Etienne Emmrich and Dimitri Puhst

Existence results in nonlinear peridynamics in comparison with local elastodynamics

Abstract: We give an overview on existence results in nonlinear peridynamics, a nonlocal elasticity theory, and compare these results with those known for classical nonlinear elastodynamics. We conclude that even though peridynamics was developed in the beginning of the current century, it is possible to obtain stronger results under equal or even less restrictive assumptions on the stress.

Etienne Emmrich: Technische Universität Berlin, Institut für Mathematik, Straße des 17. Juni 136, 10623 Berlin, emmrich@math.tu-berlin.de
Dimitri Puhst: Technische Universität Berlin, Institut für Mathematik, Straße des 17. Juni 136, 10623 Berlin, puhst@math.tu-berlin.de

1 Introduction

Although the equations of nonlinear elastodynamics are fundamental in the description of the deformation of elastic bodies and have been studied for many years, their mathematical analysis is still incomplete. Over the last decades, new alternative formulations and models have been introduced in order to overcome certain drawbacks arising from the classical model governed by nonlinear partial differential equations of second order. One of the new models is due to Silling [64] who introduced peridynamics, a continuum theory avoiding spatial derivatives and incorporating spatial nonlocality. Peridynamics allows us to model nonlocal interaction and long-range forces and is believed to be suited for the description of material failure.

In this work, we focus on the question of existence of (generalized) solutions to nonlinear peridynamics in comparison with the nowadays classical model of nonlinear elastodynamics. Indeed, existence in peridynamics has been proven under reasonable assumptions on the function describing the constitutive law, in particular without any convexity or monotonicity assumption whatsoever. This is in contrast to classical elasticity theory in which much effort has been made in order to avoid unphysical convexity assumptions. Nevertheless relaxed assumptions such as polyconvexity of the underlying potential or an Andrews–Ball-type condition on the stress-strain relation seem to be essential in classical nonlinear elastodynamics.

This paper is organized as follows. In Section 2, results on the existence of solutions to the governing equation in local nonlinear elastodynamics are discussed in chronological order. In Section 3, the two main models in nonlocal elastodynamics are described. Subsequently, Sections 4 and 5 are devoted to existence results in nonlinear peridynamics. These results cover existence of Young-measure-valued, weak and classical solutions. Finally in Section 6, the different results in local and nonlocal elastodynamics are juxtaposed.

1.1 Notation

Vectors in $\mathbb{R}^d$ and vector-valued functions are denoted by boldfaced letters. We rely on the standard notation for Lebesgue and Sobolev spaces and function spaces of continuous functions. In particular, we denote by $\|\cdot\|_{s,p}$ and $|\cdot|_{s,p}$ norm and seminorm of $W^{s,p}(\Omega)$. In the sequel, we identify an abstract function y mapping the time interval $(0,T) \ni t$ into a function space X over the spatial domain $\Omega \ni x$ with the corresponding function defined over $\Omega \times (0,T)$ via $[y(t)](x) = y(x,t)$. For a Banach space X, we denote by $L^r(0,T;X)$ $(1 \leq r \leq \infty)$ the usual Bochner–Lebesgue space (see, e.g., Diestel and Uhl [22]) and by $W^{1,r}(0,T;X)$ the Banach space of functions $v \in L^r(0,T;X)$ whose distributional time derivative

v' is again in $L^r(0,T;X)$ (see, e.g., Roubíček [63]). By c, we denote a generic positive constant. Other notation is standard or is shortly explained throughout the text.

2 Local nonlinear elastodynamics

Nonlinear elastodynamics is, from a mathematical point of view, a very challenging field to work on. The aim of this section is to show its difficulty by summarizing the few important results, which have contributed to a better understanding of the existence of solutions in what sense soever. For more details and further references, we also refer to Antman [9, Chapter 13 and 18] as well as to Marsden and Hughes [55, Section 6.5].

Let $\Omega \subset \mathbb{R}^d$ (typically, $d = 3$) be a bounded domain describing an elastic body in its reference configuration. Throughout this paper, we assume that the boundary $\partial\Omega$ is sufficiently smooth. Under possible loading, the body deforms and the particle at position $\boldsymbol{x} \in \Omega$ has at time t the new position $\boldsymbol{y}(\boldsymbol{x}, t)$. The mapping $\boldsymbol{y} : \overline{\Omega} \times [0, T] \to \mathbb{R}^d$ is called deformation and is determined as a solution of the equation of nonlinear elastodynamics, which reads as

$$\boldsymbol{y}_{tt}(\boldsymbol{x}, t) - \operatorname{div} \boldsymbol{\sigma}(\nabla \boldsymbol{y}(\boldsymbol{x}, t)) = \boldsymbol{b}(\boldsymbol{x}, t)\,, \quad (\boldsymbol{x}, t) \in \Omega \times (0, T)\,. \tag{2.1}$$

The stress $\boldsymbol{\sigma} : \mathbb{R}^{d\times d} \to \mathbb{R}^{d\times d}$ can – in case of hyperelasticity – be obtained as the derivative of a potential $\phi : \mathbb{R}^{d\times d} \to \mathbb{R}$ and is nonlinear in general. The right-hand side $\boldsymbol{b} : \overline{\Omega} \times [0, T] \to \mathbb{R}^d$ corresponds to external forces and is assumed to be zero in most articles. A detailed derivation of (2.1) can be found in Marsden and Hughes [55].

Concerning the analysis of this equation, not much is known. Early results have been obtained for the one-dimensional case, where we start our review at. With the transformation $v := y_t$, $w := y_x$, the nonlinear equation of one-dimensional elastodynamics in absence of external forces can be transferred into the system

$$v_t - (\sigma(w))_x = 0\,, \quad w_t = v_x\,, \tag{2.2}$$

and hence, the theory of conservation laws

$$\boldsymbol{u}_t + (\boldsymbol{f}(\boldsymbol{u}))_x = \boldsymbol{0} \tag{2.3}$$

applies with $\boldsymbol{u} = (v, w)^T$. Therefore, nonlinear elastodynamics was settled in the class of nonlinear (to be precise, quasilinear) hyperbolic partial differential equations.

Using the transformation above, Lax [49] showed 1964 the nonexistence of global smooth solutions to (2.2) when σ'' is always nonzero. However, local existence of classical solutions is known (see also below). For a more detailed discussion under which assumptions classical solutions cannot exist for all time, we refer again to Antman [9, Chapter 13]. In 1965, Glimm [46] proved the existence of weak solutions to (2.3) when the initial value $\boldsymbol{u}(\cdot, 0)$ is of bounded variation and $\boldsymbol{f}$ is smooth, strictly hyperbolic, and genuinely nonlinear (see also LeFloch [50]). These conditions correspond in (2.2) to the case that $\sigma \in \mathcal{C}^2(\mathbb{R})$ with $\sigma' > 0$, i.e., σ is strictly monotonically increasing, and σ'' does not vanish. Unfortunately, the assumption $\sigma'' \neq 0$ is unphysical (see again [50, 55]).

Note that a weak solution is not unique and one needs additional criteria to identify the *physically relevant* solution. One condition is given by an additional conservation law connected to the entropy of a solution (see DiPerna [23] and the references cited therein, in particular see Friedrichs and Lax [43] for symmetric hyperbolic systems). In 1983, DiPerna [23] proved the existence of a so-called entropy solution to (2.3), i.e., a weak solution satisfying an additional conservation law. The proof is based on a regularization via the viscosity method and uses the theory of Young measures and compensated compactness. Applied to (2.2), the assumptions on the stress $\sigma \in \mathcal{C}^2(\mathbb{R})$ are $\sigma' > 0$ and that σ'' vanishes at just one point.

First results on multi-dimensional elastodynamics are due to Hughes, Kato and Marsden [47] in 1977 and Dafermos and Hrusa [17] in 1985, where an abstract framework was developed to show the existence of local smooth solutions. In both the works the corresponding quasilinear evolution equation $\boldsymbol{u}''+A(\boldsymbol{u})\boldsymbol{u}=\boldsymbol{b}$ (or its first-order system, respectively) is considered, where for fixed $\boldsymbol{w}$ the operator $A(\boldsymbol{w})$ is linear. Under the assumption that the stress is a smooth function in the gradient of the displacement and that the derivative of the stress is a uniformly strongly elliptic tensor of fourth order (i.e., it satisfies the strong Legendre–Hadamard condition), existence of a unique smooth solution is obtained via existence for the linear problem $\boldsymbol{u}''+A(\boldsymbol{w})\boldsymbol{u}=\boldsymbol{b}$ together with a fixed-point argument.

In 1992, Lin [51] as well as in 2000, Demoulini, Stuart and Tzavaras [20] presented (under additional assumptions on σ) generalizations to DiPerna's result [23], i.e., existence of weak solutions to (2.2) in one dimension satisfying an entropy condition, where the main difference to [23] is the method of approximation.

Further in 1997, Demoulini [18] faced the problem of convexity in elasticity. Indeed, convexity of the elastic energy (i.e., σ being monotone, which is the case if σ' is positive) is known to be an unphysical assumption in elasticity theory (see Antman [9, Chapter 13]). It conflicts with the axiom of material frame-indifference as well as with the behaviour of the material when the volume is compressed to zero. Hence, the *realistic* system of conservation laws (2.2) is in fact not hyperbolic, and more general assumptions on the stress should be considered. In [18], Young-measure-valued solutions are obtained by the method of time discretization in the scalar case, i.e., for a deformation y defined on $\Omega\subset\mathbb{R}^d$ but with values in $\mathbb{R}$. The equation under consideration is of type (2.1) but with σ being replaced by the derivative of the convexification ϕ^{**} of a nonconvex energy ϕ, which is assumed to have quadratic growth from above and from below.

