivity:

\[ \frac{d}{dt} u = mu^{m-1} \Delta u + m(m-1)u^{m-2}|\nabla u|^2. \]

The diffusivity coefficient vanishes for \( u \to 0 \).
Recall: The deterministic porous medium equation

\[
\frac{d}{dt} u = \Delta (u^m),
\]  

(1)

for non-negative initial conditions \( u_0 \geq 0 \) [Vázquez, 2006].

In the superlinear case \( m > 1 \), (1) has degenerate diffusivity:

\[
\frac{d}{dt} u = mu^{m-1} \Delta u + m(m - 1)u^{m-2}|\nabla u|^2.
\]

The diffusivity coefficient vanishes for \( u \to 0 \).
Finite speed of propagation

- Limited regularity: $\nabla u$ discontinuous. E.g. Barenblatt solutions:

![Graph showing finite speed of propagation](image)

- Regularity is limited precisely at the free boundary.
- Comparison to Barenblatt solutions yields finite speed of propagation.
Finite speed of propagation

- Limited regularity: $\nabla u$ discontinuous. E.g. Barenblatt solutions:

- Regularity is limited precisely at the free boundary.
- Comparison to Barenblatt solutions yields finite speed of propagation.
The deterministic case

Finite speed of propagation

- Limited regularity: $\nabla u$ discontinuous. E.g. Barenblatt solutions:

- Regularity is limited precisely at the free boundary.
- Comparison to Barenblatt solutions yields finite speed of propagation.
Optimal estimates

Finite speed of propagation is a local property.

Finite speed of hole-filling:
Let $u_0$ vanish in $B_R(x_0)$. Then $u$ vanishes in $B_{R(t)}(x_0)$ with

$$R(t) \geq R - \left( \frac{\|u\|_{\infty}^{m-1}}{C_{det}} t \right)^{1/2}.$$ 

This is optimal.

Finite speed of propagation:
Let $S(t) := \{ x \in \mathbb{R}^d \mid u(t, x) > 0 \}$ be the positivity set. Then

$$S(t + h) \subseteq B_{ch^{1/2}}(S(t)).$$
Finite speed of propagation is a local property.

*Finite speed of hole-filling:*
Let $u_0$ vanish in $B_R(x_0)$. Then $u$ vanishes in $B_{R(t)}(x_0)$ with

$$R(t) \geq R - \left( \frac{\|u\|_{\infty}^{m-1}}{C_{det}} t \right)^{1/2}.$$ 

This is optimal.

*Finite speed of propagation:*
Let $S(t) := \{ x \in \mathbb{R}^d | u(t, x) > 0 \}$ be the positivity set. Then

$$S(t + h) \subseteq B_{ch^{1/2}}(S(t)).$$
The deterministic case

Optimal estimates

- Finite speed of propagation is a local property.
- **Finite speed of hole-filling:**
  Let \( u_0 \) vanish in \( B_R(x_0) \). Then \( u \) vanishes in \( B_{R(t)}(x_0) \) with

  \[
  R(t) \geq R - \left( \frac{\|u\|_{\infty}^{m-1}}{C_{det}} t \right)^{1/2}.
  \]

  This is optimal.

- **Finite speed of propagation:**
  Let \( S(t) := \{x \in \mathbb{R}^d | u(t, x) > 0\} \) be the positivity set. Then

  \[
  S(t + h) \subseteq B_{c h^{1/2}}(S(t)).
  \]
The stochastic case

The stochastic case
Stochastic porous medium equation

\[ dX_t = \Delta X^m_t \, dt + \sum_{k=1}^N f_k X_t \circ d\beta^{(k)}_t. \]  

(SPME)

on bounded domains, homogeneous Dirichlet boundary conditions.

\( \beta^k \) Brownian motion, \( f_k \in C^\infty(\mathcal{O}) \).
The stochastic case

Stochastic porous medium equation

\[ dX_t = \Delta X_t^m dt + \sum_{k=1}^{N} f_k X_t \circ d\beta_t^{(k)}. \]  

(SPME)

on bounded domains, homogeneous Dirichlet boundary conditions.

- \( \beta^k \) Brownian motion, \( f_k \in C^\infty(\mathcal{O}) \).
Known results

- **Finite speed of hole-filling** [Barbu, Röckner, EJP 2012]: Let $X_0$ vanish in $B_R(x_0)$. Then $X$ vanishes in $B_{R(t,\omega)}(x_0)$ for some function $R(\cdot, \omega) : [0, T] \to (0, R)$.

