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FULL DISCRETIZATION OF THE POROUS MEDIUM/FAST DIFFUSION EQUATION BASED ON ITS VERY WEAK FORMULATION*

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Abstract. The very weak formulation of the porous medium/fast diffusion equation yields an evolution problem in a Gelfand triple with the pivot space H^{-1}. This allows to employ methods of the theory of monotone operators in order to study fully discrete approximations combining a Galerkin method (including conforming finite element methods) with the backward Euler scheme. Convergence is shown even for rough initial data and right-hand sides. The theoretical results are illustrated for the piecewise constant finite element approximation of the porous medium equation with the δ-distribution as initial value. As a byproduct, L^p-stability of the H^{-1}-orthogonal projection onto the space of piecewise constant functions is shown.

Key words. Porous medium equation, fast diffusion equation, very weak solution, time discretization, Galerkin method, monotone operator, convergence, rough initial data

AMS subject classifications. 65M12, 35K65, 76S05, 47J35, 47H05

1. Introduction. We will be studying fully discrete approximations to a class of nonlinear second order degenerate parabolic PDEs including the porous medium equation, heat equation and fast diffusion equation. Consider

$$u_t - \Delta(|u|^{p-2}u) = f, \quad p > 1, \quad u = u(x,t), \quad u(\cdot,0) = u_0, \quad (x,t) \in \Omega \times (0,T),$$

where $\Omega \subset \mathbb{R}^d$ is a "nice" bounded domain, $-\Delta$ is the Laplace operator acting on the spatial variables, u_0 are given initial data, f is a given right-hand side and we assume some boundary condition on the boundary of Ω. We immediately see that for $p = 2$ this is the classical heat equation. For $p > 2$, it is referred to as the porous medium equation. In this case u is the density of the gas at a given point and time. For $1 < p < 2$, the equation above is known as the fast diffusion equation.

The porous medium equation has many applications in natural sciences. As the name suggests it can model flow of gas through porous medium but it also models nonlinear heat transfer, groundwater flow (Boussinesq's model), population dynamics, thin liquid film spreading under gravity. It can be used for contour enhancement in image processing. For further applications and more details see Vázquez [22, Chapters 2 and 21].

Like in the case of heat equation, symmetry arguments can be used to derive exact solutions. We mention, in particular, the Barenblatt solution: an exact solution when the initial condition is Dirac mass at zero. For $p > 2$, in the one dimensional case, the solution is given by (5.2). We will later use this in numerical experiments. The solution is also known in higher dimensions, see again Vázquez [22, Chapter 4].

Throughout the article, we focus on the following generalization of the above

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equation. For given data f, g, u_0, we consider the initial-boundary value problem

$$\begin{cases} u_t - \Delta\alpha(u) = f \text{ in } \Omega \times (0,T)\,, \\ \alpha(u) = g \text{ on } \partial\Omega \times (0,T)\,,\ u(\cdot,0) = u_0 \text{ in } \Omega\,, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^d$ $(d \geq 1)$ is assumed to be a bounded domain of class $\mathscr{C}^{1,1}$.

With respect to the function $\alpha : \mathbb{R} \to \mathbb{R}$, we assume that α is continuous, is monotonically increasing, fulfils a growth condition, and is coercive. In particular, there are numbers $p > 1, c > 0, \mu > 0, \lambda \geq 0$ such that for all $z \in \mathbb{R}$

$$|\alpha(z)| \leq c\left(|z|^{p-1} + 1\right), \quad \alpha(z)z \geq \mu|z|^p - \lambda\,. \tag{1.2}$$

Often, α is supposed to fulfil the stronger monotonicity relation (sometimes called the "d-monotonicity", see Gajewski et al. [10, Chapter 3, Definition 1.2]).

$$(\alpha(y) - \alpha(z))(y - z) \geq \mu\left(|y|^{p-1} - |z|^{p-1}\right)(|y| - |z|)\,, \quad y, z \in \mathbb{R}\,, \tag{1.3}$$

from which strict monotonicity as well as coercivity follow. The even stronger assumption

$$(\alpha(y) - \alpha(z))(y - z) \geq \mu|y - z|^p\,, \quad y, z \in \mathbb{R}\,, \tag{1.4}$$

leads to uniform monotonicity. Note, however, that (1.4) requires $p \geq 2$.

We shall remark that one arrives at (1.1) from the degenerate nonlinear diffusion equation

$$u_t - \nabla \cdot (\psi(u)\nabla u) = f$$

when taking $\alpha(z) = \int_0^z \psi(y)dy$. A typical example is $\alpha(z) = |z|^{p-2}z$ with $\psi(z) = (p-1)\,|z|^{p-2}$ for $p > 1$. For this standard example, relation (1.4) is fulfilled with $\mu = 2^{-(p-2)}$ if $p \geq 2$.

For sufficiently smooth functions $u = u(x,t)$ and arbitrary $\tilde{v} = \tilde{v}(x)$ vanishing on the boundary $\partial\Omega$, we obtain from (1.1) by multiplying with $\tilde{v}$ and carrying out integration by parts twice

$$\frac{d}{dt}\int_\Omega u\tilde{v}dx - \int_\Omega \alpha(u)\Delta\tilde{v}dx = \int_\Omega f\tilde{v}dx - \int_{\partial\Omega} g\partial_\nu\tilde{v}d\Gamma\,,$$

where $\partial_\nu \equiv \nu \cdot \nabla$ denotes the derivative in the direction of the outer normal ν on $\partial\Omega$. If $\tilde{v} =: (-\Delta)^{-1}v$ is the solution to the homogeneous Dirichlet problem for the Poisson equation

$$-\Delta\tilde{v} = v \text{ in } \Omega\,,\ \tilde{v} = 0 \text{ on } \partial\Omega\,, \tag{1.5}$$

we formally arrive at

$$\begin{aligned} &\frac{d}{dt}\int_\Omega u(-\Delta)^{-1}vdx + \int_\Omega \alpha(u)vdx \\ &= \int_\Omega f(-\Delta)^{-1}vdx - \int_{\partial\Omega} g\partial_\nu(-\Delta)^{-1}vd\Gamma\,. \end{aligned}$$

We, finally, interpret the time derivative in the weak sense such that for all $\varphi \in \mathscr{C}_c^\infty(0,T)$

$$\begin{aligned} &-\int_0^T \int_\Omega u(-\Delta)^{-1} v\varphi'(t)dxdt + \int_0^T \int_\Omega \alpha(u) v\varphi(t)dxdt \\ &= \int_0^T \int_\Omega f(-\Delta)^{-1} v\varphi(t)dxdt - \int_0^T \int_{\partial\Omega} g\partial_\nu(-\Delta)^{-1} v\varphi(t)d\Gamma dt\,, \end{aligned} \tag{1.6}$$

supplemented by the initial condition.

This gives rise to study *very weak solutions* $u \in L^p(\Omega \times (0,T))$ to (1.1), where $\int_\Omega u(-\Delta)^{-1}vdx$ is to be interpreted as the $H^{-1}(\Omega)$-inner product (see Lions [13, pp. 191ff.] for the case $\alpha(z) = |z|^{p-2}z/(p-1)$ as well as Gajewski et al. [10, pp. 72f.]).

In this paper, we consider the very weak formulation (1.6) and propose a convergent full discretization combining a piecewise constant finite element approximation with the backward Euler scheme.

To be precise, we firstly show that (1.6) is equivalent to an evolution equation governed by a monotone and coercive operator. Next we consider the full discretization of the abstract evolution equation by combining a Galerkin method with the backward Euler scheme. We prove weak and strong convergence results in this abstract setting. These convergence results apply to the finite element/backward Euler approximation of (1.6). Numerical examples support the theoretical results (see Section 5). Let us emphasize that these results are not restricted to the porous medium equation and fast diffusion equation but in fact apply to any first order evolution equation governed by a monotone and coercive operator.

For the piecewise constant finite element approximation in one spatial dimension, we also establish the stability of the orthogonal projection of $H^{-1}(\Omega)$ onto the finite element space with respect to the $L^p(\Omega)$-operator norm. Such stability property of the projection operator has also been studied in other function spaces by other authors. See, e.g., Crouzeix & Thomée [5] for the stability of the $L^2(\Omega)$-orthogonal projection onto finite element spaces in $W^{1,p}(\Omega)$.

Spatial approximations have been studied by Mizutani et. al. [14] in $L^1(\Omega)$ using nonlinear semigroup theory. They prove convergence of the semidiscrete finite difference scheme. Furthermore they show that their numerical scheme preserves positivity for $L^1(\Omega)$ initial data. Finally they present results of numerical experiments using explicit Euler approximation in time. These and other results are also presented in a wider context in Fujita et. al. [9, Chapter 6].

Numerical approximations using finite element methods in space and Euler and Runge–Kutta methods in time have been studied previously by others. See [2], [6], [16] and [18]. For two-dimensional problems with initial data in $L^1(\Omega)$, Rulla & Walkington [20] derive optimal convergence rate of order $h+\tau$ (for space mesh width h and time step τ) in the space $L^\infty(0,T;H^{-1}(\Omega))$, using nonlinear semigroup theory, for discretizations based on continuous piecewise linear finite elements and implicit Euler method. Time and space approximation of the porous medium equation (not covering the fast diffusion equation) has recently been studied by Hansen & Ostermann [12]. For initial data $u_0 \in L^\infty(\Omega)$ and such that $\alpha(u_0) \in W^{2,1}(\Omega)$, they prove a rate of convergence estimate for Runge–Kutta methods. The estimate is proved in the norm of the dual of the space of finite element approximation of $H_0^1(\Omega)$, which is a very weak norm. Furthermore this is assuming much higher regularity of the initial data then what is assumed in this article.

The functional framework with $H^{-1}(\Omega)$ as the pivot space, which is employed in this article, allows us to apply methods and results known for evolution problems governed by monotone operators.

For abstract evolution problems with monotone operators, results on the convergence of the Galerkin method can already be found in the monographs Gajewski et al. [10] and Zeidler [24] and the references cited therein.

The convergence of the Rothe method, i.e., the implicit Euler method, for the approximation of evolution problems governed by (pseudo-) monotone operators has been dealt with, e.g., in the monograph Roubíček [19]. In Emmrich [7], convergence as well as stability and error estimates are provided for the two-step BDF applied to evolution problems governed by monotone operators with a strongly continuous perturbation, see also, e.g., Emmrich & Thalhammer [8] for Runge–Kutta methods.

The paper is organized as follows: In Section 2 the notation is introduced, the definition of the very weak solution is presented and it is shown that if the very weak solution has sufficient regularity then it is the classical solution. In Section 3 weak and strong convergence results for the full discretization of the abstract problem (2.6) are derived under rather general assumptions. In Section 4 a particular conforming finite element approximation is studied. This is then used in Section 5 for numerical experiments supporting the theoretical results.

2. Continuous problem. Throughout this paper, we assume $p \in \Pi$ where $\Pi = (1,\infty)$ for $d \in \{1,2\}$ and $\Pi = [2d/(d+2),\infty)$ for $d \geq 3$. Moreover, we denote by $q = p/(p-1)$ the exponent conjugated to p. Note that $p \geq 2$ refers indeed to the generalized porous medium equation whereas $p < 2$ corresponds to the fast diffusion equation.

We rely upon the usual notation for Lebesgue, Sobolev, Bochner-Lebesgue spaces, and spaces of continuous functions. In particular, we denote by $\|\cdot\|_{m,p,D}$ (resp. $|\cdot|_{m,p,D}$) the usual norm (resp. seminorm) in $W^{m,p}(D)$ ($p \in [1,\infty], m \in \mathbb{R}$), and we omit the subscript D if $D = \Omega$. The $L^2(\Omega)$-inner product is denoted by $(\cdot,\cdot)_{0,2}$, the usual $H_0^1(\Omega)$-inner product by $(\cdot,\cdot)_{1,2} = (\nabla\cdot,\nabla\cdot)_{0,2}$.