In the general multi-dimensional case, a Young-measure-valued solution is a pair $(\boldsymbol{y},\nu)$ such that

$$\int_0^T\int_\Omega\left(-\boldsymbol{y}_t(\boldsymbol{x},t)\cdot\boldsymbol{z}_t(\boldsymbol{x},t)+\nabla\boldsymbol{z}(\boldsymbol{x},t):\int_{\mathbb{R}^{d\times d}}\boldsymbol{\sigma}(\boldsymbol{\lambda})\,\mathrm{d}\nu_{\boldsymbol{x},t}(\boldsymbol{\lambda})\right)\mathrm{d}\boldsymbol{x}\,\mathrm{d}t=0$$

for all $\boldsymbol{z}\in\mathcal{C}_c^\infty(\Omega\times(0,T))^d$ as well as

$$\nabla\boldsymbol{y}(\boldsymbol{x},t)=\int_{\mathbb{R}^{d\times d}}\boldsymbol{\lambda}\,\mathrm{d}\nu_{\boldsymbol{x},t}(\boldsymbol{\lambda})\quad\text{a.e. on }\Omega\times(0,T)\,,$$

where $\mathcal{C}_c^\infty(\Omega\times(0,T))$ denotes the function space of infinitely times differentiable functions with compact support in $\Omega\times(0,T)$ and where $\boldsymbol{A}:\boldsymbol{B}$ denotes the Frobenius inner product of $\boldsymbol{A},\boldsymbol{B}\in\mathbb{R}^{d\times d}$. A Young measure $\nu=\{\nu_{\boldsymbol{x},t}\}_{(\boldsymbol{x},t)\in\Omega\times(0,T)}$ is a family of probability measures and by definition an element of the space $L^\infty_{w^*}(\Omega\times(0,T);\mathcal{M}_b(\mathbb{R}^d))$ that is the dual of $L^1(\Omega\times(0,T);\mathcal{C}_0(\mathbb{R}^d))$, where $\mathcal{C}_0(\mathbb{R}^d)$ is the Banach space of continuous functions vanishing at infinity and $\mathcal{M}_b(\mathbb{R}^d)=(\mathcal{C}_0(\mathbb{R}^d))^*$ is the space of bounded Radon measures, see Málek, Nečas, Rokyta and Růžička [54], Fonseca and Leoni [41], Pedregal [58], or Roubíček [62] for further details.

Demoulini, Stuart and Tzavaras [21] showed 2001 the existence of such a Young-measure-valued solution with $\boldsymbol{y}\in W^{1,\infty}(0,T;L^2(\Omega)^d)\cap L^\infty(0,T;W^{1,p}(\Omega)^d)$ under the assumption that the elastic energy functional ϕ is strictly polyconvex, i.e., there exists a function ψ such that $\phi(\boldsymbol{F})=\psi(\boldsymbol{F},\operatorname{cof}\boldsymbol{F},\det\boldsymbol{F})$, $\boldsymbol{F}\in\mathbb{R}^{d\times d}$, and ψ is strictly convex in each of its three arguments. In addition, ψ is assumed to be twice continuously differentiable and to satisfy polynomial growth conditions from below and above (of degree $p>4$ in the deformation gradient and of degree at least 2 in the cofactor and determinant of the deformation gradient). The assumption of polyconvexity does not conflict with the physics of elastic materials, see Ciarlet [15, Section 4.9].

Moreover, existence of Young-measure-valued solutions with $\boldsymbol{y}$ in the same space as above is proven 2003 by Rieger [61] if $\boldsymbol{\sigma}=\phi'$ satisfies an Andrews–Ball-type condition such that $\boldsymbol{\sigma}+\lambda\boldsymbol{I}$ is monotone for some $\lambda>0$ and if the underlying potential ϕ is twice continuously differentiable with polynomial growth of degree $p\geq 2$ in the deformation gradient. This Andrews–Ball-type condition is suitable, e.g., for double-well potentials ϕ and corresponds to the convexity of the functional $\phi_\lambda(\boldsymbol{F})=\phi(\boldsymbol{F})+\frac{\lambda}{2}|\boldsymbol{F}|^2$ (this property is also called λ-convexity). The existence of solutions is obtained via studying the limit of weak solutions $\boldsymbol{y}^\varepsilon$ to the regularized problem of viscoelasticity (elasticity with additional damping)

$$\boldsymbol{y}^\varepsilon_{tt}(\boldsymbol{x},t) - \varepsilon\Delta\boldsymbol{y}^\varepsilon_t(\boldsymbol{x},t) - \operatorname{div}\boldsymbol{\sigma}(\nabla\boldsymbol{y}^\varepsilon(\boldsymbol{x},t)) = \mathbf{0}$$

as $\varepsilon \to 0$.

Indeed, the latter equation is easier to handle and hence, opposed to (2.1), much more is known. Existence of weak solutions under the above assumptions has been shown by, e.g., Friesecke and Dolzmann [44] and in a more general setting of abstract evolution equations by Emmrich and Šiška [30]. Numerical results on the solution of the viscoelastic equation as well as on the Young measure solutions of (2.1) are given by Carstensen and Dolzmann [13], Carstensen and Rieger [14], and Prohl [60]. Further results on nonlinear viscoelastic problems are (without being complete) due to Andrews [7], Andrews and Ball [8], Clements [16], Demoulini [19], Engler [32], Friedmann and Nečas [42], and Pecher [57]. The original Andrews–Ball condition (monotonicity of σ in the large) goes back to [7, 8].

3 Models in nonlocal elastodynamics

As seen in the previous section, the equation of local nonlinear elastodynamics is very hard to deal with and a complete existence theory meeting the demands of the broad range of applications is not yet at hand. Additionally, if one likes to incorporate damage and fracture of the material under investigation, one is left with the question of the correct choice of function spaces and a justification of the governing equation on spatial discontinuities. Altan [5] discusses that local theories fail in explaining phenomena arising at atomic scale as well as show "absurd" behaviour at crack problems. It is concluded that nonlocal elasticity theory is able to incorporate these aspects and therefore seems to be a more appropriate ansatz for describing complex elastic behaviour.

3.1 Eringen model

The idea of describing elastic behaviour with long range forces goes back to Kröner [48] and is mainly developed by Eringen [34, 35] (see also Eringen and Edelen [39]) in the early 1970s. In the so-called Eringen model, the equation of motion reads as (2.1) but with a nonlocal representation of the stress, which averages the local stress over the whole body,

$$[\boldsymbol{\sigma}_{NL}(\nabla\boldsymbol{y})](\boldsymbol{x},t) = \int\limits_\Omega \alpha(\hat{\boldsymbol{x}},\boldsymbol{x})\boldsymbol{\sigma}(\nabla\boldsymbol{y}(\hat{\boldsymbol{x}},t))\,\mathrm{d}\hat{\boldsymbol{x}}\,, \quad (\boldsymbol{x},t)\in\Omega\times(0,T)\,. \tag{3.1}$$

In most cases, the kernel α is of convolution type. A more general model considers a convex combination of the local with the nonlocal stress (see, e.g., Eringen [36], Eringen and Kim [40]). There are numerous survey papers and monographs on the Eringen model in continuum mechanics pointing out its relevance and studying applications such as Bažant and Jirásek [10], and Eringen [37, 38].

Less is known on existence and uniqueness for the (generalized) Eringen model in nonlocal elastodynamics. Altan [3, 4] as well as Whang and Dhaliwal [65, 66] proved uniqueness in various situations. Altan [5] showed existence and uniqueness but only in the linear stationary case under the assumption that α is a symmetric, continuous, square-integrable kernel of positive type. Of course, this result can be generalized to the nonstationary case using the well-known existence result on linear evolution problems of second order provided in Lions and Magenes [52, Chapter 3, Section 8]. A rigorous mathematical development of variational principles is given in Polizzotto [59], again for the stationary case. However, all of the references cited above consider the linear problem. To the best knowledge of the authors, the only results on existence for a nonstationary nonlinear Eringen-type model is given in the works of Duruk, Erbay and Erkip [25, 26] in the one-dimensional case based on Banach's fixed point principle.

3.2 Peridynamic model

A different approach to modelling elastic behaviour is due to Silling [64], who introduced peridynamic theory in 2000. Its main difference to local and other nonlocal models in elastodynamics is the assumption that the stress depends on (divided) differences of the deformation instead of the deformation gradient. No spatial derivatives appear in this new theory. An immediate consequence is that less regularity of the deformation is needed. In particular, spatial discontinuities may be incorporated. The peridynamic equation of motion reads as

$$\boldsymbol{y}_{tt}(\boldsymbol{x},t) - \int_\Omega \boldsymbol{f}(\hat{\boldsymbol{x}} - \boldsymbol{x}, \boldsymbol{y}(\hat{\boldsymbol{x}},t) - \boldsymbol{y}(\boldsymbol{x},t))\,\mathrm{d}\hat{\boldsymbol{x}} = \boldsymbol{b}(\boldsymbol{x},t)\,, \quad (\boldsymbol{x},t) \in \Omega \times (0,T)\,, \tag{3.2a}$$

and is supplemented by initial conditions

$$\boldsymbol{y}(\cdot,0) = \boldsymbol{y}_0\,, \quad \boldsymbol{y}_t(\cdot,0) = \boldsymbol{v}_0\,. \tag{3.2b}$$

Observe that no extra boundary conditions are imposed since no spatial derivatives occur. In particular, by integrating over Ω instead of $\mathbb{R}^d$ in the stress term, nonlocal Neumann boundary conditions are implicitly encoded, see also Andreu-Vaillo, Mazón, Rossi and Toledo-Melero [6]. Nevertheless, it is possible to prescribe values of $\boldsymbol{y}$ on a set of nonzero measure, e.g., in a strip along the boundary of Ω, see also Aksoylu and Mengesha [2].