- No uniform control on $R(t, \omega)$ in $x_0 \to$ cannot deduce finite speed of propagation

- No information about optimality of the bounds

- Two aims:
  - show finite speed of propagation
  - deduce (locally) optimal bounds
The stochastic case

Known results

- **Finite speed of hole-filling** [Barbu, Röckner, EJP 2012]:
  Let $X_0$ vanish in $B_R(x_0)$. Then $X$ vanishes in $B_{R(t,\omega)}(x_0)$ for some function $R(\cdot, \omega) : [0, T] \rightarrow (0, R)$.

- No uniform control on $R(t, \omega)$ in $x_0 \rightarrow$ cannot deduce finite speed of propagation

- No information about optimality of the bounds

- Two aims:
  - show finite speed of propagation
  - deduce (locally) optimal bounds
Known results

- **Finite speed of hole-filling** [Barbu, Röckner, EJP 2012]: Let $X_0$ vanish in $B_R(x_0)$. Then $X$ vanishes in $B_{R(t,\omega)}(x_0)$ for some function $R(\cdot, \omega) : [0, T] \rightarrow (0, R)$.
- No uniform control on $R(t, \omega)$ in $x_0 \rightarrow$ cannot deduce finite speed of propagation
- No information about optimality of the bounds
- Two aims:
  - show finite speed of propagation
  - deduce (locally) optimal bounds
Known results

- **Finite speed of hole-filling** [Barbu, Röckner, EJP 2012]:
  Let $X_0$ vanish in $B_R(x_0)$. Then $X$ vanishes in $B_{R(t,\omega)}(x_0)$ for some function $R(\cdot, \omega) : [0, T] \to (0, R)$.

- No uniform control on $R(t, \omega)$ in $x_0 \to$ cannot deduce finite speed of propagation

- No information about optimality of the bounds

- Two aims:
  - show finite speed of propagation
  - deduce (locally) optimal bounds
The stochastic case

Transformation

- Recall:

\[ dX_t = \Delta (X^m_t) dt + \sum_{k=1}^{N} f_k X_t \circ d\beta^{(k)}_t. \]

- Set \( Y_t := e^{\mu_t} X_t \), where \( \mu_t = - \sum_{k=1}^{N} f_k \beta^{(k)}_t \). Then

\[ \partial_t Y(t, x) = e^{\mu(t, x)} \Delta \left( e^{-\mu(t, x)} Y(t, x) \right)^m. \]

Recall:

\[ dX_t = \Delta (X_t^m) \, dt + \sum_{k=1}^{N} f_k X_t \circ d\beta_t^{(k)}. \]

Set \( Y_t := e^{\mu_t} X_t \), where \( \mu_t = -\sum_{k=1}^{N} f_k \beta_t^{(k)} \). Then

\[ \partial_t Y(t, x) = e^{\mu(t,x)} \Delta \left( e^{-\mu(t,x)} Y(t, x) \right)^m. \]

Recall:

\[ dX_t = \Delta(X^m_t) dt + \sum_{k=1}^{N} f_k X_t \circ d\beta_t^{(k)}. \]

Set \( Y_t := e^{\mu_t} X_t \), where \( \mu_t = -\sum_{k=1}^{N} f_k \beta_t^{(k)} \). Then

\[ \partial_t Y(t, x) = e^{\mu(t,x)} \Delta \left( e^{-\mu(t,x)} Y(t, x) \right)^m. \]

Non-spatially distributed noise

- Assume $f_k$ constant. Then
  \[ \partial_t Y_t = e^{(1-m)\mu_t} \Delta Y_t^m. \]

- Let $F' := e^{(1-m)\mu_t}$, $g = F^{-1}$. Then $u_t := Y_{g(t)}$ solves
  \[ \frac{d}{dt} u = \Delta (u^m) \]

- Finite speed of propagation follows from the deterministic case and the estimates are optimal.
- **Finite speed of hole-filling:**
  Let $X_0$ vanish in $B_R(x_0)$. Then $X_t$ vanishes in $B_{R_{stoch}(t, \omega)}(x_0)$ with
  \[
  R_{stoch}(t) = R - \left( \frac{H^{m-1}}{C_{det}} F(t) \right)^{\frac{1}{2}}
  = R - \left( \frac{H^{m-1}}{C_{det}} \int_0^t e^{-(m-1)\mu_\omega dr} \right)^{\frac{1}{2}}.
  \]
The stochastic case