The dual V^* of a Banach space V is equipped with the usual norm $\|f\|_{V^*} := \sup_{v\in V\setminus\{0\}} \langle f,v\rangle_{V^*\times V}/\|v\|_V$, where $\langle\cdot,\cdot\rangle_{V^*\times V}$ denotes the duality pairing.

By c, we denote a generic positive constant.

In what follows, let $V := L^p(\Omega)$ with $\|\cdot\|_V := \|\cdot\|_{0,p}$ and $H := H^{-1}(\Omega) := (H_0^1(\Omega))^*$ with $\|\cdot\|_H := \|\cdot\|_{-1,2}$. Denoting by $\tilde{v} = (-\Delta)^{-1}v \in H_0^1(\Omega)$ the unique weak solution to (1.5) for the right-hand side $v \in H^{-1}(\Omega)$, we have that $(-\Delta)^{-1} : H^{-1}(\Omega) \to H_0^1(\Omega)$ is linear and bijective with $|(-\Delta)^{-1}v|_{1,2} = \|v\|_{-1,2}$ for all $v \in H^{-1}(\Omega)$. This shows that H is a Hilbert space with the inner product

$$\begin{aligned}(v,w)_H &= \frac{1}{4}\left(\|v+w\|_H^2 - \|v-w\|_H^2\right) \\ &= \frac{1}{4}\left(|(-\Delta)^{-1}(v+w)|_{1,2}^2 - |(-\Delta)^{-1}(v-w)|_{1,2}^2\right) \\ &= \left((-\Delta)^{-1}v, (-\Delta)^{-1}w\right)_{1,2} \\ &= \langle v, (-\Delta)^{-1}w\rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)}, \quad v,w \in H\,.\end{aligned} \tag{2.1}$$

Since $H_0^1(\Omega) \subset L^q(\Omega)$ for $p \in \Pi$, the integral $\int_\Omega vw dx$ is well-defined for all $v \in V$, $w \in H_0^1(\Omega)$. The continuous embedding $H_0^1(\Omega) \hookrightarrow L^q(\Omega)$ then shows that, for any $v \in V$, the mapping $w \mapsto \int_\Omega vw dx$ is linear and bounded on $H_0^1(\Omega)$ and, thus,

$j_1 : V \to H$ with

$$\langle j_1(v), w\rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)} = \int_\Omega vwdx\,, \quad v \in V,\, w \in H_0^1(\Omega)\,,$$

is a well-defined linear bounded injection. This proves the continuous embedding $V \overset{j_1}{\hookrightarrow} H$. Moreover, the image $j_1(V)$ is dense in H and V is reflexive. Then, we also have the continuous and dense embedding $H^* \overset{j_{-1}}{\hookrightarrow} V^*$, where j_{-1} is the dual of j_1 (see Wloka [23, pp. 261f.]). Furthermore, we can identify H and H^* by means of the Riesz isomorphism $j_0 : H \to H^*$ which is given here by

$$\langle j_0(v), w\rangle_{H^*\times H} = \langle w, (-\Delta)^{-1}v\rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)}\,, \quad v, w \in H\,,$$

see also (2.1). So, we come up with the Gelfand triple $V \overset{j_1}{\hookrightarrow} H \overset{j_0}{\equiv} H^* \overset{j_{-1}}{\hookrightarrow} V^*$.

In view of the properties of α, see (1.2), the integral $\int_\Omega \alpha(v)wdx$ is well-defined for all $v, w \in V$ and

$$\left|\int_\Omega \alpha(v)wdx\right| \le c\left(\|v\|_V^{p-1} + 1\right)\|w\|_V\,,$$

see also (1.2). This shows that, for any $v \in V$, the mapping $w \mapsto \int_\Omega \alpha(v)wdx$ is linear and bounded on V. With α, we can thus associate the operator $A : V \to V^*$, defined via

$$\langle Av, w\rangle_{V^*\times V} = \int_\Omega \alpha(v)wdx,\ v, w \in V\,.$$

The operator $A : V \to V^*$ then is continuous (see Zeidler [24, pp. 561f.]), monotone, and, in view of (1.2), bounded with

$$\|Av\|_{V^*} \le c\left(\|v\|_V^{p-1} + 1\right),\ v \in V\,. \tag{2.2}$$

We, further, have the coercivity relation

$$\langle Av, v\rangle_{V^*\times V} \ge \mu\|v\|_V^p - \lambda|\Omega|,\ v \in V\,. \tag{2.3}$$

If α fulfils (1.3) then Hölder's inequality yields

$$\langle Av - Aw, v - w\rangle_{V^*\times V} \ge \mu\left(\|v\|_V^{p-1} - \|w\|_V^{p-1}\right)(\|v\|_V - \|w\|_V)\,,\ v, w \in V\,, \tag{2.4}$$

which implies strict monotonicity as well as coercivity of $A : V \to V^*$ since V is uniformly convex (see Gajewski et al. [10, pp. 62f.]). Under the stronger assumption (1.4), the operator $A : V \to V^*$ is uniformly monotone such that

$$\langle Av - Aw, v - w\rangle_{V^*\times V} \ge \mu\|v - w\|_V^p\,. \tag{2.5}$$

Via $(Au)(t) := Au(t)$, the operator $A : V \to V^*$ can be extended to an operator $A : L^p(0,T;V) \to (L^p(0,T;V))^* \equiv L^q(0,T;V^*)$. Note that $L^p(0,T;V) = L^p(\Omega \times (0,T))$.

The duality pairing between $L^p(0,T;V)$ and $L^q(0,T;V^*) = (L^p(0,T;V))^*$ is given by

$$\langle f, v\rangle_{L^q(0,T;V^*)\times L^p(0,T;V)} = \int_0^T \langle f(t), v(t)\rangle_{V^*\times V}dt\,.$$

The solution to our problem is now sought in the Banach space

$$\mathscr{W} = \{v \in \mathscr{X} : v' \in \mathscr{X}^*\}, \quad \|v\|_{\mathscr{W}} = \|v\|_{\mathscr{X}} + \|v'\|_{\mathscr{X}^*},$$

with v' being the distributional time derivative and where

$$\mathscr{X} = L^p(0,T;V) \cap L^2(0,T;H), \quad \|v\|_{\mathscr{X}} = \|v\|_{L^p(0,T;V)} + \|v\|_{L^2(0,T;H)},$$

is a reflexive, separable Banach space. Its dual $\mathscr{X}^*$ can be identified with the sum $L^q(0,T;V^*) + L^2(0,T;H)$, equipped with the norm

$$\|f\|_{\mathscr{X}^*} = \inf_{\substack{f_1 \in L^q(0,T;V^*),\, f_2 \in L^2(0,T;H) \\ f = f_1 + f_2}} \max\left(\|f_1\|_{L^q(0,T;V^*)}, \|f_2\|_{L^2(0,T;H)}\right).$$

The duality pairing between $f = f_1 + f_2 \in L^q(0,T;V^*) + L^2(0,T;H)$ and $v \in \mathscr{X}$ is given by

$$\langle f, v\rangle_{\mathscr{X}^* \times \mathscr{X}} = \int_0^T \left(\langle f_1(t), v(t)\rangle_{V^* \times V} + (f_2(t), v(t))_H\right) dt = \int_0^T \langle f(t), v(t)\rangle_{V^* \times V} dt,$$

see, e.g., Gajewski et al. [10] for more details. Note that $\mathscr{X} \subseteq L^2(0,T;H) \subseteq \mathscr{X}^*$ forms a Gelfand triple again and that $\mathscr{W}$ is continuously embedded in $\mathscr{C}([0,T];H)$. If $p \geq 2$ then we can just take $\mathscr{X} = L^p(0,T;V)$, $\mathscr{X}^* = L^q(0,T;V^*)$.

For what follows, let $W := W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ and $f \in L^q(0,T;W^*)$. (Note that V, H, W^* form an admissible triplet in the sense of Zeidler [24, pp. 784, 598].) Due to the famous result by Agmon–Douglis–Nirenberg, there is for each $v \in V$ a unique solution $(-\Delta)^{-1}v \in W$ to (1.5) that depends continuously on v (see, e.g., Gilbarg & Trudinger [11, Thm. 9.15, Lemma 9.17]). Recall here that $\partial\Omega \in \mathscr{C}^{1,1}$. So, the mapping

$$b_f : v \mapsto \int_0^T \langle f(t), (-\Delta)^{-1} v(t)\rangle_{W^* \times W} dt$$

is linear and bounded on $L^p(0,T;V)$ and thus an element of $(L^p(0,T;V))^*$.

With respect to the boundary condition in (1.1), we observe that the trace of the derivative in direction of the outer normal (still denoted by ∂_ν) is a linear bounded map of W onto $W^{1/q,p}(\partial\Omega)$ (see Nečas [15, Thm. 5.5 on p. 99]). For boundary values $g \in L^q(0,T;(W^{1/q,p}(\partial\Omega))^*)$, we thus have that

$$b_g : v \mapsto -\int_0^T \langle g(t), \partial_\nu(-\Delta)^{-1} v(t)\rangle_{(W^{1/q,p}(\partial\Omega))^* \times W^{1/q,p}(\partial\Omega)} dt$$

is an element of $(L^p(0,T;V))^*$.

The initial value problem

$$u' + Au = b, \; u(0) = u_0. \tag{2.6}$$

with $b = b_f + b_g$ can thus be seen as a generalization of (1.1), and a solution to (2.6) with $b = b_f + b_g$ will be called a *very weak solution* to (1.1).

With respect to the solvability of (2.6) (and thus of (1.6)), we have the following result.

THEOREM 2.1 (Well-posedness). *For any $b \in \mathscr{X}^*$, $u_0 \in H$, there is a unique solution $u \in \mathscr{W}$ to (2.6). The solution u depends continuously on the data b, u_0.*

Since $A : V \to V^*$ fulfils the growth condition (2.2), is continuous, coercive, and monotone, Theorem 2.1 follows directly from Gajewski et al. [10, Satz 1.1 on p. 201, Bsp. 3 on p. 215], see also Barbu [1, Thm. 2.6 on p. 140f.] (for $p \geq 2$ only), Zeidler [24, Thm. 30.A] (with $\tilde{\mathscr{W}} := \{v \in L^p(0,T;V) : v' \in L^q(0,T;V^*)\}$ instead of $\mathscr{W}$ if $p < 2$), Roubíček [19, Thm. 8.28] (again with $\tilde{\mathscr{W}}$ instead of $\mathscr{W}$ if $p < 2$). Note that the initial condition makes sense in view of the continuous embedding $\mathscr{W} \hookrightarrow \mathscr{C}([0,T];H)$. (For $\alpha(z) = |z|^{p-2}z/(p-1)$, Theorem 2.1 then is in accordance with Lions [13, Thm. 3.1 on p. 192], allowing a slightly more general right-hand side f.)

Sufficiently regular very weak solutions to (1.1) are classical solutions as is shown in the following.

THEOREM 2.2 (Classical solution). *Let the data $u_0 \in H^{-1}(\Omega)$, $f \in L^q(0,T;W^*)$, and $g \in L^q(0,T;(W^{1/q,p}(\partial\Omega))^*)$ be given by corresponding functions $u_0 \in \mathscr{C}(\overline{\Omega})$, $f \in \mathscr{C}(\overline{\Omega} \times [0,T])$, and $g \in \mathscr{C}(\partial\Omega \times [0,T])$, such that*

$$
\begin{aligned}
\langle u_0, w\rangle_{H^{-1}(\Omega)\times H^1_0(\Omega)} &= \int_\Omega u_0(x)w(x)dx \quad \forall w \in H^1_0(\Omega)\,,\\
\langle f(t), w\rangle_{W^*\times W} &= \int_\Omega f(x,t)w(x)dx \quad \forall w \in W,\, t \in [0,T]\,,\\
\langle g(t), \partial_\nu w\rangle_{(W^{1/q,p}(\partial\Omega))^*\times W^{1/q,p}(\partial\Omega)} &= \int_{\partial\Omega} g(x,t)\partial_\nu w(x)d\Gamma \quad \forall w \in W,\, t \in [0,T]\,.
\end{aligned}
$$

Then every very weak solution u to (1.1) with

$$u \in \mathscr{D} := \{v \in \mathscr{C}(\overline{\Omega} \times [0,T]) : \exists v_t, \Delta\alpha(v) \in \mathscr{C}(\overline{\Omega} \times [0,T])\}$$

satisfies (1.1) in the classical sense.

Proof. The function u fulfils the initial condition since for all $v \in H^1_0(\Omega)$

$$
\begin{aligned}
&\int_\Omega u_0(x)v(x)dx = \langle u_0, v\rangle_{H^{-1}(\Omega)\times H^1_0(\Omega)}\\
&= \langle j_1(u(0)), v\rangle_{H^{-1}(\Omega)\times H^1_0(\Omega)} = \int_\Omega u(x,0)v(x)dx\,.
\end{aligned}
$$

By density and the properties of functions in $\mathscr{W}$, the evolution equation in (2.6) is equivalent to require for all $v \in V, \phi \in \mathscr{C}^1_c((0,T))$

$$
\begin{aligned}
&-\int_0^T (j_1(u(\cdot,t)), j_1(v))_H \phi'(t)dt + \int_0^T \int_\Omega \alpha(u(x,t))v(x)\phi(t)dxdt\\
&= \int_0^T \langle f(t), (-\Delta)^{-1}v\rangle_{W^*\times W}\phi(t)dt\\
&\quad - \int_0^T \langle g(t), \partial_\nu(-\Delta)^{-1}v\rangle_{(W^{1/q,p}(\partial\Omega))^*\times W^{1/q,p}(\partial\Omega)}\phi(t)dt\,.\\
&= \int_0^T \int_\Omega f(x,t)(-\Delta)^{-1}v(x)\phi(t)dxdt - \int_0^T \int_{\partial\Omega} g(x,t)\partial_\nu(-\Delta)^{-1}v(x)\phi(t)d\Gamma dt\,.
\end{aligned}
$$

With (2.1), the definition of j_1, $(-\Delta)^{-1}$ and A, and with integration by parts, we find for almost all $t \in (0,T)$

$$(j_1(u(\cdot,t)), j_1(v))_H = \big((-\Delta)^{-1}u(\cdot,t), (-\Delta)^{-1}v\big)_{1,2} = \int_\Omega u(x,t)(-\Delta)^{-1}v(x)dx\,.$$

Substituting $(-\Delta)^{-1}v$ by $\tilde{v} \in W$ (remember that $(-\Delta)^{-1} : V \to W$ is bijective), we obtain with integration by parts

$$\int_0^T \left(\int_\Omega (u_t(x,t) - \Delta\alpha(u(x,t)) - f(x,t))\, \tilde{v}(x)dx \right.$$
$$\left. + \int_{\partial\Omega} (g(x,t) - \alpha(u(x,t)))\partial_\nu \tilde{v}(x) d\Gamma \right) \phi(t)dt = 0\,,$$

which finally shows that u fulfils (1.1). ☐

3. Fully discrete approximation and its convergence. In this section, we derive, under rather general assumptions, weak and strong convergence results for the full discretization of the abstract problem (2.6). This abstract setting applies, but is not restricted, to the very weak formulation of the porous medium/fast diffusion equation.

For $N \in \mathbb{N}$, let $\tau := T/N$, $t_n := n\tau$ $(n = 0, 1, \ldots, N)$. Moreover, let $\{V_m\}_{m\in\mathbb{N}}$ be a Galerkin scheme for the separable Banach space V consisting of finite dimensional subspaces V_m with $V_m \subset V_{m+1}$ that satisfy the property of limited completeness

$$\mathrm{clos}_{\|\cdot\|_V} \bigcup_{m=1}^{\infty} V_m = V\,.$$

For given approximations $u^0 \in V_m$ of the initial value and $\{b^n\}_{n=1}^N \subset V^*$ of the right-hand side, we then look for approximations $\{u^n\}_{n=1}^N \subset V_m$, $u^n \approx u(t_n)$, of the exact solution to (2.6) such that for all $n = 1, 2, \ldots, N$

$$\left(\frac{u^n - u^{n-1}}{\tau}, v \right)_H + \langle Au^n, v\rangle_{V^*\times V} = \langle b^n, v\rangle_{V^*\times V} \quad \forall v \in V_m\,. \tag{3.1}$$

Recall that $A : V \to V^*$ is monotone, hemicontinuous, fulfils the growth condition (2.2) and the coercivity condition (2.3). In this abstract setting we can now derive the following results.

THEOREM 3.1 (Discrete problem and a priori estimates I). *For any $u^0 \in V_m$ and $\{b^n\}_{n=1}^N \subset V^*$ there exists a unique solution $\{u^n\}_{n=1}^N \subset V_m$ to (3.1). Let $b^n = b_1^n + b_2^n$ with $b_1^n \in V^*$ and $b_2^n \in H$ $(n = 1, 2, \ldots, N)$. Then the discrete solution $\{u^n\}_{n=1}^N$ fulfils the a priori estimate*

$$\begin{aligned} &\max_{n=1,2,\ldots,N} \|u^n\|_H^2 + \sum_{n=1}^N \|u^n - u^{n-1}\|_H^2 + \tau \sum_{n=1}^N \|u^n\|_V^p \\ &\quad \le c \left(\|u^0\|_H^2 + \tau \sum_{n=1}^N \|b_1^n\|_{V^*}^q + \tau \sum_{n=1}^N \|b_2^n\|_H^2 + 1 \right). \end{aligned} \tag{3.2}$$

Proof. Existence and uniqueness follow step by step from the Browder–Minty theorem since the operator appearing in each step is $\frac{1}{\tau}I + A$, which is strictly monotone, hemicontinuous and coercive. The a priori estimates can be proved similarly as in Emmrich [7, Sect. 4] by employing the algebraic relation

$$2(a-b)a = a^2 - b^2 + (a-b)^2\,, \quad a, b \in \mathbb{R}\,, \tag{3.3}$$

together with the coercivity of A, and Young's inequality. □

We may also derive an a priori estimate for the discrete time derivative, which later will allow us to apply the Lions–Aubin theorem in order to prove a strong convergence result.

THEOREM 3.2 (A priori estimates II). *Let $u^0 \in V_m$ and $b^n = b_1^n + b_2^n$ with $b_1^n \in V^*$ and $b_2^n \in H$ $(n = 1, 2, \ldots, N)$. The discrete solution $\{u^n\}_{n=1}^N \subset V_m$ to (3.1) then fulfils the a priori estimate*

$$\begin{aligned} &\tau \sum_{n=1}^{N} \left\| \frac{u^n - u^{n-1}}{\tau} \right\|_{V^*}^q \\ &\leq c \|P_m\|_{V \leftarrow V}^q \left(\|u^0\|_H^2 + \tau \sum_{n=1}^{N} \|b_1^n\|_{V^*}^q + \tau \sum_{n=1}^{N} \|b_2^n\|_{V^*}^q + \tau \sum_{n=1}^{N} \|b_2^n\|_H^2 + 1 \right). \end{aligned} \tag{3.4}$$

Here, $P_m : H \to V_m$ $(m \in \mathbb{N})$ is the orthogonal projection of H onto V_m and $\|P_m\|_{V \leftarrow V} := \sup_{v \in V \setminus \{0\}} \|P_m v\|_V / \|v\|_V$ denotes the operator norm of P_m as an operator in V.

Proof. By definition and inserting the scheme (3.1) , we have

$$\begin{aligned} \left\| \frac{u^n - u^{n-1}}{\tau} \right\|_{V^*} &= \sup_{v \in V \setminus \{0\}} \frac{1}{\|v\|_V} \left(\frac{u^n - u^{n-1}}{\tau}, v \right)_H \\ &= \sup_{v \in V \setminus \{0\}} \frac{1}{\|v\|_V} \left(\frac{u^n - u^{n-1}}{\tau}, P_m v \right)_H \\ &= \sup_{v \in V \setminus \{0\}} \frac{1}{\|v\|_V} \langle b^n - A u^n, P_m v \rangle_{V^* \times V} \\ &\leq \|b^n - A u^n\|_{V^*} \|P_m\|_{V \leftarrow V} . \end{aligned}$$

With the growth condition (2.2) for A, we find

$$\|b^n - A u^n\|_{V^*} \leq \|b^n\|_{V^*} + c \left(\|u^n\|_V^{p-1} + 1 \right)$$

and thus

$$\tau \sum_{n=1}^{N} \|b^n - A u^n\|_{V^*}^q \leq c\tau \sum_{n=1}^{N} \|b^n\|_{V^*}^q + c\tau \sum_{n=1}^{N} \|u^n\|_V^p + c .$$

This proves, together with (3.2), the assertion. □

The appearance of the term $\tau \sum_{n=1}^N \|b_2^n\|_{V^*}^q$ on the right-hand side in (3.4) is no problem if $p \geq 2$ or if $b \in L^q(0, T; V^*)$ (even for $p < 2$). It could also be avoided if the part b_2 of the right-hand side b is also approximated in V_m.

In what follows, we consider a sequence of time grids and finite dimensional approximations of V and prove convergence of the corresponding sequence of numerical solutions.

Let $\{m_\ell\}_{\ell \in \mathbb{N}} \subset \mathbb{N}$ and $\{N_\ell\}_{\ell \in \mathbb{N}} \subset \mathbb{N}$ be two nondecreasing sequences tending to infinity as $\ell \to \infty$. From the discrete solution $\{u^n\}_{n=0}^N$ that corresponds to the partition of $[0, T]$ with the step size $\tau_\ell = T/N_\ell$ and the finite dimensional space V_{m_ℓ}, we construct the piecewise constant interpolant u_ℓ with $u_\ell(t) = u^n$ for $t \in (t_{n-1}, t_n]$ $(n = 1, 2, \ldots, N_\ell)$, $u_\ell(0) = u^1$. Moreover, let $\hat{u}_\ell$ be the piecewise linear interpolation of the points (t_n, u^n) $(n = 0, 1, \ldots, N_\ell)$. Note that $\hat{u}_\ell$ is piecewise differentiable with $\hat{u}_\ell'(t) = (u^n - u^{n-1})/\tau_\ell$ for $t \in (t_{n-1}, t_n)$ $(n = 1, 2, \ldots, N_\ell)$. For simplicity, the approximation of the right-hand side shall be given by

$$b^n := \frac{1}{\tau_\ell} \int_{t_{n-1}}^{t_n} b(t)dt\,, \quad n = 1, 2, \ldots, N_\ell\,. \tag{3.5}$$

By b_ℓ, we denote the corresponding piecewise constant interpolant (again taking the values at the right endpoints of the subintervals). For $b = b_1 + b_2$, we also work with the corresponding approximations b_1^n, b_2^n and $b_{1,\ell}$, $b_{2,\ell}$.

The following convergence results are general results for nonlinear evolution problems governed by a monotone and coercive operator and are not only restricted to the current context of the porous medium/fast diffusion equation.

Theorem 3.3 (Weak convergence). *Let $u_0 \in H$ and $b \in \mathscr{X}^*$ be given and assume $u_\ell^0 \to u_0$ in H as $\ell \to \infty$. The sequence of piecewise constant interpolants u_ℓ corresponding to the discrete solutions to (3.1) then converges weakly in $L^p(0,T;V)$ and weakly* in $L^\infty(0,T;H)$ towards the exact solution $u \in \mathscr{W}$ to (2.6) as $\ell \to \infty$. The sequence of the corresponding piecewise linear interpolants $\hat{u}_\ell$ converges weakly* in $L^\infty(0,T;H)$ towards the exact solution u.*

Furthermore, if there is a constant $c > 0$ such that for all $\ell \in \mathbb{N}$

$$\tau_\ell \|u_\ell^0\|_V^p \le c \tag{3.6}$$

then the sequence of piecewise linear interpolants $\hat{u}_\ell$ converges weakly in $L^p(0,T;V)$ towards the exact solution u.

Remark 3.4. *The assumption (3.6) can always be fulfilled since V is dense in H. Depending on the initial datum, however, (3.6) may result in a coupling of the time step and spatial mesh size.*

The proof of the preceding convergence result will be prepared by the following preliminary results.

Lemma 3.5. *Under the assumptions of Theorem 3.3 there is an element $u \in \mathscr{X}$ and a subsequence, denoted by ℓ', such that*

$$u_{\ell'}\,,\ \hat{u}_{\ell'} \overset{*}{\rightharpoonup} u \text{ in } L^\infty(0,T;H)\,, \quad u_{\ell'} \rightharpoonup u \text{ in } L^p(0,T;V) \text{ as } \ell' \to \infty\,.$$

Moreover, there is an element $\xi \in H$ such that

$$\hat{u}_{\ell'}(T) \rightharpoonup \xi \text{ in } H \text{ as } \ell' \to \infty\,.$$

Finally, there is an element $a \in L^q(0,T;V^*)$ such that

$$Au_{\ell'} \rightharpoonup a \text{ in } L^q(0,T;V^*) \text{ as } \ell' \to \infty\,.$$

If (3.6) is fulfilled then also

$$\hat{u}_{\ell'} \rightharpoonup u \text{ in } L^p(0,T;V) \text{ as } \ell' \to \infty\,.$$

Proof. A straightforward calculation shows that

$$\|u_\ell\|_{L^\infty(0,T;H)} = \max_{n=1,2,\ldots,N_\ell} \|u^n\|_H\,, \quad \|u_\ell\|_{L^p(0,T;V)}^p = \tau_\ell \sum_{n=1}^{N_\ell} \|u^n\|_V^p$$

as well as

$$\|\hat{u}_\ell\|_{L^\infty(0,T;H)} = \max_{n=0,1,\dots,N_\ell} \|u^n\|_H \,.$$

Since the sequence $\{u_\ell^0\}_{\ell\in\mathbb{N}} \subset H$ is convergent, it is also bounded. As $b \in \mathscr{X}^*$, there holds $b = b_1 + b_2$ with $b_1 \in L^q(0,T;V^*)$ and $b_2 \in L^2(0,T;H)$. With Hölder's inequality, we find for the approximation (3.5) that

$$\tau_\ell \sum_{n=1}^{N_\ell} \|b_1^n\|_{V^*}^q \le \|b_1\|_{L^q(0,T;V^*)}^q \,, \quad \tau_\ell \sum_{n=1}^{N_\ell} \|b_2^n\|_H^2 \le \|b_2\|_{L^2(0,T;H)}^2 \,.$$

From the a priori estimate given in Theorem 3.1, we hence conclude that $\{u_\ell\}_{\ell\in\mathbb{N}}$ is bounded in $L^\infty(0,T;H)$ as well as in $L^p(0,T;V)$. Moreover, $\{\hat{u}_\ell\}_{\ell\in\mathbb{N}}$ is bounded in $L^\infty(0,T;H)$. By standard compactness (see, e.g., Brézis [4, Cor. III.26, Thm. III.27]) and density arguments, we thus have elements $u \in L^p(0,T;V) \cap L^\infty(0,T;H)$ and $\hat{u} \in L^\infty(0,T;H)$ and a subsequence, denoted by ℓ', such that

$$u_{\ell'} \rightharpoonup u \text{ in } L^p(0,T;V)\,,\; u_{\ell'} \overset{*}{\rightharpoonup} u \text{ in } L^\infty(0,T;H)\,,\; \hat{u}_{\ell'} \overset{*}{\rightharpoonup} \hat{u} \text{ in } L^\infty(0,T;H)$$

as $\ell' \to \infty$.

Because of

$$\|\hat{u}_\ell - u_\ell\|_{L^2(0,T;H)}^2 = \frac{\tau_\ell}{3} \sum_{n=1}^{N_\ell} \|u^n - u^{n-1}\|_H^2$$

and the priori estimate (3.2), we have the strong convergence

$$\hat{u}_\ell - u_\ell \to 0 \text{ in } L^2(0,T;H) \text{ as } \ell \to \infty\,. \tag{3.7}$$

As the weak* convergence in $L^\infty(0,T;H)$ yields the weak convergence in $L^2(0,T;H)$, u and $\hat{u}$ must coincide.

Again from the priori estimate (3.2), we have the uniform boundedness of $\hat{u}_\ell(T) = u^{N_\ell}$ in H, which implies the weak convergence of a subsequence.

An immediate consequence of the growth condition (2.2) and the a priori estimate (3.2) is the uniform boundedness of

$$\|Au_\ell\|_{L^q(0,T;V^*)}^q \le c(1 + \|u_\ell\|_{L^p(0,T;V)}^p)\,,$$

which shows the weak convergence of a subsequence of $\{Au_\ell\}_{\ell\in\mathbb{N}}$ in $L^q(0,T;V^*)$.

Since

$$\|\hat{u}_\ell\|_{L^p(0,T;V)}^p \le c\tau_\ell \sum_{n=0}^{N_\ell} \|u^n\|_V^p\,,$$

we get with (3.6) also the weak convergence of a subsequence $\hat{u}_{\ell'}$ in $L^p(0,T;V)$, and the limit can, in view of density arguments, only be u. □

Proof. [of Theorem 3.3] We start by rewriting the numerical scheme as

$$\langle \hat{u}_\ell'(t), v\rangle_{V^*\times V} + \langle Au_\ell(t), v\rangle_{V^*\times V} = \langle b_\ell(t), v\rangle_{V^*\times V} \quad \forall v \in V_{m_\ell}\,. \tag{3.8}$$

This equation holds pointwise for all $t \in (t_{n-1}, t_n)$ $(n = 1, 2, \dots, N_\ell)$ as well as in the distributional sense on $(0,T)$.

Let $k \in \mathbb{N}$ be arbitrary but fixed. Then (3.8) implies that for all $\ell \geq k$

$$-\int_0^T (\hat{u}_\ell(t), v)_H \phi'(t)dt + \int_0^T \langle Au_\ell(t), v\rangle_{V^*\times V}\phi(t)dt = \int_0^T \langle b_\ell(t), v\rangle_{V^*\times V}\phi(t)dt \qquad \forall v \in V_{m_k}\,,\ \phi \in \mathscr{C}_c^\infty(0,T)$$

since, in particular, $V_{m_k} \subset V_{m_\ell}$. In view of Lemma 3.5 and since (by standard arguments)

$$b_\ell \to b \text{ in } \mathscr{X}^* \text{ as } \ell \to \infty\,,$$

we find in the limit, passing to a subsequence (still denoted by ℓ) if necessary,

$$-\int_0^T (u(t), v)_H \phi'(t)dt + \int_0^T \langle a(t), v\rangle_{V^*\times V}\phi(t)dt = \int_0^T \langle b(t), v\rangle_{V^*\times V}\phi(t)dt \qquad \forall v \in V_{m_k}\,,\ \phi \in \mathscr{C}_c^\infty(0,T)\,.$$

Because of the limited completeness of the Galerkin scheme and the definition of the weak derivative, this shows that $u \in \mathscr{X}$ possesses the weak derivative

$$u' = b - a \in \mathscr{X}^*\,.$$

Therefore, we also have $u \in \mathscr{W}$, and it remains to prove $u(0) = u_0$ as well as $a = Au$.

Let $k \in \mathbb{N}$ be arbitrary but fixed. Then, for all $\ell \geq k$, we observe the following: Since $u, \hat{u}_\ell \in \mathscr{W}$, we can employ integration by parts and obtain for all $v \in V_{m_k}$ and $\phi \in \mathscr{C}^1([0,T])$

$$\begin{aligned}
&(u(T), v)_H\phi(T) - (u(0), v)_H\phi(0)\\
&= \int_0^T \Big(\langle u'(t), v\rangle_{V^*\times V}\phi(t) + \langle u(t), v\rangle_{V^*\times V}\phi'(t)\Big)dt\\
&= \int_0^T \Big(\langle b(t) - a(t), v\rangle_{V^*\times V}\phi(t) + \langle u(t), v\rangle_{V^*\times V}\phi'(t)\Big)dt\\
&= \int_0^T \Big(\langle b(t) - b_\ell(t) + \hat{u}_\ell'(t) + Au_\ell(t) - a(t), v\rangle_{V^*\times V}\phi(t) + \langle u(t), v\rangle_{V^*\times V}\phi'(t)\Big)dt\\
&= \int_0^T \Big(\langle b(t) - b_\ell(t) + Au_\ell(t) - a(t), v\rangle_{V^*\times V}\phi(t) + \langle u(t) - \hat{u}_\ell(t), v\rangle_{V^*\times V}\phi'(t)\Big)dt\\
&\qquad + (\hat{u}_\ell(T), v)_H\phi(T) - (\hat{u}_\ell(0), v)_H\phi(0)\,.
\end{aligned}$$

Taking the limit on the right-hand side and recalling $\hat{u}_\ell(0) = u_\ell^0 \to u_0$ in H, we come up with

$$(u(T), v)_H\phi(T) - (u(0), v)_H\phi(0) = (\xi, v)_H\phi(T) - (u_0, v)_H\phi(0) \quad \forall v \in V_{m_k}\,.$$

Choosing $\phi(T) = 0$ and $\phi(0) = 0$, respectively, we find that $u(0) = u_0$ and $u(T) = \xi$ in H due to the limited completeness of the Galerkin scheme in V and the density of V in H.

From (3.8), we find

$$\int_0^T \langle \hat{u}_\ell'(t), u_\ell(t)\rangle_{V^*\times V}dt + \int_0^T \langle Au_\ell(t), u_\ell(t)\rangle_{V^*\times V}dt = \int_0^T \langle b_\ell(t), u_\ell(t)\rangle_{V^*\times V}dt\,. \tag{3.9}$$

For the first term, a short calculation employing (3.3) shows that

$$
\begin{aligned}
\int_0^T \langle \hat u_\ell'(t), u_\ell(t)\rangle_{V^*\times V}\, dt &= \sum_{n=1}^{N_\ell} \int_{t_{n-1}}^{t_n} \left(\frac{u^n - u^{n-1}}{\tau}, u^n \right)_H dt \\
&= \frac{1}{2} \sum_{n=1}^{N_\ell} \left(\|u^n\|_H^2 - \|u^{n-1}\|_H^2 + \|u^n - u^{n-1}\|_H^2 \right) \\
&\geq \frac{1}{2} \|u^{N_\ell}\|_H^2 - \frac{1}{2} \|u_\ell^0\|_H^2 \, .
\end{aligned}
$$

Upon noting that $u^{N_\ell} = \hat u_\ell(T)$, the weak convergence of $\hat u_\ell(T)$ towards $\xi = u(T)$ in H and the strong convergence of u_ℓ^0 towards $u_0 = u(0)$ in H yields, together with an integration by parts,

$$
\int_0^T \langle u'(t), u(t)\rangle_{V^*\times V}\, dt = \frac{1}{2}\|u(T)\|_H^2 - \frac{1}{2}\|u_0\|_H^2 \leq \liminf_{\ell\to\infty} \int_0^T \langle \hat u_\ell'(t), u_\ell(t)\rangle_{V^*\times V}\, dt \, . \tag{3.10}
$$

For the second term on the left-hand side in (3.9), we employ the monotonicity of A. For arbitrary $w \in L^p(0,T;V)$, we have

$$
\begin{aligned}
&\int_0^T \langle Au_\ell(t), u_\ell(t)\rangle_{V^*\times V}\, dt \\
&\geq \int_0^T \left(\langle Au_\ell(t), u_\ell(t)\rangle_{V^*\times V}\, dt - \langle Au_\ell(t) - Aw(t), u_\ell(t) - w(t)\rangle_{V^*\times V} \right) dt \\
&= \int_0^T \langle Au_\ell(t), w(t)\rangle_{V^*\times V}\, dt + \int_0^T \langle Aw(t), u_\ell(t) - w(t)\rangle_{V^*\times V}\, dt \, .
\end{aligned}
$$

In the limit, we thus come up with

$$
\begin{aligned}
&\liminf_{\ell\to\infty} \int_0^T \langle Au_\ell(t), u_\ell(t)\rangle_{V^*\times V}\, dt \\
&\geq \int_0^T \langle a(t), w(t)\rangle_{V^*\times V}\, dt + \int_0^T \langle Aw(t), u(t) - w(t)\rangle_{V^*\times V}\, dt \, .
\end{aligned}
$$

Because of the strong convergence of b_ℓ towards b in $\mathscr{X}^*$ and the weak convergence of u_ℓ towards u in $\mathscr{X}$, we now obtain from (3.9)

$$
\begin{aligned}
&\int_0^T \langle u'(t) + a(t), u(t)\rangle_{V^*\times V}\, dt = \int_0^T \langle b(t), u(t)\rangle_{V^*\times V}\, dt = \lim_{\ell\to\infty} \int_0^T \langle b_\ell(t), u_\ell(t)\rangle_{V^*\times V}\, dt \\
&= \lim_{\ell\to\infty} \left(\int_0^T \langle \hat u_\ell'(t), u_\ell(t)\rangle_{V^*\times V}\, dt + \int_0^T \langle Au_\ell(t), u_\ell(t)\rangle_{V^*\times V}\, dt \right) \\
&\geq \int_0^T \langle u'(t), u(t)\rangle_{V^*\times V}\, dt + \int_0^T \langle a(t), w(t)\rangle_{V^*\times V}\, dt + \int_0^T \langle Aw(t), u(t) - w(t)\rangle_{V^*\times V}\, dt \, .
\end{aligned}
$$

This leads to

$$
\int_0^T \langle Aw(t), w(t) - u(t)\rangle_{V^*\times V}\, dt \geq \int_0^T \langle a(t), w(t) - u(t)\rangle_{V^*\times V}\, dt,
$$

and the hemicontinuity of A shows that $Au = a$ in $L^q(0,T;V^*)$ such that finally

$$u' + Au = b \text{ in } \mathscr{X}^* . \tag{3.11}$$

Under the additional assumption (3.6), we obtain, in view of Lemma 3.5, the additional convergence of the piecewise linear interpolants in time. Finally, by contradiction, we may prove that the whole sequence (and not only a subsequence) converges because of the uniqueness of the exact solution.□

Under additional assumptions, we may also prove strong convergence results. We firstly derive a result of strong convergence if the governing operator fulfils a stronger monotonicity assumption (without requiring additional compactness of the embedding of V in H).

Theorem 3.6 (Strong convergence I). *Let $u_0 \in H$ and $b \in \mathscr{X}^*$ be given, assume $u_\ell^0 \to u_0$ in H as $\ell \to \infty$ and let $A : V \to V^*$ satisfy the stronger monotonicity assumption (2.4) or even (2.5). Then, in addition to the assertions of Theorem 3.3, u_ℓ converges strongly in $L^p(0,T;V)$ towards the exact solution. Moreover, u_ℓ and $\hat{u}_\ell$ converge strongly in $L^r(0,T;H)$ for any $r \in [1,\infty)$ towards the exact solution u.*

Proof. We firstly observe with Hölder's inequality that (2.4) implies for any $v, w \in \mathscr{X}$

$$\begin{aligned}
&\int_0^T \langle Av(t) - Aw(t), v(t) - w(t)\rangle_{V^*\times V}\, dt \\
&\geq \mu \left(\|v\|_{L^p(0,T;V)}^{p-1} - \|w\|_{L^p(0,T;V)}^{p-1}\right)\left(\|v\|_{L^p(0,T;V)} - \|w\|_{L^p(0,T;V)}\right).
\end{aligned}$$

Hence, we find with (3.9)

$$\begin{aligned}
0 &\leq \mu \left(\|u_\ell\|_{L^p(0,T;V)}^{p-1} - \|u\|_{L^p(0,T;V)}^{p-1}\right)\left(\|u_\ell\|_{L^p(0,T;V)} - \|u\|_{L^p(0,T;V)}\right) \\
&\leq \int_0^T \langle Au_\ell(t) - Au(t), u_\ell(t) - u(t)\rangle_{V^*\times V}\, dt \\
&= \int_0^T \langle b_\ell(t), u_\ell(t)\rangle_{V^*\times V}\, dt - \int_0^T \langle \hat{u}_\ell'(t), u_\ell(t)\rangle_{V^*\times V}\, dt \\
&\quad - \int_0^T \langle Au_\ell(t), u(t)\rangle_{V^*\times V}\, dt - \int_0^T \langle Au(t), u_\ell(t) - u(t)\rangle_{V^*\times V}\, dt .
\end{aligned}$$

Because of (3.10) and (3.11), the limes superior of the right-hand side of the foregoing estimate can be estimated from above by

$$\int_0^T \langle b(t), u(t)\rangle_{V^*\times V}\, dt - \int_0^T \langle u'(t), u(t)\rangle_{V^*\times V}\, dt - \int_0^T \langle Au(t), u(t)\rangle_{V^*\times V}\, dt = 0 .$$

This shows that

$$\|u_\ell\|_{L^p(0,T;V)} \to \|u\|_{L^p(0,T;V)} \text{ as } \ell \to \infty .$$

Since $L^p(0,T;V)$ is uniformly convex and since u_ℓ converges already weakly towards u in $L^p(0,T;V)$, this implies (see, e.g., Brézis [4, Prop. III.30]) the strong convergence of u_ℓ towards u in $L^p(0,T;V)$.

In case of the even stronger assumption (2.5) of uniform monotonicity, we immediately have

$$\mu \|u_\ell - u\|^p_{L^p(0,T;V)} \le \int_0^T \langle Au_\ell(t) - Au(t), u_\ell(t) - u(t)\rangle_{V^*\times V}\, dt$$

and the proof above applies again.

Since $V \hookrightarrow H$, the strong convergence in $L^p(0,T;V)$ also implies the strong convergence in $L^p(0,T;H)$. Since $\{u_\ell\}$ is also bounded in $L^\infty(0,T;H)$, we arrive at the strong convergence in $L^r(0,T;H)$ for any $r \in [1,\infty)$.

The strong convergence of the linear interpolants $\hat{u}_\ell$ towards u in $L^r(0,T;H)$ for any $r \in [1,\infty)$ now follows from (3.7) together with the boundedness of $\{\hat{u}_\ell\}$ in $L^\infty(0,T;H)$. ☐

We now prove strong convergence by employing a priori estimates for the discrete time derivative (and requiring additional compactness of the embedding of V in H).

Theorem 3.7 (Strong convergence II). *Let $u_0 \in H$ and $b \in L^q(0,T;V^*)$ be given, assume $u^0_\ell \to u_0$ in H as $\ell \to \infty$ and let (3.6) be satisfied. If V is compactly embedded in H and if there is a constant $c > 0$ such that for all $\ell \in \mathbb{N}$*

$$\|P_{m_\ell}\|_{V\leftarrow V} \le c \tag{3.12}$$

then, in addition to the assertions of Theorem 3.3, u_ℓ and $\hat{u}_\ell$ converge strongly in $L^r(0,T;H)$ for any $r \in [1,\infty)$ towards the exact solution u, and the sequence of time derivatives $\hat{u}'_\ell$ converges weakly in $\mathscr{X}^*$ towards the time derivative u' of the exact solution.

Remark 3.8. *The assumption (3.12) is an additional approximation property of the underlying function spaces and is known to be satisfied in special situations by appropriate finite element approximations in $W^{1,p}(\Omega)$ with $L^2(\Omega)$ as the pivot space (see, e.g., Crouzeix & Thomée [5]). We later give an example of a finite element approximation of the porous medium equation in our setting with $L^p(\Omega) \subset H^{-1}(\Omega)$ and show that it fulfils (3.12) (see Lemma 4.2).*

Proof. [of Theorem 3.7] Theorem 3.7 is an immediate consequence of the results of Theorem 3.3 together with the following results that hold in addition to Lemma 3.5.

Under the additional assumption (3.12), Theorem 3.2 provides the uniform boundedness of the sequence of time derivatives $\hat{u}'_\ell$ in $\mathscr{X}^*$ since

$$\|\hat{u}'_\ell\|_{\mathscr{X}^*} \le \|\hat{u}'_\ell\|_{L^q(0,T;V^*)}$$

and since

$$\|\hat{u}'_\ell\|^q_{L^q(0,T;V^*)} = \tau_\ell \sum_{n=1}^{N_\ell} \left\| \frac{u^n - u^{n-1}}{\tau_\ell} \right\|^q_{V^*}.$$

It follows the weak convergence of a subsequence of the time derivatives $\hat{u}'_\ell$ towards u' in $\mathscr{X}^*$.

If $V \overset{c}{\hookrightarrow} H$ then, in view of the Lions–Aubin theorem (see, e.g., [13, Thm. 5.1 on p. 58]), $\mathscr{W} \overset{c}{\hookrightarrow} L^2(0,T;H)$, and because of $\mathscr{W} \hookrightarrow L^\infty(0,T;H)$, we also have $\mathscr{W} \overset{c}{\hookrightarrow} L^r(0,T;H)$ for any $r \in [1,\infty)$. Since $\{\hat{u}_\ell\}$ is now bounded in $\mathscr{W}$, we thus conclude with the strong convergence of a subsequence in $L^r(0,T;H)$ for any $r \in [1,\infty)$. The limit can only be the weak limit u, i..e., the exact solution. Because of the uniqueness of the exact solution, again the whole sequence must converge.

The strong convergence of the piecewise constant interpolants follows from (3.7) and the boundedness of $\{u_\ell\}$ in $L^\infty(0,T;H)$. □

REMARK 3.9. *The techniques employed for proving the preceding results, together with the techniques developed in, e.g., Emmrich [7] and Emmrich & Thalhammer [8], may allow to derive analogous convergence results also for other time discretization methods such as the two-step backward differentiation formula, the ϑ- and Crank–Nicolson scheme, stiffly accurate Runge–Kutta methods, or the discontinuous Galerkin method in time, even when using variable time grids.*

4. Approximating $L^p(\Omega) \subset H^{-1}(\Omega)$. In this section, we study a particular conforming finite element approximation in the one-dimensional case. So let $\Omega = (-L, L)$ for $L > 0$. For $m \in \mathbb{N}$ given, let Ω be equidistantly partitioned into $M := 2^m$ subintervals $(x_{j-1}, x_j]$ with $x_j = -L + jh$ ($h = 2L/M$, $j = 0, 1, \ldots, M$).

The finite element space V_m now consists of all piecewise constant functions, i.e., $v \in V_m$ if and only if $v(x) = v_j \in \mathbb{R}$ for $x \in (x_{j-1}, x_j]$ ($j = 1, 2, \ldots, M$). Besides the basis $\{\chi_i\}_{i=1}^M$ of V_m given by the characteristic functions $\chi_i := \chi_{(x_{i-1},x_i]}$, we propose to use the special basis $\{\phi_i\}_{i=1}^M$ given by

$$\phi_1 = \frac{3}{2}\chi_1 - \frac{1}{2}\chi_2\,, \quad \phi_i = -\frac{1}{2}\chi_{i-1} + \chi_i - \frac{1}{2}\chi_{i+1}\ (i = 2, 3, \ldots, M-1)\,, \tag{4.1}$$

$$\phi_M = -\frac{1}{2}\chi_{M-1} + \frac{3}{2}\chi_M\,. \tag{4.2}$$

Note the different definitions for ϕ_1 and ϕ_M.

The advantage of this second basis lies in the fact that the basis function ϕ_i as well as the solution $(-\Delta)^{-1}\phi_i$ to the corresponding homogeneous Dirichlet problem both have small support. This leads to a tridiagonal structure of both the stiffness and mass matrix. To be precise, there holds

$$\operatorname{supp}(-\Delta)^{-1}\phi_i = \operatorname{supp}\phi_i = [x_{i-2}, x_{i+1}]\,, \quad i = 2, 3, \ldots, M-1\,,$$

with obvious modifications for $i = 1$ and $i = M$. Let

$$\psi_i(x) := (-\Delta)^{-1}\phi_i(x).$$

A straightforward calculation employing the Green function shows that

$$\psi_1(x) = \begin{cases} -\frac{3}{4}(x - x_0)^2 + h(x - x_0) & \text{if } x \in [x_0, x_1]\,, \\ \frac{1}{4}(x - x_1)^2 - \frac{h}{2}(x - x_1) + \frac{h^2}{4} & \text{if } x \in (x_1, x_2]\,, \\ 0 & \text{otherwise,} \end{cases} \tag{4.3a}$$

$$\psi_i(x) = \begin{cases} \frac{1}{4}(x - x_{i-2})^2 & \text{if } x \in (x_{i-2}, x_{i-1}]\,, \\ -\frac{1}{2}\left(x - x_{i-1} - \frac{h}{2}\right)^2 + \frac{3h^2}{8} & \text{if } x \in (x_{i-1}, x_i]\,, \\ \frac{1}{4}(x_{i+1} - x)^2 & \text{if } x \in (x_i, x_{i+1}]\,, \\ 0 & \text{otherwise,} \end{cases} \tag{4.3b}$$

for $i = 2, 3, \ldots, M-1$, and

$$\psi_M(x) = \begin{cases} \frac{1}{4}(x_{M-1} - x)^2 - \frac{h}{2}(x_{M-1} - x) + \frac{h^2}{4} & \text{if } x \in (x_{M-2}, x_{M-1}]\,, \\ -\frac{3}{4}(x_M - x)^2 + h(x_M - x) & \text{if } x \in [x_{M-1}, x_M]\,, \\ 0 & \text{otherwise,} \end{cases} \tag{4.3c}$$

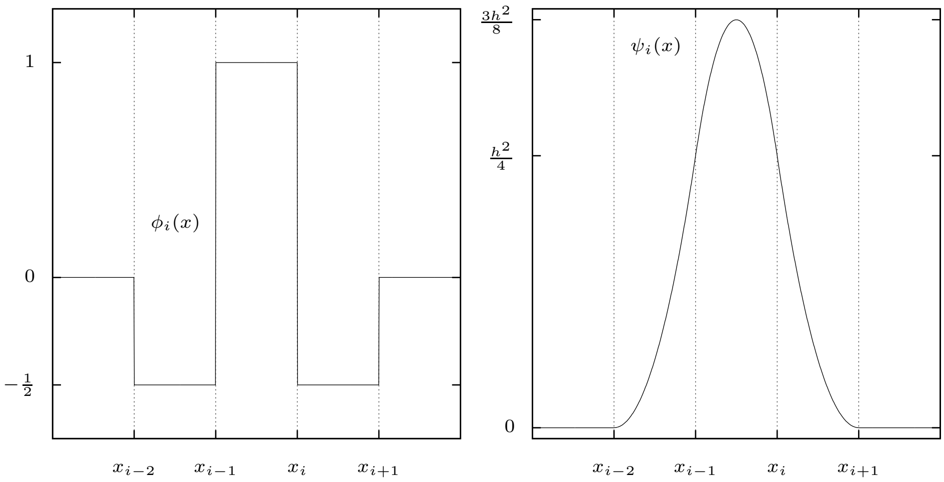

FIG. 4.1. *Special piecewise constant basis function ϕ_i and its Dirichlet solution $\psi_i := (-\Delta)^{-1}\phi_i$*

see also Fig. 4. Note that $\psi_i \in \mathscr{C}^1(\overline{\Omega})$ for $i = 1, 2, \ldots, M$.

We first analyse some approximation properties of our finite element spaces.

LEMMA 4.1. *The sequence $\{V_m\}_{m\in\mathbb{N}}$ of finite element spaces is a Galerkin scheme for $V = L^p(\Omega)$.*

Proof. Obviously, we have $V_m \subset V_{m+1} \subset V$. The limited completeness immediately follows from the following estimate of the approximation error together with the density of $W^{1,p}(\Omega)$ in $L^p(\Omega)$.

Let the restriction operator $R_m : V \to V_m$ be defined by

$$(R_m v)(x) := \frac{1}{h}\int_{x_{j-1}}^{x_j} v(\xi)d\xi \text{ for } x \in (x_{j-1}, x_j] \; (j = 1, 2, \ldots, M)\,. \tag{4.4}$$

Then there holds for any $v \in W^{1,p}(\Omega)$

$$\|v - R_m v\|_{0,p} \le h\,\|v'\|_{0,p}\,.$$

This is seen as follows: By definition, we have with Hölder's inequality that

$$\begin{aligned}
\|v - R_m v\|_{0,p}^p &= \sum_{j=1}^M \int_{x_{j-1}}^{x_j} \left| v(x) - \frac{1}{h} \int_{x_{j-1}}^{x_j} v(\xi) d\xi \right|^p dx \\
&\le \sum_{j=1}^M \int_{x_{j-1}}^{x_j} \left(\frac{1}{h} \int_{x_{j-1}}^{x_j} |v(x) - v(\xi)| d\xi \right)^p dx \\
&\le \frac{1}{h} \sum_{j=1}^M \int_{x_{j-1}}^{x_j} \int_{x_{j-1}}^{x_j} |v(x) - v(\xi)|^p d\xi dx \\
&\le h^{p/q-1} \sum_{j=1}^M \int_{x_{j-1}}^{x_j} \int_{x_{j-1}}^{x_j} \int_{x_{j-1}}^{x_j} |v'(\zeta)|^p d\zeta d\xi dx \\
&= h^p \|v'\|_{0,p}^p \,.
\end{aligned}$$

For every $v \in W^{1,p}(\Omega)$, we, therefore, obtain

$$\inf_{v_h \in V_m} \|v - v_h\|_{0,p} \le \|v - R_m v\|_{0,p} \le h \|v'\|_{0,p} \,.$$

Since $W^{1,p}(\Omega)$ is dense in V, we have for every $v \in V$

$$\inf_{v_h \in V_m} \|v - v_h\|_{0,p} \to 0 \text{ as } m \to \infty \,,$$

which is the limited completeness. □

In view of the strong convergence result in Theorem 3.7, we shall now prove that the finite element approximation above possesses the property (3.12), i.e., that the $H^{-1}(\Omega)$-orthogonal projection onto the finite dimensional space of piecewise constant functions is $L^p(\Omega)$-stable.

Due to different Lebesgue exponents (2 for $H^{-1}(\Omega)$ but p for $L^p(\Omega)$) and the strongly nonlocal character of the $H^{-1}(\Omega)$-norm, the proof cannot be based upon an estimate of the approximation order of the restriction together with an inverse inequality. Instead, we directly calculate the projection relying on our two different bases. This result is of interest for its own and is, to the best knowledge of the authors, the first result in this direction for $H^{-1}(\Omega)$-orthogonal projections.

Lemma 4.2. *The sequence $\{V_m\}_{m \in \mathbb{N}}$ of finite element spaces fulfils (3.12), i.e, the $H^{-1}(\Omega)$-orthogonal projection onto the space of piecewise constant finite elements is $L^p(\Omega)$-stable.*

Proof. As previously, we denote the H^{-1}-orthogonal projection onto V_m by P_m. For arbitrary $v \in V = L^p(\Omega)$, we find with the restriction (4.4)

$$\|P_m v\|_{0,p} \le \|P_m v - R_m v\|_{0,p} + \|R_m v\|_{0,p} \,.$$

With Hölder's inequality, we immediately find

$$\|R_m v\|_{0,p} = \left(\sum_{k=1}^M \int_{x_{k-1}}^{x_k} \left| \frac{1}{h} \int_{x_{k-1}}^{x_k} v(\xi) d\xi \right|^p dx \right)^{1/p} \le \|v\|_{0,p} \,.$$

Moreover, $P_m R_m v = R_m v$ for all $v \in L^p(\Omega)$. We thus find

$$\|P_m v\|_{0,p} \le \|P_m (v - R_m v)\|_{0,p} + \|v\|_{0,p} \,,$$

and it remains to prove that $\|P_m(v - R_m v)\|_{0,p} \le c\,\|v\|_{0,p}$.

Let $z = P_m(v - R_m v) \in V_m$. Then z is uniquely determined by

$$(z, w)_{-1,2} = (v - R_m v, w)_{-1,2} \quad \forall w \in V_m \,. \tag{4.5}$$

Since $z = \sum_{j=1}^{M} z_j \chi_j$ for some vector $\boldsymbol{z} := [z_1, \ldots, z_M]^T \in \mathbb{R}^M$, (4.5) is equivalent to the linear system

$$\sum_{j=1}^{M} (\chi_j, \phi_i)_{-1,2}\, z_j = (v - R_m v, \phi_i)_{-1,2}\,, \quad i = 1, 2, \ldots, M\,. \tag{4.6}$$

Here we have used our first basis as trial but our second basis as test functions. Because of (2.1) and together with $\psi_i := (-\Delta)^{-1}\phi_i$ (recall (4.3)), we see that (4.6) is equivalent to the linear system

$$\boldsymbol{G}\boldsymbol{z} = \boldsymbol{y}\,, \quad \text{where } \boldsymbol{G}_{ij} = (\psi_i, \chi_j)_{0,2}\,,\ y_i = (v - R_m v, \psi_i)_{0,2}\,.$$

A straightforward calculation shows that

$$\boldsymbol{G}_{ij} = \int_{x_{j-1}}^{x_j} \psi_i(x)dx = \frac{h^3}{12}\begin{cases} 3 & \text{if } j = i = 1 \text{ or } j = i = M \\ 1 & \text{if } j = i - 1 \\ 4 & \text{if } j = i \text{ and } i \ne 1 \text{ and } i \ne M \\ 1 & \text{if } j = i + 1 \\ 0 & \text{otherwise.} \end{cases} \tag{4.7}$$

Denoting by $\|\cdot\|_r$ ($r \in [1, \infty]$) the usual r-vector norm on $\mathbb{R}^M$ as well as the induced matrix norm, we now find

$$\|P_m(v - R_m v)\|_{0,p} = \|z\|_{0,p} = \left(h \sum_{i=1}^{M} |z_i|^p\right)^{1/p} = h^{1/p}\|\boldsymbol{z}\|_p \le h^{1/p}\|\boldsymbol{G}^{-1}\|_p \|\boldsymbol{y}\|_p\,. \tag{4.8}$$

Since $\boldsymbol{G}$ is symmetric, also $\boldsymbol{G}^{-1}$ is symmetric, and the maximum row and column sum norms coincide. By interpolation (Riesz–Thorin theorem, see, e.g., Bergh & Löfström [3, Thm. 1.1.1]), we find

$$\|\boldsymbol{G}^{-1}\|_p \le \|\boldsymbol{G}^{-1}\|_1^{1/p}\,\|\boldsymbol{G}^{-1}\|_\infty^{1/q} = \|\boldsymbol{G}^{-1}\|_\infty\,.$$

Since $\boldsymbol{G}$ is strictly diagonal dominant, the ∞-norm of $\boldsymbol{G}^{-1}$ can easily be estimated (see, e.g., [17, Lemma 2.13 on p. 32]), and we obtain

$$\|\boldsymbol{G}^{-1}\|_\infty \le 6h^{-3}$$

and thus

$$\|\boldsymbol{G}^{-1}\|_p \le \|\boldsymbol{G}^{-1}\|_\infty \le ch^{-3}\,. \tag{4.9}$$

We now estimate $\|\boldsymbol{y}\|_p$. Since with (4.3) and (4.4) for $i = 2, 3, \ldots, M-1$

$$
\begin{aligned}
y_i = (v - R_m v, \psi_i)_{0,2} &= \frac{1}{4}\int_{x_{i-2}}^{x_{i-1}} \left(v(x) - \frac{1}{h}\int_{x_{i-2}}^{x_{i-1}} v(\xi)d\xi \right) (x - x_{i-2})^2 dx \\
&+ \int_{x_{i-1}}^{x_i} \left(v(x) - \frac{1}{h}\int_{x_{i-1}}^{x_i} v(\xi)d\xi \right) \left(-\frac{1}{2}\left(x - x_{i-1} - \frac{h}{2}\right)^2 + \frac{3h^2}{8} \right) dx \\
&+ \frac{1}{4}\int_{x_i}^{x_{i+1}} \left(v(x) - \frac{1}{h}\int_{x_{i-2}}^{x_{i-1}} v(\xi)d\xi \right) (x_{i+1} - x)^2 dx \\
&= \frac{1}{4}\int_{x_{i-2}}^{x_{i-1}} v(x) \left((x - x_{i-2})^2 - \frac{h^2}{3} \right) dx \\
&+ \int_{x_{i-1}}^{x_i} v(x) \left(-\frac{1}{2}\left(x - x_{i-1} - \frac{h}{2}\right)^2 + \frac{3h^2}{8} - \frac{h^2}{3} \right) dx \\
&+ \frac{1}{4}\int_{x_i}^{x_{i+1}} v(x) \left((x_{i+1} - x)^2 - \frac{h^2}{3} \right) dx \,,
\end{aligned}
$$

we obtain with Hölder's inequality

$$
|y_i| \le ch^2 \int_{x_{i-2}}^{x_{i+1}} |v(x)|\, dx \le ch^{2+1/q} \left(\int_{x_{i-2}}^{x_{i+1}} |v(x)|^p\, dx \right)^{1/p} ,
$$

with obvious modifications for $i = 1$ and $i = M$. It immediately follows that

$$
\|\boldsymbol{y}\|_p \le ch^{2+1/q} \|v\|_{0,p} \,. \tag{4.10}
$$

Altogether, we thus have from (4.8), (4.9), (4.10)

$$
\|P_m(v - R_m v)\|_{0,p} \le ch^{1/p-3+2+1/q} \|v\|_{0,p} = c\,\|v\|_{0,p} \,,
$$

which finally proves the assertion. □

Finally, we can state a result about the convergence of the full discretization. We only focus on the piecewise constant in time approximation; results for the piecewise linear in time approximation can readily be gathered from Theorem 3.3, 3.6, and 3.7. Note here that the assumption (3.6) may lead to a coupling of the time step and spatial mesh size, especially for rough initial data.

Corollary 4.3. *Let $\Omega = (-L, L)$ and let $u_0 \in H^{-1}(\Omega)$, $f \in L^q(0,T;(W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega))^*)$, $g \in L^q(0,T;(W^{1/q,p}(\partial\Omega))^*)$ be given. Let (1.1) be approximated by the piecewise constant finite element method combined with the backward Euler method as described above. The piecewise constant in time and space numerical solution then converges weakly in $L^p(\Omega \times (0,T))$ and weakly* in $L^\infty(0,T;H^{-1}(\Omega))$ towards the exact very weak solution as the time step and mesh size goes to zero.*

If one of the stronger monotonicity assumptions (1.3) or (1.4) are fulfilled then the numerical solution converges also strongly in $L^p(\Omega \times (0,T))$ and in $L^r(0,T;H^{-1}(\Omega))$ for any $r \in [1,\infty)$.

If (3.6) holds then the numerical solution converges strongly in $L^r(0,T;H^{-1}(\Omega))$ for any $r \in [1,\infty)$.

Proof. The weak convergence of the numerical solution immediately results from Theorem 3.3 applied to the very weak formulation of (1.1) with $V = L^p(\Omega)$ and

$H = H^{-1}(\Omega)$. Note in particular that $L^p(0,T;V) = L^p(\Omega \times (0,T))$ for $V = L^p(\Omega)$ and recall that f, g lead to the right-hand side $b \in L^q(0,T;V^*)$ in the abstract formulation.

The strong convergence in case of one of the stronger monotonicity assumptions (1.3) or (1.4) is a direct consequence of Theorem 3.6.

If (3.6) is fulfilled then strong convergence follows from Theorem 3.7 together with Lemma 4.2, which shows that assumption (3.12) is satisfied. Recall here that, in the one-dimensional case, $L^p(\Omega)$ is compactly embedded in $H^{-1}(\Omega)$ for any $p \in \Pi$.$\square$

For a discussion of a possible step size restriction due to (3.6) see Remark 5.1.

5. Numerical results for the Barenblatt solution. We rely upon the notation of the preceding section. In particular, recall that we are working with two types of basis functions: $\chi_i := \chi_{(x_{i-1},x_i]}$ and ϕ_i, which are given by (4.1). Recall that the matrix $\boldsymbol{G}$ is given by (4.7).

For all $n = 0, 1, \ldots, N$, let $u^n \in V_m$ be given by

$$u^n(x) = \sum_{i=1}^{M} u_i^n \phi_i(x)\,, \quad x \in [-L, L]\,.$$

As test functions, we take the basis functions χ_i ($i = 1, 2, \ldots, M$). Let $\boldsymbol{u}^n := [u_1^n, \ldots, u_M^n]^T$. The numerical scheme (3.1) then is equivalent to

$$\boldsymbol{G}\,\frac{\boldsymbol{u}^n - \boldsymbol{u}^{n-1}}{\tau} + \boldsymbol{A}(\boldsymbol{u}^n) = \boldsymbol{b}^n\,, \quad n = 1, 2, \ldots, N\,,$$

where

$$\boldsymbol{A}(\boldsymbol{u}^n) := [\alpha_1^n, \ldots, \alpha_M^n]^T \quad \text{with} \quad \alpha_j^n := \int_{x_{j-1}}^{x_j} \alpha\left(\sum_{i=1}^{M} u_i^n \phi_i(x)\right) dx\,,$$

such that

$$\begin{aligned}
\alpha_1^n &= h\alpha\left(\frac{3}{2}\,u_1^n - \frac{1}{2}\,u_2^n\right),\\
\alpha_j^n &= h\alpha\left(-\frac{1}{2}\,u_{j-1}^n + u_j^n - \frac{1}{2}\,u_{j+1}^n\right) \quad \text{for } j = 2, 3, \ldots, M-1\,,\\
\alpha_M^n &= h\alpha\left(-\frac{1}{2}\,u_{M-1}^n + \frac{3}{2}\,u_M^n\right).
\end{aligned}$$

Moreover, $\boldsymbol{u}^0$ and

$$\boldsymbol{b}^n := [b_1^n, \ldots, b_M^n]^T \quad \text{with} \quad b_j^n := \int_{x_{j-1}}^{x_j} b^n(x)dx$$

are given. We recall here that $\boldsymbol{G}$ is symmetric.

If we take χ_i ($i = 1, 2, \ldots, M$) as trial and ϕ_i ($i = 1, 2, \ldots, M$) as test functions, the resulting numerical scheme reads as

$$\boldsymbol{G}\,\frac{\tilde{\boldsymbol{u}}^n - \tilde{\boldsymbol{u}}^{n-1}}{\tau} + \tilde{\boldsymbol{A}}(\tilde{\boldsymbol{u}}^n) = \tilde{\boldsymbol{b}}^n\,, \quad n = 1, 2, \ldots, N\,, \tag{5.1}$$

where now $\tilde{\boldsymbol{u}} := [\tilde{u}_1^n, \ldots, \tilde{u}_M^n]^T$ such that

$$u^n(x) = \sum_{i=1}^{M} \tilde{u}_i^n \chi_i(x) , \quad x \in [-L, L] ,$$

with $\tilde{\boldsymbol{u}}^0$ given. Moreover, we have $\tilde{\boldsymbol{A}}(\tilde{\boldsymbol{u}}^n) := [\tilde{\alpha}_1^n, \ldots, \tilde{\alpha}_M^n]^T$ with

$$\begin{aligned}
\tilde{\alpha}_1^n &= \frac{3h}{2}\,\alpha(u_1^n) - \frac{h}{2}\,\alpha(u_2^n) , \\
\tilde{\alpha}_j^n &= -\frac{h}{2}\,\alpha(u_{j-1}^n) + h\alpha(u_j^n) - \frac{h}{2}\,\alpha(u_{j+1}^n) \quad \text{for } j = 2, 3, \ldots, M-1 , \\
\tilde{\alpha}_M^n &= -\frac{h}{2}\,\alpha(u_{M-1}^n) + \frac{3h}{2}\alpha(u_M^n) .
\end{aligned}$$

Finally, we have $\tilde{\boldsymbol{b}}^n := [\tilde{b}_1^n, \ldots, \tilde{b}_M^n]^T$ with

$$\begin{aligned}
\tilde{b}_1^n &:= \frac{3}{2}\int_{x_0}^{x_1} b^n(x)dx - \frac{1}{2}\int_{x_1}^{x_2} b^n(x)dx , \\
\tilde{b}_j^n &:= -\frac{1}{2}\int_{x_{j-2}}^{x_{j-1}} b^n(x)dx + \int_{x_{j-1}}^{x_j} b^n(x)dx - \frac{1}{2}\int_{x_j}^{x_{j+1}} b^n(x)dx \text{ for } j = 2, 3, \ldots, M-1 , \\
\tilde{b}_M^n &:= -\frac{1}{2}\int_{x_{M-2}}^{x_{M-1}} b^n(x)dx + \frac{3}{2}\int_{x_{M-1}}^{x_M} b^n(x)dx .
\end{aligned}$$

The mass matrix here is again $\boldsymbol{G}$ since if $\boldsymbol{M}$ denotes the matrix for the basis transformation then $\boldsymbol{G} = \boldsymbol{M}\boldsymbol{G}\boldsymbol{M}^{-1}$.

Note that in both the cases, the unknown coefficient vector can be determined step-by-step without any additional information about the first and last coefficient (there are as many equations as unknowns). In what follows, we restrict our considerations to the second representation.

In what follows, we describe our test problem. Let u_0 be the δ-distribution, which is indeed an element of $H^{-1}(\Omega)$ since Ω is one-dimensional. Moreover, let $f = g = 0$ such that $b = 0$. Finally, we consider $\alpha(z) = |z|^{p-2}z$.

The exact (very weak) solution then is the Barenblatt solution given by

$$u(x,t) = t^{-1/p}\left(C - \frac{p-2}{2p(p-1)}\,x^2 t^{-2/p}\right)_+^{1/(p-2)} , \tag{5.2}$$

where $(\cdot)_+$ denotes the positive part of a real number and where $C > 0$ is chosen such that

$$\int_\Omega u(t,x) = 1, \quad \text{i.e.,} \quad C = \frac{1}{\sqrt[3]{15}^2}\left(\frac{\Gamma(\frac{7}{4})}{\Gamma(\frac{1}{2})\Gamma(\frac{5}{4})}\right)^{4/3} .$$

See, e.g., Vázquez [22], for a multidimensional formula and further details.

On the current grid, the initial condition is approximated by

$$u^0(x) = \frac{1}{2h}\begin{cases} 1 & \text{for } x \in (x_{M/2-1}, x_{M/2+1}] , \\ 0 & \text{otherwise} \end{cases}$$

such that $\tilde{\boldsymbol{u}}^0 = [0, \ldots, 0, 1, 1, 0, \ldots, 0]^T/2h$ and $\boldsymbol{u}^0 = \boldsymbol{M}\tilde{\boldsymbol{u}}^0$.

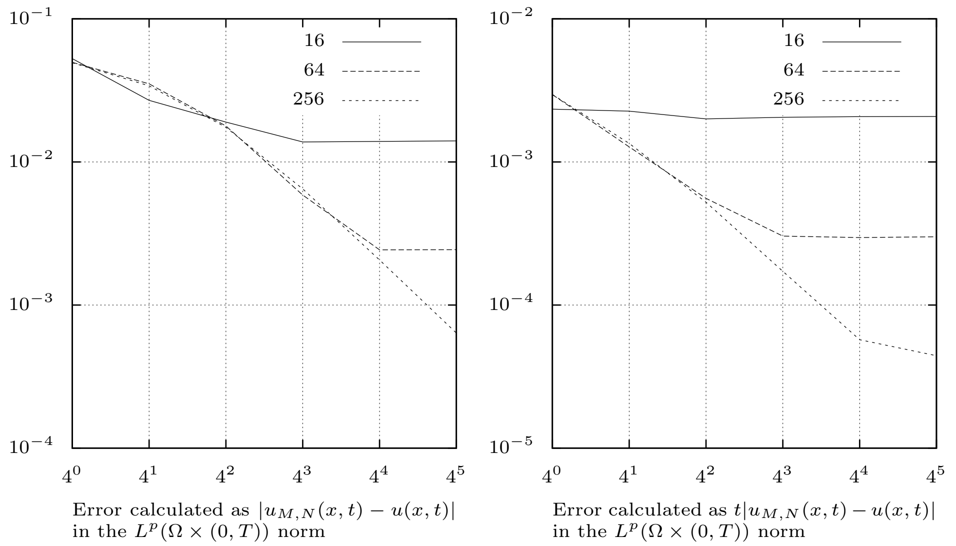

FIG. 5.1. *Plot of convergence of the numerical solution to the Barenblatt solution for $p = 3$. The scales are logarithmic. Different lines correspond to different number of finite elements (from 16 to 256).*

$\tau \Downarrow \quad h \Rightarrow$	**0.750**	**0.186**	**0.047**	**0.0118**
1.0E-01	1.5E-01	5.3E-02	5.0E-02	4.9E-02
2.5E-02	1.4E-01	2.7E-02	3.5E-02	3.4E-02
6.3E-03	1.3E-01	1.9E-02	1.8E-02	1.8E-02
1.5E-03	1.3E-01	1.4E-02	5.9E-03	6.5E-03
3.9E-04	1.3E-01	1.4E-02	2.4E-03	2.1E-03
9.8E-05	1.3E-01	1.4E-02	2.4E-03	6.4E-04

FIG. 5.2. *The table of convergence errors measured as $|u_{M,N}(x,t) - u(x,t)|$ in the $L^3((-1.5, 1.5) \times (0.01, 0.11))$ norm. Each column quarters the spatial mesh size h and each row quarters the time step τ.*

Note that, to the best knowledge of the authors, this is the first paper in which the analytical framework allows to deal with the δ-distribution as initial condition.

We now describe and present the results of numerical tests. We choose the domain $(-1.5, 1.5) \times (0, 0.11)$. We only consider the case $p = 3$. We solve the approximation (5.1). Each time step solves a nonlinear problem using the Newton's iterative method. We present the output of the numerical calculation, the exact solution and the difference between the exact solution and the numerical calculation as 3D plots in Figures 5.4 and 5.5. The time range of the plot is $[0.01, 0.11]$.

Let us denote by $u_{M,N}$ the piecewise constant interpolation of the numerical solution for the time grid with N subintervals and the spatial grid with M subintervals. We calculate two error measures: the first error measure is $|u_{M,N}(x,t) - u(x,t)|$ in the $L^3((-1.5, 1.5) \times (0.01, 0.11))$ norm and the second is the "time-weighted" error $t|u_{M,N}(x,t) - u(x,t)|$, again in the $L^3((-1.5, 1.5) \times (0.01, 0.11))$ norm. Note that while the scheme runs well when started from an approximation of the δ-distribution, the

$\tau \Downarrow \quad h \Rightarrow$	**0.750**	**0.186**	**0.047**	**0.0118**
1.0E-01	5.6E-03	2.3E-03	3.0E-03	3.0E-03
2.5E-02	1.3E-02	2.3E-03	1.3E-03	1.3E-03
6.3E-03	1.3E-02	2.0E-03	5.6E-04	5.2E-04
1.5E-03	1.3E-02	2.1E-03	3.0E-04	1.7E-04
3.9E-04	1.3E-02	2.1E-03	3.0E-04	5.7E-05
9.8E-05	1.3E-02	2.1E-03	3.0E-04	4.4E-05

FIG. 5.3. *The table of convergence errors measured as* $t|u_{M,N}(x,t) - u(x,t)|$ *in the* $L^3((-1.5, 1.5) \times (0.01, 0.11))$ *norm. Each column quarters the spatial mesh size* h *and each row quarters the time step* τ*.*

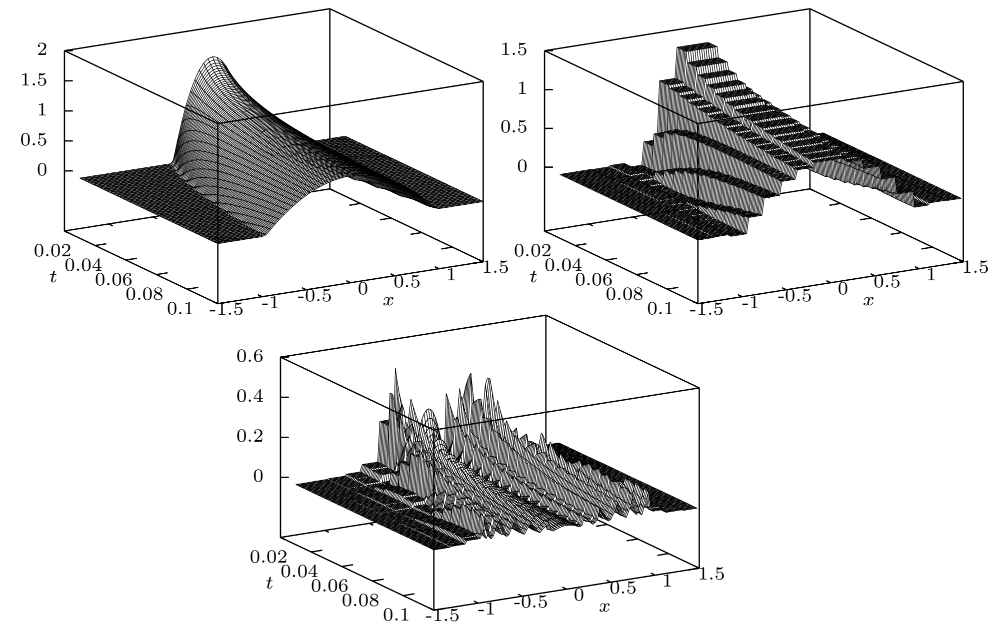

FIG. 5.4. *Exact solution, numerical solution and the respective errors for* $p = 3$ *on interval* $[-1.5, 1.5]$ *with* 16 *finite elements and time steps. Vertical axes have different scales.*

exact solution is not calculated well for very small values of t on the computer. This is the reason for starting the error measurement slightly away from zero.

Due to Corollary 4.3 we have strong convergence in energy space because the uniform monotonicity condition (1.4) is satisfied. Numerical tests illustrate and support the theoretical results, see Figures 5.1, 5.2 and 5.3. Notice that, again because of uniform monotonicity, we do not need the coupling of time and space grid sizes (3.6).

With respect to the condition (3.6) in combination with the δ-distribution as the initial datum, we have the following remark.

REMARK 5.1. *The assumption (3.6) in general means a coupling of the time step and spatial mesh size. Let* $u_0 = \delta$ *be approximated as above. Then*

$$\frac{\tau}{h^{p-1}} \leq c$$

is required for satisfying (3.6), which is rather a severe restriction.

It is interesting to mention that the time-weighted error seems to be of first order even for the δ-distribution as initial value (see Figure 5.3 and the right-hand side

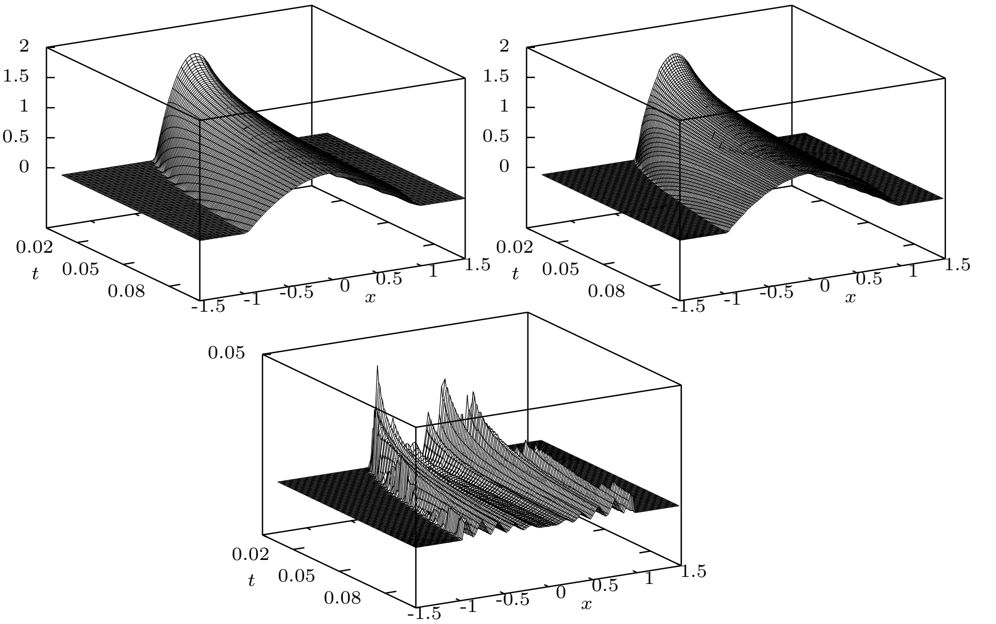

FIG. 5.5. *Exact solution, numerical solution and the respective errors for* $p = 3$ *on interval* $[-1.5, 1.5]$. *Used* 256 *finite elements and* 1024 *time steps. Vertical axes have different scales.*

plot in Figure 5.1). This resembles non-smooth data error estimates and leads to the question whether the porous medium equation exhibits a parabolic smoothing property.

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