The structure of the so-called pairwise force function $\boldsymbol{f}$ depends on the material under consideration. A common example is the bondstretch or proportional microelastic material model given by

$$\boldsymbol{f}(\boldsymbol{\xi},\boldsymbol{\zeta}) = \begin{cases} c_{d,\delta} \dfrac{|\boldsymbol{\zeta}| - |\boldsymbol{\xi}|}{|\boldsymbol{\xi}|} \dfrac{\boldsymbol{\zeta}}{|\boldsymbol{\zeta}|} & \text{if } |\boldsymbol{\xi}| < \delta\,, \\ 0 & \text{otherwise,} \end{cases}$$

where $|\cdot|$ denotes the Euclidean norm. Here, $c_{d,\delta}$ is a positive constant depending on the dimension d and the parameter δ, which describes the radius of nonlocal interaction and is called peridynamic horizon. Of course, δ can be chosen such that $\delta \geq \operatorname{diam}\Omega$ so that the case of nonlocal interaction over the whole domain is included in what follows. So far there is no mathematical analysis available for the bondstretch model. Recent numerical simulations of material failure, however, often rely on a variant of the bondstretch model in which possible breaking of bonds is inherited, see, e.g, [1, 12, 45, 56].

In Silling [64], it is shown that by the balance of angular momentum and by Newton's law *actio et reactio*, the pairwise force function must have the form

$$\boldsymbol{f}(\boldsymbol{\xi},\boldsymbol{\zeta}) = a(|\boldsymbol{\xi}|, |\boldsymbol{\zeta}|)\boldsymbol{\zeta}\,, \tag{3.3}$$

where a is a suitable scalar function (that again depends on d and δ). The possible singularity of a in its first argument determines the choice of the basic function space and hence which concept of solution is appropriate.

In this work, we overview results on the existence of solutions to the peridynamic initial value problem (3.2). For a more detailed description of peridynamics, we refer the reader to Emmrich, Lehoucq and Puhst [27]. We shall remark that the convergence of the nonlinear peridynamic model towards local elastodynamics as $\delta \to 0$ is a question of ongoing research.

4 Weak and measure-valued solutions to peridynamics

In this section, we present existence results corresponding to a weak formulation of the multi-dimensional nonlinear peridynamic initial value problem (3.2) with (3.3). To the best knowledge of the authors, this problem has not been considered in any work other than [29]. We therefore survey these results in detail.

The weak formulation of (3.2) with (3.3) we consider is to find $\boldsymbol{y} : [0,T] \to X$ (satisfying the initial conditions) such that for all ϕ in $\mathcal{C}_c^\infty(0,T)$ and $\boldsymbol{z} \in X$ there holds

$$-\int_0^T (\boldsymbol{y}'(t), \boldsymbol{z})\, \phi'(t)\,\mathrm{d}t - \int_0^T \langle K\boldsymbol{y}(t), \boldsymbol{z}\rangle\, \phi(t)\,\mathrm{d}t = \int_0^T (\boldsymbol{b}(t), \boldsymbol{z})\ \phi(t)\,\mathrm{d}t\,, \tag{4.1}$$

where $(\cdot,\cdot)$ denotes the inner product in $L^2(\Omega)^d$. Here, $X \subseteq L^2(\Omega)^d$ is a suitable Banach space, whose choice depends on the assumptions on the pairwise force function, and $K : X \to X^*$ is the energetic extension of the classical peridynamic operator given by

$$\langle K\boldsymbol{y}, \boldsymbol{z}\rangle = -\frac{1}{2}\iint_{\Omega\times\Omega} a(|\hat{\boldsymbol{x}} - \boldsymbol{x}|, |\boldsymbol{y}(\hat{\boldsymbol{x}}) - \boldsymbol{y}(\boldsymbol{x})|)\big(\boldsymbol{y}(\hat{\boldsymbol{x}}) - \boldsymbol{y}(\boldsymbol{x})\big)\cdot\big(\boldsymbol{z}(\hat{\boldsymbol{x}}) - \boldsymbol{z}(\boldsymbol{x})\big)\,\mathrm{d}(\hat{\boldsymbol{x}}, \boldsymbol{x})\,.$$

The choice of the basic function space X depends on the degree of singularity of $a = a(\xi,\zeta)$ at $\xi = 0$. If there is no or only a weak singularity, one works with $X = L^p(\Omega)^d$ $(2 \le p < \infty)$. In this case, the peridynamic operator $K : L^p(\Omega)^d \to L^{p'}(\Omega)^d$ (p' denotes the exponent conjugated to p) possesses the representation

$$(K\boldsymbol{y})(\boldsymbol{x}) = \int_\Omega a(|\hat{\boldsymbol{x}} - \boldsymbol{x}|, |\boldsymbol{y}(\hat{\boldsymbol{x}}) - \boldsymbol{y}(\boldsymbol{x})|)\,\big(\boldsymbol{y}(\hat{\boldsymbol{x}}) - \boldsymbol{y}(\boldsymbol{x})\big)\,\mathrm{d}\hat{\boldsymbol{x}} \tag{4.2}$$

since the nonlocal integration-by-parts formula

$$\int_\Omega (K\boldsymbol{y})(\boldsymbol{x})\cdot\boldsymbol{z}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = -\frac{1}{2}\iint_{\Omega\times\Omega} a(|\hat{\boldsymbol{x}} - \boldsymbol{x}|, |\boldsymbol{y}(\hat{\boldsymbol{x}}) - \boldsymbol{y}(\boldsymbol{x})|)\big(\boldsymbol{y}(\hat{\boldsymbol{x}}) - \boldsymbol{y}(\boldsymbol{x})\big)\cdot\big(\boldsymbol{z}(\hat{\boldsymbol{x}}) - \boldsymbol{z}(\boldsymbol{x})\big)\,\mathrm{d}(\hat{\boldsymbol{x}}, \boldsymbol{x})$$

applies for all $\boldsymbol{y}, \boldsymbol{z} \in L^p(\Omega)^d$.

The benefit of taking $X = L^p(\Omega)^d$ is that the class of solutions is larger, the drawback is that only existence of measure-valued solutions can be provided. In case of a strong singularity, the basic function space is the Sobolev–Slobodetskii space $X = W^{\sigma,p}(\Omega)^d$ $(0 < \sigma < 1,\ 2 \le p < \infty)$. By working with a smaller set of functions with more regularity and a coercivity assumption on the potential corresponding to $a = a(\xi,\zeta)$ that is stronger for ξ close to 0, it is possible to show existence of weak solutions to the peridynamic initial value problem.

We start with the case of no or only weak singularity of $a = a(\xi,\zeta)$ at $\xi = 0$. Suppose that the pairwise force function fulfills the following assumptions, where $2 \le p < \infty$ is fixed.

Assumption A

(A1) The function $a(\cdot,\zeta) : (0,\infty) \to \mathbb{R}$ is Lebesgue-measurable for all $\zeta \in [0,\infty)$.

(A2) The function $\boldsymbol{\zeta} \mapsto a(\xi, |\boldsymbol{\zeta}|)\boldsymbol{\zeta} : \mathbb{R}^d \to \mathbb{R}^d$ is continuous for almost all $\xi \in (0,\infty)$.

(A3) There holds $a(\xi,\zeta) = 0$ if $\xi \ge \delta$ and $\zeta \in [0,\infty)$.

(A4) There exist $c_0, c_1, c_2 \ge 0$ and $\gamma_1, \gamma_2 \in [0,d)$ such that for all $\zeta \in [0,\infty)$ and almost all $\xi \in (0,\delta)$ there holds

$$|a(\xi,\zeta)|\,\zeta \le c_0 + c_1\xi^{-\gamma_1} + c_2\xi^{-\gamma_2}\zeta^{p-1}\,.$$

(A5) There exist $\mu > 0$ and a nonnegative function $\alpha \in L^1(0,\delta)$ such that for all $\zeta \in [0,\infty)$ and almost all $\xi \in (0,\delta)$

$$\int_0^\zeta a(\xi,s)\,s\,\mathrm{d}s \ge \mu\zeta^p - \alpha(\xi)\xi^{-d+1}\,.$$

Under these assumptions, the peridynamic operator maps $L^p(\Omega)^d$ into $L^{p'}(\Omega)^d$ and is bounded and strong-weak continuous.

Unfortunately, similarly as in local elastodynamics, it is not known whether a weak solution to the problem under these assumptions exists. However, it is possible to prove the existence of Young-measure-valued solutions.

Theorem 4.1 ([29]). *Suppose that $\boldsymbol{b} \in L^1(0,T;L^2(\Omega)^d)$ and initial data $\boldsymbol{y}_0 \in L^p(\Omega)^d$, $\boldsymbol{v}_0 \in L^2(\Omega)^d$ are given and let the pairwise force function* (3.3) *fulfill Assumption A. Then there exists a function $\boldsymbol{y} : [0,T] \to L^p(\Omega)^d$ with*

$$\boldsymbol{y} \in \mathcal{C}_w([0,T];L^p(\Omega)^d)\,,$$
$$\boldsymbol{y}' \in \mathcal{C}_w([0,T];L^2(\Omega)^d)\,,$$
$$\boldsymbol{y}'' \in L^1(0,T;L^{p'}(\Omega)^d)\,,$$

and a Young measure $\nu = \{\nu_{\boldsymbol{x},t}\}_{(\boldsymbol{x},t)\in\Omega\times(0,T)}$ such that

$$\boldsymbol{y}'' - \boldsymbol{K}_\nu = \boldsymbol{b} \quad \text{in } L^1(0,T;L^{p'}(\Omega)^d)\,,$$

where $\boldsymbol{K}_\nu \in L^{p'}(0,T;L^{p'}(\Omega)^d)$ with

$$[\boldsymbol{y}(t)](\boldsymbol{x}) = \boldsymbol{y}(\boldsymbol{x},t) = \int_{\mathbb{R}^d} \boldsymbol{\lambda}\,\mathrm{d}\nu_{\boldsymbol{x},t}(\boldsymbol{\lambda}) \quad \text{a.e. in } \Omega\times(0,T)\,,$$

$$[\boldsymbol{K}_\nu(t)](\boldsymbol{x}) := \int_\Omega\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} a(|\hat{\boldsymbol{x}}-\boldsymbol{x}|,|\hat{\boldsymbol{\lambda}}-\boldsymbol{\lambda}|)\big(\hat{\boldsymbol{\lambda}}-\boldsymbol{\lambda}\big)\,\mathrm{d}\nu_{\hat{\boldsymbol{x}},t}(\hat{\boldsymbol{\lambda}})\,\mathrm{d}\nu_{\boldsymbol{x},t}(\boldsymbol{\lambda})\,\mathrm{d}\hat{\boldsymbol{x}}\,,$$

and $\boldsymbol{y}(0) = \boldsymbol{y}_0$ in $L^p(\Omega)^d$ as well as $\boldsymbol{y}'(0) = \boldsymbol{v}_0$ in $L^2(\Omega)^d$.

For a Banach space X, we denote by $\mathcal{C}_w([0,T];X)$ the space of functions $v : [0,T] \to X$ being continuous with respect to the weak convergence in X.

The proof of Theorem 4.1 relies upon a Galerkin approximation. As always for nonlinear problems, the essential difficulty is to obtain good a priori estimates and to pass to the limit in the approximate equation. Let $\boldsymbol{y}_\ell : [0,T] \to V_\ell$ denote a Galerkin solution corresponding to the finite dimensional subspace $V_\ell \subset L^p(\Omega)^d$. One can prove global-in-time existence of $\boldsymbol{y}_\ell$ and arrives at

$$\|\boldsymbol{y}_\ell'(t)\|_{0,2}^2 + \iint\limits_{\substack{\Omega\times\Omega\\ |\hat{\boldsymbol{x}}-\boldsymbol{x}|<\delta}} |\boldsymbol{y}_\ell(\hat{\boldsymbol{x}},t) - \boldsymbol{y}_\ell(\boldsymbol{x},t)|^p\,\mathrm{d}(\hat{\boldsymbol{x}},\boldsymbol{x}) \le c\,, \quad t \in (0,T)\,,$$

which already shows that the sequence $\{\boldsymbol{y}_\ell'\}$ is bounded in $L^\infty(0,T;L^2(\Omega)^d)$ and thus $\{\boldsymbol{y}_\ell\}$ is bounded in $\mathcal{C}([0,T];L^2(\Omega)^d)$. Using the inequality

$$|\boldsymbol{a}-\boldsymbol{b}|^p \ge \frac{1}{2}\big(|\boldsymbol{a}|^p + |\boldsymbol{b}|^p\big) - \frac{p}{2}\left(|\boldsymbol{a}|^{p-1}|\boldsymbol{b}| + |\boldsymbol{b}|^{p-1}|\boldsymbol{a}|\right)\,,$$

which is valid for $\boldsymbol{a},\boldsymbol{b} \in \mathbb{R}^d$ and $p \in [2,\infty)$ (also proven in [29]), it follows that

$$\iint\limits_{\substack{\Omega\times\Omega\\ |\hat{\boldsymbol{x}}-\boldsymbol{x}|<\delta}} |\boldsymbol{y}_\ell(\hat{\boldsymbol{x}},t) - \boldsymbol{y}_\ell(\boldsymbol{x},t)|^p\,\mathrm{d}(\hat{\boldsymbol{x}},\boldsymbol{x}) \ge \lambda\|\boldsymbol{y}_\ell(t)\|_{0,p}^p - \kappa\|\boldsymbol{y}_\ell(t)\|_{0,1}^p\,,$$

with constants $\lambda,\kappa > 0$. Therefore, the sequence $\{\boldsymbol{y}_\ell\}$ is bounded in $L^\infty(0,T;L^p(\Omega)^d)$ and thus possesses a weakly* convergent subsequence. Moreover, Young measure theory can be applied. To pass to the limit in the nonlinear term, one needs to use a result of Bellido and Carlos-Corral [11] expressing the Young measure of the sequence $\{\tilde{\boldsymbol{y}}_\ell\}$ given by $\tilde{\boldsymbol{y}}_\ell(\hat{\boldsymbol{x}},\boldsymbol{x},t) := \boldsymbol{y}_\ell(\hat{\boldsymbol{x}},t) - \boldsymbol{y}_\ell(\boldsymbol{x},t)$ in terms of the corresponding Young measure generated by $\{\boldsymbol{y}_\ell\}$. Note that since the peridynamic operator $K : L^p(\Omega)^d \to L^{p'}(\Omega)^d$ is not compact, strong convergence of $\{\boldsymbol{y}_\ell\}$ in $L^p(0,T;L^p(\Omega)^d)$ would be needed to obtain weak solutions. This, however, cannot be concluded from the a priori estimates.

The situation differs when the pairwise force function possesses a strong singularity at $\xi = 0$. Then the basic function space is $W^{\sigma,p}(\Omega)^d$, and it is possible to get strong convergence in a larger space to finally obtain the existence of weak solutions. Suppose that the pairwise force function fulfills the following assumptions, where $0 < \sigma < 1$ and $2 \le p < \infty$.

Assumption B
(B1) The function $a(\cdot,\zeta) : (0,\delta) \to \mathbb{R}$ is Lebesgue-measurable for all $\zeta \in [0,\infty)$.
(B2) The function $\boldsymbol{\zeta} \mapsto a(\xi,|\boldsymbol{\zeta}|)\boldsymbol{\zeta} : \mathbb{R}^d \to \mathbb{R}^d$ is continuous for almost all $\xi \in (0,\delta)$.
(B3) There holds $a(\xi,\zeta) = 0$ if $\xi \ge \delta$ and $\zeta \in [0,\infty)$.
(B4) There exist $c_0, c_1, c_2 \ge 0$, $\gamma_1 \in [0, d+\sigma)$ and $\gamma_2 \in [0, d+\sigma p]$ such that for all $\zeta \in [0,\infty)$ and almost all $\xi \in (0,\delta)$

$$|a(\xi,\zeta)|\,\zeta \le c_0 + c_1\xi^{-\gamma_1} + c_2\xi^{-\gamma_2}\zeta^{p-1} .$$

(B5) There exist $\mu > 0$ and a nonnegative function $\alpha \in L^1(0,\delta)$ such that for all $\zeta \in [0,\infty)$ and almost all $\xi \in (0,\delta)$

$$\int_0^\zeta a(\xi,s)\,s\,\mathrm{d}s \ge \mu\xi^{-(d+\sigma p)}\zeta^p - \alpha(\xi)\xi^{-d+1} .$$

Compared to the previous setting, (B4) and (B5) have changed taking into account now the strong singularity. Note that under the assumptions above, the peridynamic operator does not necessarily possess the representation (4.2). For the following, it is convenient to introduce the nonlinear form k via

$$k(\boldsymbol{y},\boldsymbol{z}) := \langle K\boldsymbol{y},\boldsymbol{z}\rangle = -\frac{1}{2}\iint_{\Omega\times\Omega} a(|\hat{\boldsymbol{x}}-\boldsymbol{x}|,|\boldsymbol{y}(\hat{\boldsymbol{x}})-\boldsymbol{y}(\boldsymbol{x})|)\big(\boldsymbol{y}(\hat{\boldsymbol{x}})-\boldsymbol{y}(\boldsymbol{x})\big)\cdot\big(\boldsymbol{z}(\hat{\boldsymbol{x}})-\boldsymbol{z}(\boldsymbol{x})\big)\,\mathrm{d}(\hat{\boldsymbol{x}},\boldsymbol{x}) . \tag{4.4}$$

The form $k : W^{\sigma,p}(\Omega)^d \times W^{\sigma,p}(\Omega)^d \to \mathbb{R}$ is linear and bounded in its second argument and is continuous in both of its arguments. Again, $K : W^{\sigma,p}(\Omega) \to (W^{\sigma,p}(\Omega)^d)^*$ is well-defined, bounded and strong-weak continuous.

Under Assumption B, there exists a weak solution to the peridynamic initial value problem.

Theorem 4.2 ([29])**.** *Suppose that $\boldsymbol{b} \in L^1(0,T;L^2(\Omega)^d)$ as well as initial data $\boldsymbol{y}_0 \in W^{\sigma,p}(\Omega)^d$ and $\boldsymbol{v}_0 \in L^2(\Omega)^d$ are given and the pairwise force function* (3.3) *fulfills Assumption B. Then there exists a function $\boldsymbol{y} : [0,T] \to W^{\sigma,p}(\Omega)^d$ with*

$$\begin{aligned}\boldsymbol{y} &\in \mathcal{C}_w([0,T];W^{\sigma,p}(\Omega)^d) ,\\ \boldsymbol{y}' &\in \mathcal{C}_w([0,T];L^2(\Omega)^d) ,\\ \boldsymbol{y}'' &\in L^1(0,T;(W^{\sigma,p}(\Omega)^d)^*)\end{aligned}$$

such that

$$\boldsymbol{y}'' - K\boldsymbol{y} = \boldsymbol{b} \quad \text{in } L^1(0,T;(W^{\sigma,p}(\Omega)^d)^*)$$

and $\boldsymbol{y}(0) = \boldsymbol{y}_0$ in $W^{\sigma,p}(\Omega)^d$ as well as $\boldsymbol{y}'(0) = \boldsymbol{v}_0$ in $L^2(\Omega)^d$.

Again the proof relies upon a Galerkin approximation and has its main features in obtaining a priori estimates and passing to the limit in the nonlinear term. For a Galerkin solution $\boldsymbol{y}_\ell$, we can derive the a priori estimate

$$\|\boldsymbol{y}_\ell'(t)\|_{0,2}^2 + \iint_{\substack{\Omega\times\Omega\\ |\hat{\boldsymbol{x}}-\boldsymbol{x}|<\delta}} \frac{|\boldsymbol{y}_\ell(\hat{\boldsymbol{x}},t)-\boldsymbol{y}_\ell(\boldsymbol{x},t)|^p}{|\hat{\boldsymbol{x}}-\boldsymbol{x}|^{d+\sigma p}}\,\mathrm{d}(\hat{\boldsymbol{x}},\boldsymbol{x}) \le c , \quad t \in (0,T) ,$$

which again shows the boundedness of $\{\boldsymbol{y}_\ell'\}$ in $L^\infty(0,T;L^2(\Omega)^d)$ and therefore boundedness of $\{\boldsymbol{y}_\ell\}$ in $\mathcal{C}([0,T];L^2(\Omega)^d)$. For the integral term, it follows on the one hand

$$\iint\limits_{\substack{\Omega\times\Omega\\ |\hat{\boldsymbol{x}}-\boldsymbol{x}|<\delta}} \frac{|\boldsymbol{y}_\ell(\hat{\boldsymbol{x}},t)-\boldsymbol{y}_\ell(\boldsymbol{x},t)|^p}{|\hat{\boldsymbol{x}}-\boldsymbol{x}|^{d+\sigma p}}\,\mathrm{d}(\hat{\boldsymbol{x}},\boldsymbol{x}) \geq \lambda|\boldsymbol{y}_\ell(t)|_{\sigma,p}^p - \kappa\|\boldsymbol{y}_\ell(t)\|_{0,p}^p$$

and on the other hand

$$\iint\limits_{\substack{\Omega\times\Omega\\ |\hat{\boldsymbol{x}}-\boldsymbol{x}|<\delta}} \frac{|\boldsymbol{y}_\ell(\hat{\boldsymbol{x}},t)-\boldsymbol{y}_\ell(\boldsymbol{x},t)|^p}{|\hat{\boldsymbol{x}}-\boldsymbol{x}|^{d+\sigma p}}\,\mathrm{d}(\hat{\boldsymbol{x}},\boldsymbol{x}) \geq \lambda\|\boldsymbol{y}_\ell(t)\|_{0,p}^p - \kappa\|\boldsymbol{y}_\ell(t)\|_{0,1}^p$$

for constants $\lambda,\kappa>0$. Combining these estimates, we deduce that the sequence $\{\boldsymbol{y}_\ell\}$ of approximate solutions is bounded in $L^\infty(0,T;W^{\sigma,p}(\Omega)^d)$ and thus possesses a weakly* convergent subsequence, which we shall not relabel. In order to pass to the limit in the peridynamic operator, weak (weak*) convergence is not enough – strong convergence is needed. However, thanks to the Lions–Aubin lemma, strong convergence can be obtained in the larger space $L^p(0,T;W^{\sigma-\eta,p}(\Omega)^d)$ for appropriately chosen $\eta>0$ since $W^{\sigma,p}(\Omega)$ is compactly embedded in $W^{\sigma-\eta,p}(\Omega)$. It turns out that this is sufficient for passing to the limit. Indeed, we want to show that

$$\int_0^T k(\boldsymbol{y}_\ell(t),\boldsymbol{z}_m)\,\phi(t)\,\mathrm{d}t \to \int_0^T k(\boldsymbol{y}(t),\boldsymbol{z}_m)\,\phi(t)\,\mathrm{d}t \tag{4.5}$$

for all $\boldsymbol{z}_m\in V_m$ ($m\in\mathbb{N}$, $m\leq\ell$) and $\phi\in\mathcal{C}_c^\infty(0,T)$. Here, V_m denotes the finite-dimensional Galerkin subspace of $W^{\sigma,p}(\Omega)^d$, which is chosen such that its elements are smooth. Combining the strong convergence of $\{\boldsymbol{y}_\ell\}$ with the smoothness of $\boldsymbol{z}_m$, Lebesgue's theorem on dominated convergence and Proposition 4.1 below, the limit relation (4.5) holds.

Proposition 4.1 ([29])**.** *Let $0<\eta<\min\left\{\frac{1-\sigma}{p-1},\sigma\right\}$. Then the form k given by (4.4) is also well-defined and bounded on $W^{\sigma-\eta,p}(\Omega)^d\times W^{\sigma+\eta(p-1),p}(\Omega)^d$. Furthermore, $k: W^{\sigma-\eta,p}(\Omega)^d\times W^{\sigma+\eta(p-1),p}(\Omega)^d\to\mathbb{R}$ is continuous in its first and second argument.*

The proof of this proposition strongly relies upon the nonlocal structure of the form k. First note that $\sigma-\eta>0$ and $\sigma+\eta(p-1)<1$. For $\boldsymbol{y}\in W^{\sigma-\eta,p}(\Omega)^d$ and $\boldsymbol{z}\in W^{\sigma+\eta(p-1),p}(\Omega)^d$, we observe that

$$\begin{aligned} I :=& \iint\limits_{\substack{\Omega\times\Omega\\ |\hat{\boldsymbol{x}}-\boldsymbol{x}|<\delta}} \frac{|\boldsymbol{y}(\hat{\boldsymbol{x}})-\boldsymbol{y}(\boldsymbol{x})|^{p-1}|\boldsymbol{z}(\hat{\boldsymbol{x}})-\boldsymbol{z}(\boldsymbol{x})|}{|\hat{\boldsymbol{x}}-\boldsymbol{x}|^{\gamma_2}}\,\mathrm{d}(\hat{\boldsymbol{x}},\boldsymbol{x}) \\ =& \iint\limits_{\substack{\Omega\times\Omega\\ |\hat{\boldsymbol{x}}-\boldsymbol{x}|<\delta}} |\hat{\boldsymbol{x}}-\boldsymbol{x}|^\beta \frac{|\boldsymbol{y}(\hat{\boldsymbol{x}})-\boldsymbol{y}(\boldsymbol{x})|^{p-1}}{|\hat{\boldsymbol{x}}-\boldsymbol{x}|^{(d+(\sigma-\eta)p)/p'}} \frac{|\boldsymbol{z}(\hat{\boldsymbol{x}})-\boldsymbol{z}(\boldsymbol{x})|}{|\hat{\boldsymbol{x}}-\boldsymbol{x}|^{(d+(\sigma+\eta(p-1))p)/p}}\,\mathrm{d}(\hat{\boldsymbol{x}},\boldsymbol{x}) \end{aligned}$$

with

$$\begin{aligned} \beta :&= -\gamma_2 + (d+(\sigma-\eta)p)/p' + (d+(\sigma+\eta(p-1))p)/p \\ &= -\gamma_2 + d + \sigma p \\ &\geq 0\,. \end{aligned}$$

Let us recall that $\gamma_2\in[0,d+\sigma p]$. Hence, by Hölder's inequality, we find $I\leq c\,|\boldsymbol{y}|_{\sigma-\eta,p}^{p-1}|\boldsymbol{z}|_{\sigma+\eta(p-1),p}$, and thus

$$|k(\boldsymbol{y},\boldsymbol{z})| \leq c\left(1+|\boldsymbol{y}|_{\sigma-\eta,p}^{p-1}\right)|\boldsymbol{z}|_{\sigma+\eta(p-1),p}\,.$$

Continuity of $k : W^{\sigma-\eta,p}(\Omega)^d \times W^{\sigma+\eta(p-1),p}(\Omega)^d \to \mathbb{R}$ in its first argument is shown as follows. Assume $\boldsymbol{y}_n \to \boldsymbol{y}$ in $W^{\sigma-\eta,p}(\Omega)^d$ and let $\boldsymbol{z} \in W^{\sigma+\eta(p-1),p}(\Omega)^d$. Then, passing to a subsequence, there is $h \in L^1(\Omega \times \Omega)$ such that

$$\boldsymbol{y}_n(\boldsymbol{x}) \to \boldsymbol{y}(\boldsymbol{x}) \quad \text{a.e. in } \Omega\,, \quad \frac{|\boldsymbol{y}_n(\hat{\boldsymbol{x}}) - \boldsymbol{y}_n(\boldsymbol{x})|^p}{|\hat{\boldsymbol{x}} - \boldsymbol{x}|^{d+(\sigma-\eta)p}} \leq h(\hat{\boldsymbol{x}}, \boldsymbol{x}) \quad \text{a.e. in } \Omega \times \Omega\,.$$

With Assumption (B2), the integrand of $k(\boldsymbol{y}_n, \boldsymbol{z})$ converges almost everywhere towards the integrand of $k(\boldsymbol{y}, \boldsymbol{z})$. Moreover, the integrand of $k(\boldsymbol{y}_n, \boldsymbol{z})$ can be majorized in terms of h and $\boldsymbol{z}$. Lebesgue's theorem on dominated convergence thus yields

$$k(\boldsymbol{y}_n, \boldsymbol{z}) \to k(\boldsymbol{y}, \boldsymbol{z})\,,$$

and by a standard contradiction argument, the whole sequence converges.

5 Classical solutions to peridynamics

In this section, we overview the results on solutions to the nonlinear peridynamic initial value problem in the classical sense. We restrict ourselves to the nonlinear case; for classical solutions in the linear case, see Du and Zhou [24, 67], and Emmrich and Weckner [31]. As opposed to the variational setting, for a suitably chosen Banach space X, the time derivative of the deformation $\boldsymbol{y} : [0, T] \to X$ also takes values in X, and the peridynamic operator K maps X into X instead of mapping X into its dual X^* as in the previous section. Usually, compared to the variational setting, the assumptions on the pairwise force function are stronger and the results are easier to obtain. This is why – from a chronological point of view – the theorems presented in this section were established first. Most of them rely upon a fixed-point formulation and an application of Banach's fixed-point theorem. Note that an application of Schauder's fixed-point theorem does not seem to be useful because of the lack of compactness of $K : X \to X$.

The first result on a nonlinear peridynamic-type initial value problem is due to Erbay, Erkip and Muslu [33]. The authors consider the one-dimensional infinite elastic bar and pairwise force functions of convolution type separated in their variables, i.e.,

$$u_{tt}(x, t) = \int_{\mathbb{R}} \alpha(\hat{x} - x) g(u(\hat{x}, t) - u(x, t))\, \mathrm{d}\hat{x}\,, \quad x \in \mathbb{R}\,,\ t > 0\,, \tag{5.1}$$

supplemented with initial conditions. Here, the problem is formulated with respect to the displacement field $u : \overline{\Omega} \times [0, T] \to \mathbb{R}$, $u(x, t) := y(x, t) - x$. The following results on this particular model for a peridynamic elastic bar are known.

Theorem 5.1 ([33])**.** *Let $X = \mathcal{C}_b(\mathbb{R})$ or $X = L^p(\mathbb{R}) \cap L^\infty(\mathbb{R})$ with $1 \leq p \leq \infty$. Assume $\alpha \in L^1(\mathbb{R})$ and $g \in \mathcal{C}^1(\mathbb{R})$ with $g(0) = 0$ and let initial data be given in X. Then the initial value problem to* (5.1) *is locally well-posed with solution in $\mathcal{C}^2([0, T^*]; X)$, where $T^* > 0$.*

By $\mathcal{C}_b(\mathbb{R})$, we denote the space of bounded continuous functions. If $g \in \mathcal{C}^2(\mathbb{R})$ then the result also holds for $X = \mathcal{C}^1_b(\mathbb{R})$ and $X = W^{1,p}(\mathbb{R})$ $(1 \leq p \leq \infty)$. Local well-posedness is also provided for the fractional Sobolev space $X = W^{\sigma,2}(\mathbb{R}) \cap L^\infty(\mathbb{R})$ $(\sigma > 0)$ if $g(\eta) = \eta^3$. Global existence is shown when g is linearly bounded and, e.g., for the case $g(\eta) = |\eta|^{r-1}\eta$ with $r \leq 3$.

The multi-dimensional general case with (locally) Lipschitz-continuous pairwise force function and nonzero right-hand side has been treated in [28]. Note that in the following, the pairwise force function is not necessarily of the form (3.3) and therefore also includes the linear case, which is not of the form (3.3) (see [29]). Moreover, the pairwise force function does not necessarily need to have a micropotential, which follows directly from (3.3) in the previous section.

Theorem 5.2 ([28]). *Assume* $\boldsymbol{y}_0, \boldsymbol{v}_0 \in \mathcal{C}(\overline{\Omega})^d$ *and* $\boldsymbol{b} \in \mathcal{C}([0,T];\mathcal{C}(\overline{\Omega})^d)$ *are given and define* $\rho := 2r + 2\|\boldsymbol{y}_0\|_{0,\infty}$ *for* $r > 0$. *Suppose that the pairwise force function* $\boldsymbol{f} : \overline{B(\boldsymbol{0};\delta)} \times \overline{B(\boldsymbol{0};\rho)} \to \mathbb{R}^d$ *is continuous and there exists a nonnegative function* $\ell \in L^1(B(\boldsymbol{0};\delta))$ *such that for all* $\boldsymbol{\xi}, \boldsymbol{\zeta}, \hat{\boldsymbol{\zeta}} \in \mathbb{R}^d$ *with* $|\boldsymbol{\xi}| \le \delta$ *and* $|\hat{\boldsymbol{\zeta}}|, |\boldsymbol{\zeta}| \le \rho$ *there holds*

$$|\boldsymbol{f}(\boldsymbol{\xi},\hat{\boldsymbol{\zeta}}) - \boldsymbol{f}(\boldsymbol{\xi},\boldsymbol{\zeta})| \le \ell(\boldsymbol{\xi})|\hat{\boldsymbol{\zeta}} - \boldsymbol{\zeta}| .$$

Then the peridynamic operator $K : \{\boldsymbol{v} \in \mathcal{C}(\overline{\Omega})^d : \|\boldsymbol{v} - \boldsymbol{y}_0\|_{0,\infty} \le r\} \to \mathcal{C}(\overline{\Omega})^d$ *given by*

$$(K\boldsymbol{y})(\boldsymbol{x}) = \int_{\Omega \cap B(\boldsymbol{x};\delta)} \boldsymbol{f}(\hat{\boldsymbol{x}} - \boldsymbol{x}, \boldsymbol{y}(\hat{\boldsymbol{x}}) - \boldsymbol{y}(\boldsymbol{x}))\, \mathrm{d}\hat{\boldsymbol{x}} \tag{5.2}$$

is well-defined and Lipschitz-continuous, and the initial value problem

$$\boldsymbol{y}''(t) - K\boldsymbol{y}(t) = \boldsymbol{b}(t)\,, \quad t \in (0,T), \quad \boldsymbol{y}(0) = \boldsymbol{y}_0\,, \quad \boldsymbol{y}'(0) = \boldsymbol{v}_0\,, \tag{5.3}$$

is locally well-posed with solution $\boldsymbol{y} \in \mathcal{C}^2([0,T^*];\mathcal{C}(\overline{\Omega})^d)$, *where* $0 < T^* \le T$.

This solution can be extended to a maximal time interval.

Theorem 5.3 ([28]). *Assume* $\boldsymbol{y}_0, \boldsymbol{v}_0 \in \mathcal{C}(\overline{\Omega})^d$ *and let the right-hand side* $\boldsymbol{b} : [0,\infty) \to \mathcal{C}(\overline{\Omega})^d$ *be continuous with* $\sup_{t\in[0,\infty)} \|\boldsymbol{b}(t)\|_{0,\infty} < \infty$. *Suppose the pairwise force function* $\boldsymbol{f} : \overline{B(\boldsymbol{0};\delta)} \times \mathbb{R}^d \to \mathbb{R}^d$ *is continuous and for each* $\rho > 0$ *there exists a positive function* $\ell_\rho \in L^1(B(\boldsymbol{0};\delta))$ *such that for all* $\boldsymbol{\xi}, \boldsymbol{\zeta}, \hat{\boldsymbol{\zeta}} \in \mathbb{R}^d$ *with* $|\boldsymbol{\xi}| \le \delta$ *and* $|\hat{\boldsymbol{\zeta}}|, |\boldsymbol{\zeta}| \le \rho$ *there holds*

$$|\boldsymbol{f}(\boldsymbol{\xi},\hat{\boldsymbol{\zeta}}) - \boldsymbol{f}(\boldsymbol{\xi},\boldsymbol{\zeta})| \le \ell_\rho(\boldsymbol{\xi})|\hat{\boldsymbol{\zeta}} - \boldsymbol{\zeta}| .$$

Then the peridynamic initial value problem

$$\boldsymbol{y}''(t) - K\boldsymbol{y}(t) = \boldsymbol{b}(t)\,, \quad t \in (0,\infty), \quad \boldsymbol{y}(0) = \boldsymbol{y}_0\,, \quad \boldsymbol{y}'(0) = \boldsymbol{v}_0\,,$$

with $K : \mathcal{C}(\overline{\Omega})^d \to \mathcal{C}(\overline{\Omega})^d$ *given by* (5.2), *possesses a unique maximal solution* $\boldsymbol{y} \in \mathcal{C}^2([0,T^*);\mathcal{C}(\overline{\Omega})^d)$, *where* $T^* \in (0,\infty]$. *If the solution does not blow up then* $T^* = \infty$.

Both Theorem 5.2 and Theorem 5.3 can be reformulated for $X = L^\infty(\Omega)^d$ instead of $X = \mathcal{C}(\overline{\Omega})^d$ under slightly less restrictive assumptions on the pairwise force function and the right-hand side. For deformations with values in $L^p(\Omega)^d$ ($1 \le p < \infty$), results like the above are not at hand. However, under stronger assumptions on the pairwise force function, there holds the following.

Theorem 5.4 ([28]). *For* $1 \le p \le \infty$ *let* $\boldsymbol{y}_0, \boldsymbol{v}_0 \in L^p(\Omega)^d$ *and* $\boldsymbol{b} \in \mathcal{C}([0,T];L^p(\Omega)^d)$. *Suppose that the pairwise force function* $\boldsymbol{f} : B(\boldsymbol{0};\delta) \times \mathbb{R}^d \to \mathbb{R}^d$ *is Lebesgue-measurable and there exist nonnegative Lebesgue-integrable functions* $\lambda, \ell \in L^1(B(\boldsymbol{0};\delta))$ *such that for almost all* $\boldsymbol{\xi} \in \mathbb{R}^d$ *with* $|\boldsymbol{\xi}| < \delta$ *and all* $\boldsymbol{\zeta}, \hat{\boldsymbol{\zeta}} \in \mathbb{R}^d$ *there holds*

$$|\boldsymbol{f}(\boldsymbol{\xi},\hat{\boldsymbol{\zeta}}) - \boldsymbol{f}(\boldsymbol{\xi},\boldsymbol{\zeta})| \le \ell(\boldsymbol{\xi})|\hat{\boldsymbol{\zeta}} - \boldsymbol{\zeta}|\,, \quad |f(\boldsymbol{\xi},\boldsymbol{0})| \le \lambda(\boldsymbol{\xi})\,.$$

Then the peridynamic operator $K : L^p(\Omega)^d \to L^p(\Omega)^d$, *given by* (5.2), *is well-defined and Lipschitz-continuous, and the peridynamic initial value problem* (5.3) *is globally well-posed with solution* $\boldsymbol{y} \in \mathcal{C}^2([0,T];L^p(\Omega)^d)$.

We also have the following result, which can be proven similarly to the results above.

Theorem 5.5. *Let* $\boldsymbol{y}_0, \boldsymbol{v}_0 \in \mathcal{C}(\overline{\Omega})^d$ *and* $\boldsymbol{b} \in \mathcal{C}([0,T];\mathcal{C}(\overline{\Omega})^d)$ *be given and assume that the pairwise force function* $\boldsymbol{f} : \overline{B(\boldsymbol{0};\delta)} \times \mathbb{R}^d \to \mathbb{R}^d$ *is continuous and there exists a nonnegative function* $\ell \in L^1(B(\boldsymbol{0};\delta))$ *such that for all* $\boldsymbol{\xi} \in \mathbb{R}^d$ *with* $|\boldsymbol{\xi}| \le \delta$ *and* $\boldsymbol{\zeta}, \hat{\boldsymbol{\zeta}} \in \mathbb{R}^d$ *there holds*

$$|\boldsymbol{f}(\boldsymbol{\xi},\hat{\boldsymbol{\zeta}}) - \boldsymbol{f}(\boldsymbol{\xi},\boldsymbol{\zeta})| \le \ell(\boldsymbol{\xi})|\hat{\boldsymbol{\zeta}} - \boldsymbol{\zeta}| .$$

Then the peridynamic operator $K : \mathcal{C}(\overline{\Omega})^d \to \mathcal{C}(\overline{\Omega})^d$ *is well-defined and Lipschitz-continuous, and the initial value problem* (5.3) *is globally well-posed with solution* $\boldsymbol{y} \in \mathcal{C}^2([0,T];\mathcal{C}(\overline{\Omega})^d)$.

Of course, global Lipschitz-continuity of the pairwise force function is quite a restrictive assumption since it implies linearly bounded growth.

Let us mention that a result similar to that of Theorem 5.4 with $p = 2$ has been obtained in Lipton [53] under slightly different assumptions for microelastic material.

6 Comparison of results for local elastodynamics and peridynamics

Let us summarize the results known for classical elastodynamics and for peridynamics by comparing the essential assumptions on the local and nonlocal stress, respectively. The following is known for the multi-dimensional local nonlinear case in elastodynamics.

Multi-dimensional local nonlinear elastodynamics		
Solution concept	Main assumption on $\boldsymbol{\sigma}$	References
local smooth solution	smooth $\boldsymbol{\sigma}$ with strongly elliptic derivative	[17, 47]
Young-measure-valued	strictly polyconvex potential, where $\psi \in \mathcal{C}^2$ with polynomial growth	[21]
Young-measure-valued	$\boldsymbol{\sigma} = \phi'$, $\phi \in \mathcal{C}^2$ with polynomial growth, $\boldsymbol{\sigma} + \lambda \boldsymbol{I}$ is monotone for some $\lambda \geq 0$	[61]

In the one-dimensional case, existence of weak rather than of Young-measure-valued solutions can be shown.

One-dimensional local nonlinear elastodynamics		
Solution concept	Main assumption on σ	References
weak solution	$\sigma \in \mathcal{C}^2$, $\sigma'' \neq 0$, σ strictly monotone	[46]
weak entropy solution	$\sigma \in \mathcal{C}^2$, σ'' vanishes at just one point, σ strictly monotone	[23] (also [20, 51])
Young-measure-valued	$\sigma = (\phi^{**})'$, ϕ with quadratic growth	[18]

Observe that, in fact, monotonicity-type properties of σ are required in any of the results in local elastodynamics. Nevertheless, the solution theory is far from being satisfactory.

In peridynamics, stronger results are available. Without any monotonicity assumption and without restrictions on the dimension, the following results have been obtained.

Multi-dimensional nonlinear peridynamics		
Solution concept	Main assumption on $\boldsymbol{f}$	References
weak solution	Carathéodory condition, coercivity and growth conditions for $W^{\sigma,p}$ (strong singularity)	[29]
Young-measure-valued	Carathéodory condition, coercivity and growth conditions for L^p (weak or no singularity)	[29]
local/maximal classical solution	Carathéodory/continuity condition, local Lipschitz-type condition	[28]
global classical solution	Carathéodory/continuity condition, global Lipschitz-type condition	[28]

We conclude that the theory of peridynamics allows stronger results on existence under weaker assumptions on the stress. Nevertheless, the applicability of peridynamic theory and its relation to local elastodynamics in the limit of vanishing nonlocality still has to be investigated in more detail.

Finally, we interpret Theorem 4.2 (note that $W^{\sigma,p}(\Omega)^d \hookrightarrow \mathcal{C}(\overline{\Omega})^d$ if $\sigma p > d$) as well as Theorem 5.5 as follows. If the initial data are sufficiently smooth, i.e., the body in the reference configuration is not damaged, and if the pairwise force function satisfies the assumptions of Theorem 4.2 and Theorem 5.5, respectively, then there exists a global (weak and strong, respectively) solution with values in $\mathcal{C}(\overline{\Omega})^d$, which means that no cracks will evolve spontaneously. In that sense, these theorems can be seen as regularity results.

References

[1] A. Agwai, I. Guven, and E. Madenci. Predicting crack propagation with peridynamics: a comparative study. *Int. J. Fract.*, 171(1):65–78, 2011.

[2] B. Aksoylu and T. Mengesha. Results on nonlocal boundary value problems. *Numer. Funct. Anal. Optim.*, 31(12):1301–1317, 2010.

[3] B. S. Altan. Uniqueness of initial-boundary value problems in nonlocal elasticity. *Int. J. Solids Structures*, 25(11):1271–1278, 1989.

[4] B. S. Altan. Uniqueness in nonlocal thermoelasticity. *J. Thermal Stresses*, 14:121–128, 1991.

[5] S. Altan. Existence in nonlocal elasticity. *Arch. Mech.*, 41(1):25–36, 1989.

[6] F. Andreu-Vaillo, J. Mazón, J. Rossi, and J. Toledo-Melero. *Nonlocal Diffusion Problems*. AMS, Providence, Rhode Island, 2010.

[7] G. Andrews. On the existence of solutions to the equation $u_{tt} = u_{xxt} + \sigma(u_x)_x$. *J. Differential Equations*, 35(2):200–231, 1980.

[8] G. Andrews and J. M. Ball. Asymptotic behaviour and changes of phase in one-dimensional nonlinear viscoelasticity. *J. Differential Equations*, 44(2):306–341, 1982.

[9] S. S. Antman. *Nonlinear Problems of Elasticity*. Springer, New York, 2nd edition, 2005.

[10] Z. P. Bažant and M. Jirásek. Nonlocal integral formulations of plasticity and damage: survey of progress. *J. Eng. Mech.*, 128(11):1119–1149, 2002.

[11] J. C. Bellido and C. Carlos-Corral. Existence for nonlocal variational problems in peridynamics. *SIAM J. Math. Anal.*, 46(1):890–916, 2014.

[12] F. Bobaru and Y. D. Ha. Studies of dynamic crack propagation and crack branching with peridynamics. *Int. J. Fract.*, 162(1-2):229–244, 2010.

[13] C. Carstensen and G. Dolzmann. Time-space discretization of the nonlinear hyperbolic system $u_{tt} = \operatorname{div}(\sigma(Du) + Du_t)$. *SIAM J. Numer. Anal.*, 42:75–89, 2004.

[14] C. Carstensen and M. O. Rieger. Young-measure approximations for elastodynamics with non-monotone stress-strain relations. *M2AN Math. Model. Numer. Anal.*, 38:397–418, 2004.

[15] P. G. Ciarlet. *Mathematical Elasticity - Volume I: Three-Dimensional Elasticity*. Elsevier, Amsterdam, 1988.

[16] J. Clements. Existence theorems for a quasilinear evolution equation. *SIAM J. Appl. Math.*, 26:745–752, 1974.

[17] C. Dafermos and W. Hrusa. Energy methods for quasilinear hyperbolic initial-boundary value problems. Applications to elastodynamics. *Arch. Rational Mech. Anal.*, 87:267–292, 1985.

[18] S. Demoulini. Young measure solutions for nonlinear evolutionary systems of mixed type. *Ann. Inst. Henri Poincaré Non Linéaire*, 14:143–162, 1997.

[19] S. Demoulini. Weak solutions for a class of nonlinear systems of viscoelasticity. *Arch. Rational Mech. Anal.*, 155:299–334, 2000.

[20] S. Demoulini, D. Stuart, and A. Tzavaras. Construction of entropy solutions for one dimensional elastodynamics via time discretisation. *Ann. Inst. Henri Poincaré Anal. Non Linéaire*, 17:711–731, 2000.

[21] S. Demoulini, D. Stuart, and A. Tzavaras. A variational approximation scheme for three-dimensional elastodynamics with polyconvex energy. *Arch. Rational Mech. Anal.*, 157:324–344, 2001.

[22] J. Diestel and J. J. Uhl, Jr. *Vector Measures*. AMS, Providence, Rhode Island, 1977.

[23] R. DiPerna. Convergence of approximate solutions to conservation laws. *Arch. Rational Mech. Anal.*, 82:27–70, 1983.

[24] Q. Du and K. Zhou. Mathematical analysis for the peridynamic nonlocal continuum theory. *M2AN Math. Mod. Numer. Anal.*, 45(2):217–234, 2011.

[25] N. Duruk, H. A. Erbay, and A. Erkip. Global existence and blow-up for a class of nonlocal nonlinear Cauchy problems arising in elasticity. *Nonlinearity*, 23:107–118, 2010.

[26] N. Duruk, H. A. Erbay, and A. Erkip. Blow-up and global existence for a general class of nonlocal nonlinear coupled wave equations. *J. Differential Equations*, 250(3):1448–1459, 2011.

[27] E. Emmrich, R. B. Lehoucq, and D. Puhst. Peridynamics: a nonlocal continuum theory. In M. Griebel and M. A. Schweitzer, editors, *Meshfree Methods for Partial Differential Equations VI, Lect. N. Comput. Sci. Engin.*, volume 89, pages 45–65. Springer, Berlin, 2013.

[28] E. Emmrich and D. Puhst. Well-posedness of the peridynamic model with Lipschitz continuous pairwise force function. *Commun. Math. Sci.*, 11(4):1039–1049, 2013.

[29] E. Emmrich and D. Puhst. Measure-valued and weak solutions to the nonlinear peridynamic model in nonlocal elastodynamics. *Nonlinearity*, 28:285–307, 2015.

[30] E. Emmrich and D. Šiška. Evolution equations of second order with nonconvex potential and linear damping: existence via convergence of a full discretization. *J. Differential Equations*, 255:3719–3746, 2013.

[31] E. Emmrich and O. Weckner. On the well-posedness of the linear peridynamic model and its convergence towards the Navier equation of linear elasticity. *Commun. Math. Sci.*, 5(4):851–864, 2007.

[32] H. Engler. Global regular solutions for the dynamic antiplane shear problem in nonlinear viscoelasticity. *Math. Z.*, 202:251–259, 1989.

[33] H. A. Erbay, A. Erkip, and G. M. Muslu. The Cauchy problem for a one dimensional nonlinear elastic peridynamic model. *J. Differential Equations*, 252:4392–4409, 2012.

[34] A. C. Eringen. Linear theory of nonlocal elasticity and dispersion of plane waves. *Int. J. Eng. Sci.*, 10(5):425–435, 1972.

[35] A. C. Eringen. Nonlocal polar elastic continua. *Int. J. Eng. Sci.*, 10(1):1–16, 1972.

[36] A. C. Eringen. Theory of nonlocal elasticity and some applications. *Res. Mechanica*, 21:313–342, 1987.

[37] A. C. Eringen. Vistas of nonlocal continuum physics. *Int. J. Eng. Sci.*, 30(10):1551–1565, 1992.

[38] A. C. Eringen. *Nonlocal Continuum Field Theories*. Springer, New York, 2002.

[39] A. C. Eringen and D. G. B. Edelen. On nonlocal elasticity. *Int. J. Eng. Sci.*, 10(3):233–248, 1972.

[40] A. C. Eringen and B. S. Kim. Relation between non-local elasticity and lattice dynamics. *Crystal Lattice Defects*, 7:51–57, 1977.

[41] I. Fonseca and G. Leoni. *Modern Methods in the Calculus of Variations: L^p Spaces*. Springer, New York, 2007.

[42] A. Friedmann and J. Nečas. Systems of nonlinear wave equations with nonlinear viscosity. *Pacific J. Math.*, 135:29–55, 1988.

[43] K. Friedrichs and P. Lax. Systems of conservation equations with convex extension. *Proc. Nat. Acad. Sci. USA*, 68:1686–1688, 1971.

[44] G. Friesecke and G. Dolzmann. Implicit time discretization and global convergence for a quasi-linear evolution equation with nonconvex energy. *SIAM J. Math. Anal.*, 28:363–380, 1997.

[45] W. Gerstle, N. Sau, and S. A. Silling. Peridynamic modeling of plain and reinforced concrete structures. In *SMiRT18: 18th Int. Conf. Struct. Mech. React. Technol.*, Beijing, 2005.

[46] J. Glimm. Solutions in the large for nonlinear hyperbolic systems of equations. *Comm. Pure Appl. Math.*, 18:697–715, 1965.

[47] T. Hughes, T. Kato, and J. Marsden. Well-posed quasi-linear second-order hyperbolic systems with applications to nonlinear elastodynamics and general relativity. *Arch. Rational Mech. Anal.*, 63:273–294, 1977.

[48] E. Kröner. Elasticity theory of materials with long range cohesive forces. *Int. J. Solids Structures*, 3:731–742, 1967.

[49] P. Lax. Development of singularities of solutions of nonlinear hyperbolic partial differential equations. *J. Math. Phys.*, 5:611–613, 1964.

[50] P. G. LeFloch. *Hyperbolic Systems of Conservation Laws*. Birkhäuser, Basel, 2002.

[51] P. Lin. Young measures and an application of compensated compactness to one-dimensional nonlinear elastodynamics. *Trans. Amer. Math. Soc.*, 339:377–413, 1992.

[52] J. L. Lions and E. Magenes. *Non-Homogeneous Boundary Value Problems and Applications*, volume 1. Springer, Berlin, 1972.

[53] R. Lipton. Dynamic brittle fracture as a small horizon limit of peridynamics. *J. Elast.*, 117(1):21–50, 2014.

[54] J. Málek, J. Nečas, M. Rokyta, and M. Růžička. *Weak and Measure-valued Solutions to Evolutionary PDEs*. Chapman & Hall, London, 1996.

[55] J. E. Marsden and T. J. R. Hughes. *Mathematical Foundations of Elasticity*. Prentice-Hall, Englewood Cliffs, 1983.

[56] E. Oterkus, E. Madenci, O. Weckner, S. Silling, P. Bogert, and A. Tessler. Combined finite element and peridynamic analyses for predicting failure in a stiffened composite curved panel with a central slot. *Composite Structures*, 94(3):839–850, 2012.

[57] H. Pecher. On global regular solutions of third order partial differential equations. *J. Math. Anal. Appl.*, 73:278–299, 1980.

[58] P. Pedregal. *Parametrized Measures and Variational Principles*. Birkhäuser, Basel, 1997.

[59] C. Polizzotto. Nonlocal elasticity and related variational principles. *Int. J. Solids Structures*, 38(42–43):7359–7380, 2001.

[60] A. Prohl. Convergence of a finite element-based space-time discretization in elastodynamics. *SIAM J. Numer. Anal.*, 46(5):2469–2483, 2008.

[61] M. O. Rieger. Young measure solutions for nonconvex elastodynamics. *SIAM J. Numer. Anal.*, 34(6):1380–1398, 2003.

[62] T. Roubíček. *Relaxation in Optimization Theory and Variational Calculus*. De Gruyter, Berlin, 1997.

[63] T. Roubíček. *Nonlinear Partial Differential Equations with Applications*. Birkhäuser, Basel, 2013.

[64] S. A. Silling. Reformulation of elasticity theory for discontinuities and long-range forces. *J. Mech. Phys. Solids*, 48:175–209, 2000.

[65] J. Wang and R. S. Dhaliwal. On some theorems in the nonlocal theory of micropolar elasticity. *Int. J. Solids Structures*, 30(10):1331–1338, 1993.

[66] J. Wang and R. S. Dhaliwal. Uniqueness in generalized nonlocal thermoelasticity. *J. Thermal Stresses*, 16:71–77, 1993.

[67] K. Zhou and Q. Du. Mathematical and numerical analysis of linear peridynamic models with nonlocal boundary conditions. *SIAM J. Numer. Anal.*, 48(5):1759–1780, 2010.