Non-spatially distributed noise

- Assume $f_k$ constant. Then
  \[ \partial_t Y_t = e^{(1-m)\mu_t} \Delta Y_t^m. \]

- Let $F' := e^{(1-m)\mu_t}$, $g = F^{-1}$. Then $u_t := Y_{g(t)}$ solves
  \[ \frac{d}{dt} u = \Delta (u^m) \]

- Finite speed of propagation follows from the deterministic case and the estimates are optimal.

- Finite speed of hole-filling:
  Let $X_0$ vanish in $B_R(x_0)$. Then $X_t$ vanishes in $B_{R_{stoch}(t,\omega)}(x_0)$ with
  \[ R_{stoch}(t) = R - \left( \frac{H^{m-1}}{C_{det}} F(t) \right)^{\frac{1}{2}} \]
  \[ = R - \left( \frac{H^{m-1}}{C_{det}} \int_0^t e^{-(m-1)\mu_r(\omega)} dr \right)^{\frac{1}{2}}. \]
Non-spatially distributed noise

- Assume $f_k$ constant. Then

  \[ \partial_t Y_t = e^{(1-m)\mu_t} \Delta Y^m_t. \]

- Let $F' := e^{(1-m)\mu_t}$, $g = F^{-1}$. Then $u_t := Y_g(t)$ solves

  \[ \frac{d}{dt} u_t = \Delta (u^m) \]

- Finite speed of propagation follows from the deterministic case and the estimates are optimal.

- **Finite speed of hole-filling:**
  Let $X_0$ vanish in $B_R(x_0)$. Then $X_t$ vanishes in $B_{R_{stoch}(t,\omega)}(x_0)$ with

  \[ R_{stoch}(t) = R - \left( \frac{H^{m-1}}{C_{det}} F(t) \right)^{\frac{1}{2}} \]

  \[ = R - \left( \frac{H^{m-1}}{C_{det}} \int_0^t e^{-(m-1)\mu_r(\omega)} dr \right)^{\frac{1}{2}}. \]
Non-spatially distributed noise

- Assume $f_k$ constant. Then

$$\partial_t Y_t = e^{(1-m)\mu t} \Delta Y_t^m.$$  

- Let $F' := e^{(1-m)\mu t}$, $g = F^{-1}$. Then $u_t := Y_{g(t)}$ solves

$$\frac{d}{dt} u = \Delta(u^m)$$

- Finite speed of propagation follows from the deterministic case and the estimates are \textbf{optimal}.

- \textit{Finite speed of hole-filling:} Let $X_0$ vanish in $B_R(x_0)$. Then $X_t$ vanishes in $B_{R_{stoch}(t,\omega)}(x_0)$ with

$$R_{stoch}(t) = R - \left( \frac{H^{m-1}}{C_{det}} F(t) \right)^{\frac{1}{2}}$$

$$= R - \left( \frac{H^{m-1}}{C_{det}} \int_0^t e^{-(m-1)\mu r(\omega)} dr \right)^{\frac{1}{2}}.$$
The stochastic case

Spatially distributed noise

- Recall:
  \[ \partial_t Y(t,x) = e^{\mu(t,x)} \Delta \left( e^{-\mu(t,x)} Y(t,x) \right)^m. \]

- Freeze coefficients in space:
  \[ \partial_t Y(t,x) \approx e^{\mu(t,x_0)} \Delta \left( e^{-\mu(t,x_0)} Y(t,x) \right)^m, \]
  on small balls \( B_r(x_0) \).

- Freeze coefficients in time:
  \[ \partial_t Y(t,x) \approx \Delta Y(t,x)^m, \]
  for small times \( t \approx 0 \).
Spatially distributed noise

- Recall:
  \[ \partial_t Y(t, x) = e^{\mu(t, x)} \Delta \left( e^{-\mu(t, x)} Y(t, x) \right)^m. \]

- Freeze coefficients in space:
  \[ \partial_t Y(t, x) \approx e^{\mu(t, x_0)} \Delta \left( e^{-\mu(t, x_0)} Y(t, x) \right)^m, \]
  on small balls \( B_r(x_0) \).

- Freeze coefficients in time:
  \[ \partial_t Y(t, x) \approx \Delta Y(t, x)^m, \]
  for small times \( t \approx 0 \).
Spatially distributed noise

- Recall:
  \[ \partial_t Y(t,x) = e^{\mu(t,x)} \Delta \left( e^{-\mu(t,x)} Y(t,x) \right)^m. \]

- Freeze coefficients in space:
  \[ \partial_t Y(t,x) \approx e^{\mu(t,x_0)} \Delta \left( e^{-\mu(t,x_0)} Y(t,x) \right)^m, \]
  on small balls \( B_r(x_0). \)

- Freeze coefficients in time:
  \[ \partial_t Y(t,x) \approx \Delta Y(t,x)^m, \]
  for small times \( t \approx 0. \)
Hole-filling

- **Finite speed of hole-filling:**
  Let $X_0$ vanish in $B_R(x_0)$. Then $X_t$ vanishes in $B_{R_{stoch}(t,\omega)}(x_0)$ with

  $R_{stoch}(t, \omega) = R - \left( \frac{H^{m-1}}{C_{det}} F(t, \omega) \right)^{\frac{1}{2}} C_{R_{stoch}}^{-\frac{1}{2}}(\omega)$

  $$= R - \left( \frac{H^{m-1}}{C_{det}} \int_0^t e^{-(m-1)\mu_{r}(x_0, \omega)} dr \right)^{\frac{1}{2}} C_{R_{stoch}}^{-\frac{1}{2}}(\omega),$$

  with $\lim_{R \downarrow 0} C_R = 1$.

  For $R \approx 0$ we recover the optimal rate from the spatially homogeneous case with $\mu_r \equiv \mu_r(\xi_0)$.
Hole-filling

- **Finite speed of hole-filling:**
  Let $X_0$ vanish in $B_R(x_0)$. Then $X_t$ vanishes in $B_{R_{stoch}(t, \omega)}(x_0)$ with

  $$R_{stoch}(t, \omega) = R - \left( \frac{H^{m-1}}{C_{det}} F(t, \omega) \right)^{\frac{1}{2}} C_R^{-\frac{1}{2}}(\omega)$$

  $$= R - \left( \frac{H^{m-1}}{C_{det}} \int_0^t e^{-(m-1)\mu_r(x_0, \omega)} dr \right)^{\frac{1}{2}} C_R^{-\frac{1}{2}}(\omega),$$

  with $\lim_{R \downarrow 0} C_R = 1$.

- For $R \approx 0$ we recover the optimal rate from the spatially homogeneous case with $\mu_r \equiv \mu_r(\xi_0)$. 
Hole-filling

- **Finite speed of hole-filling:**
  Let $X_0$ vanish in $B_R(x_0)$. Then $X_t$ vanishes in $B_{R_{stoch}(t,\omega)}(x_0)$ with
  \[
  R_{stoch}(t,\omega) = R - \left( \frac{H^{m-1}}{C_{det}} t \right)^{\frac{1}{2}} \sqrt{C_t(\omega)}.
  \]
  with $\lim_{t \downarrow 0} C_t = 1$.

- For $t \approx 0$ we recover the optimal rate from the deterministic case.
Hole-filling

- **Finite speed of hole-filling:** Let $X_0$ vanish in $B_R(x_0)$. Then $X_t$ vanishes in $B_{R_{stoch}(t,\omega)}(x_0)$ with

$$R_{stoch}(t, \omega) = R - \left( \frac{H^{m-1}}{C_{det}} t \right)^{\frac{1}{2}} \sqrt{C_t(\omega)}.$$ 

with $\lim_{t \downarrow 0} C_t = 1$.

- For $t \approx 0$ we recover the optimal rate from the deterministic case.
Finite speed of propagation

Let $X$ be an essentially bounded, non-negative solution to (SPME). Then,

$$\text{supp}(X_t) \subseteq B_{\sqrt{t \left( \frac{H^{m-1}}{C_{\text{det}}} \right)^{\frac{1}{2}} \sqrt{C_t(\omega)}}}(\text{supp}(X_0)), \quad \forall t \in [0, T],$$

with $C_t \to 1$ for $t \to 0$. 

Finite speed of propagation: