Dear Professor Manteuffel,

Enclosed you will find the manuscript: “Towards Domain Decomposition for Nonlocal Problems.” This is jointly authored with Michael L. Parks, and we would like to submit this paper for publication in SIAM Journal on Numerical Analysis.

To the best of our knowledge, this paper represents the first work on domain decomposition methods for nonlocal models, specifically, an instance of the nonlocal p-Laplace operator. Our aim is to generalize iterative substructuring methods to a nonlocal setting and characterize the impact of nonlocality upon the scalability of these methods. We present a nonlocal variational formulation and establish spectral equivalences to bound the underlying stiffness matrix and Schur complement condition numbers by proving a nonlocal Poincaré inequality and an upper bound. We also present the supporting numerical evidence. Furthermore, we introduce a nonlocal two-domain variational formulation utilizing nonlocal transmission conditions, and prove equivalence with the single-domain formulation, a first step in establishing rigorous theory for domain decomposition methods.

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We hope that you find this paper acceptable for SINUM. Please feel free to contact me if I can be of any further assistance.

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Outline

1. Conditioning analysis of the stiffness matrix for integrable kernels
   - Summary of existing results

2. Nonlocal domain decomposition
   - Conditioning analysis of the Schur complement matrix

3. Conditioning analysis for singular kernels
   - Spectral analysis in fractional Sobolev spaces $H^s$

4. Crime(Crack) Scene Investigation (CSI) of $\lambda^{\min}$ and $\lambda^{\max}$ by $\delta$- and $h$-quantification in $H^s$
The bilinear forms with kernels from a certain family:

\[ a(u, v) := \frac{1}{2} \int_{\Omega} \int_{\Omega} C(x, x') [u(x') - u(x)][v(x') - v(x)] \, dx' \, dx \]

\[ b(u, v) := \frac{1}{2} \int_{\Omega} \int_{\Omega} C(x, x') \frac{[u(x') - u(x)][v(x') - v(x)]}{|x' - x|^{d+2s}} \, dx' \, dx \]

**Properties of the kernel function** \( C(x, x') \)

- **Radial;** \( C = C(|x - x'|) \).
- **Anti-symmetric;** \( C(x, x') \cdot [u(x') - u(x)] = -C(x', x) \cdot [u(x) - u(x')] \).
- **Positive;** \( C(r) \geq 0 \) for \( r \in [0, \infty) \) and \( C(r) > 0 \) for \( r \in (0, \delta) \).
- **Integrable;** \( C(r)r^{d-1} \in L_{loc}^1(0, \infty) \).

For **nonlocal characterization** of Sobolev spaces by Bourgain, Brezis, and Mironescu, we utilize mollifiers \( \rho_\delta \in L_{loc}^1(0, \infty) \) and \( \rho_\delta \geq 0 \) with moment conditions:

\[ \omega_d \int_0^\infty \rho_\delta(r)r^{d-1} = 1, \quad \forall \delta > 0, \quad \lim_{\delta \to 0} \int_{\delta_0}^\infty \rho_\delta(r)r^{d-1} \, dr = 0, \quad \forall \delta_0 > 0. \]
Consider \( \gamma \in L^1_{loc}(0, \infty) \) with \( \gamma \geq 0 \), \( \text{supp}(\gamma) \subset [0, 2) \),
\[ \gamma(r)r^{d-1} \in L^1_{loc}(0, \infty), \text{ and } \int_0^\infty \gamma(r)r^{d+1}dr = 1. \]
Then, the sequence
\[
\rho_\delta(r) := \frac{1}{\omega_d \delta^{d+2}} \gamma(r/\delta) r^2
\]
satisfies the moment conditions for nonlocal characterization.
If we choose,
\[
C(r) = \gamma(r/\delta),
\]
then
\[
\int\int_{\Omega} \int\int_{\Omega} \frac{|u(x) - u(x')|^2}{|x - x'|^2} \rho_\delta(|x - x'|) \, dx' \, dx = \frac{1}{\omega_d \delta^{d+2}} a(u, u).
\]
Corollary of the nonlocal Poincaré inequality

For $C$ as above $a(\cdot, \cdot)$ is coercive on $V_M$ (also $V_D$) and $V_N$. Furthermore, there exists $\delta_0 = \delta_0(\overline{\Omega}, \gamma)$ and $\lambda = \lambda(\overline{\Omega}, \delta_0)$ such that for $0 < \delta < \delta_0$ and $u \in V_M, V_N$

$$\lambda \delta^{d+2} \|u\|_{L^2(\overline{\Omega})}^2 \leq a(u, u).$$

Theorem (spectral equivalence gives well-posedness and conditioning)

$$\lambda(\overline{\Omega}, \delta_0) \delta^{d+2} \leq \frac{a(u, u)}{\|u\|_{L^2(\overline{\Omega})}^2} \leq \lambda(\overline{\Omega}, \gamma) \delta^d, \quad \delta \leq \delta_0, \quad u \in V_M, V_N.$$

The stiffness matrix $K$ produced by the discretized $a(u, u)$ has the following condition number bound:

$$\kappa(K) \lesssim \delta^{-2}.$$
The upper bound is *sharp* in 1D

Choose the canonical kernel $C(|x - x'|) = \chi_\delta(|x - x'|)$ on $\overline{\Omega} := [-1, 2]$ with the following piecewise constant function:

$$u(x) := \begin{cases} 
1, & x \in [0, \delta] \\
0, & \text{otherwise.}
\end{cases}$$

The Rayleigh quotient becomes

$$\frac{a(u, u)}{\|u\|^2_{L^2(\overline{\Omega})}} = \frac{\delta^2}{\delta} = \delta.$$
Conditioning for the nonradial kernel case

Du-Gunzburger-Lehoucq-Zhou 2012 studied this case.

Let $\gamma(x, x') \geq 0, x' \in \mathcal{H}_x(\delta), \gamma(x, x') \geq \gamma_0, x' \in \mathcal{H}_x(\delta/2)$, and $\gamma(x, x') = 0, x' \notin \mathcal{H}_x(\delta)$

Nonradial kernel bounded by radial functions

Let $s \in (0, 1)$ and $\gamma_*, \gamma^*, \gamma_1, \gamma_2 > 0$.

$$\frac{\gamma^*}{|x - x'|^{d+2s}} \leq \gamma(x, x') \leq \frac{\gamma^*}{|x - x'|^{d+2s}} \Rightarrow \kappa(K) \leq c \ h^{-2s}$$

$$\gamma_1 \leq \int_{\Omega \cap \mathcal{H}_x(\delta)} \gamma(x, x') dx', \quad \int_{\Omega} \gamma^2(x, x') dx' \leq \gamma_2 \Rightarrow \kappa(K) \leq c,$$

where $c$ is a generic constant which may depend on $\delta$. 
Conditioning in the $h \to 0$ regime

For our canonical kernel, Zhou-Du 2010 report:

$$\kappa(K) \leq c \min\{h^{-2}, \delta^{-2}\}$$

As $\delta \to 0$, the estimate recovers the classical local condition number.

Identifying $\delta$-dependence is important for $h \ll \delta$

We used:

1D experiments: $\delta = 100h, 200h, 400h$ with $h = 1/8000$.

2D experiments: $\delta = 5h, 10h, 20h$ with $h = 1/200$.

$$\kappa(K) \leq c \ \delta^{-2}.$$
Important condition number implication

Condition number of the stiffness matrix depends (weakly) on the mesh size $h$ but **bounded independently from $h$**.

Figure: Condition number for $K$ in 2D with canonical kernel. (Left) Fixed $\delta$, varying $h$. (Right) Fixed $h$, varying $\delta$. The condition number is weakly $h$-independent and varies with $\delta^{-2}$. 
The interface $\Gamma$ is 2-dimensional. Define the overlapping subdomains $\Omega^{(i)}$:

$$\Omega^{(i)} := \Omega_i \cup \Gamma \cup \Gamma_i,$$

where $\Gamma_i$ is the open line segment adjacent to $\Omega_i$ and $\Gamma$. 
We define the spaces, $i = 1, 2,$

$$\mathcal{V}^{(i)} := \left\{ v \in L^2(\overline{\Omega^{(i)}}) : v|_{B\Omega^{(i)}} = 0 \right\},$$

$$\mathcal{V}^{(i), 0} := \left\{ v \in L^2(\overline{\Omega^{(i)}}) : v|_{B\Omega^{(i)} \cup \Gamma} = 0 \right\},$$

$$\Lambda := \left\{ \mu \in L^2(\Gamma) : \mu = v|_{\Gamma} \text{ for some suitable } v \in L^2(\Omega) \right\}.$$

Define a bilinear form: $a_{\Omega^{(i)}}(u, v) : \mathcal{V} \times \mathcal{V} \to \mathbb{R}$ as follows:

$$a_{\Omega^{(i)}}(u, v) := - \int_{\Omega_i} \left\{ \int_{\Omega^{(i)} \cup B\Omega^{(i)}} \chi_{\delta}(\mathbf{x} - \mathbf{x}') \left[ u(\mathbf{x}') - u(\mathbf{x}) \right] d\mathbf{x}' \right\} v(\mathbf{x}) d\mathbf{x} \quad - \frac{1}{2} \int_{\Gamma} \left\{ \int_{\overline{\Omega}} \chi_{\delta}(\mathbf{x} - \mathbf{x}') \left[ u(\mathbf{x}') - u(\mathbf{x}) \right] d\mathbf{x}' \right\} v(\mathbf{x}) d\mathbf{x}.$$
Nonlocal domain decomposition equivalence

The two-domain weak formulation

Find \( u^{(i)} \in V^{(i)} \), \( i = 1, 2 \):

\[
\begin{align*}
    a_{\Omega^{(i)}}(u^{(i)}, v^i) &= (b, v^i)_{\Omega^i} \quad \forall v^i \in V^{(i),0}, \quad (1a) \\
    u^{(1)} &= u^{(2)} \quad \text{on } \bar{\Gamma}, \quad (1b) \\
    \sum_{i=1,2} a_{\Omega^{(i)}}(u^{(i)}, R^{(i)} \mu) &= (b, \mu)_\Gamma + \sum_{i=1,2} (b, R^{(i)} \mu)_{\Omega^i} \quad \forall \mu \in \Lambda. \quad (1c)
\end{align*}
\]

where \( R^{(i)} \) denotes any possible extension operator from \( \Gamma \cup \Gamma_i \) to \( V^{(i)} \).

Theorem

The single domain and two-subdomain weak (1) formulations are equivalent.
Conditioning of the Schur complement

Energy minimizing extension (analog of the local harmonic extension)

\[ E_i : \Gamma_h \subset L_2(\Gamma) \rightarrow V_h^{(i)} \] is the discrete energy minimizing extension into \( \Omega_i \)

\[ E_i(q)|_{\Gamma} = q, \]
\[ a(E_i(q), v) = 0, \quad v \in V_h^{(i), 0}. \]

Energy minimizing property of \( E_i(u_\Gamma) \), among \( u \in V_h^{(i)} \) with \( u|_{\Gamma} = u_\Gamma \):

\[ a_i(E_i(u_\Gamma), E_i(u_\Gamma)) \leq a_i(u, u). \]

\[ s_i(u_\Gamma, u_\Gamma) \leq a_i(u, u) \leq \lambda \delta^d \frac{\| u \|^2_{L_2(\Omega^{(i)})}}{\| u \|^2_{L_2(\Gamma)}}, \quad \forall u \in V_h^{(i)}, \text{ in particular, } u = u_\Gamma \]

\[ s_i(u_\Gamma, u_\Gamma) \leq \lambda \delta^d \| u \|^2_{L_2(\Gamma)}. \]

For the lower bound:

\[ \lambda \delta^{d+2} \| u \|^2_{L_2(\Gamma)} \leq \lambda \delta^{d+2} \| E_i(u_\Gamma) \|^2_{L_2(\Omega^{(i)})} \leq a_i(E_i(u_\Gamma), E_i(u_\Gamma)) = s_i(u_\Gamma, u_\Gamma). \]
Spectral equivalence for the Schur complement matrix

\[ \lambda \delta^{d+2} \leq \frac{s_i(q, q)}{\|q\|_{L^2(\Gamma)}^2} \leq \bar{\lambda} \delta^d, \quad q \in L^2(\Gamma). \]

The condition number of the Schur complement matrix \( S_\Gamma := S^{(1)} + S^{(2)} \) has the following bound:

\[ \kappa(S_\Gamma) \approx \delta^{-2}. \]
Condition number summary for $a(u,u)$

$$
\lambda_K \delta^{d+2} \leq \frac{a(u,u)}{\|u\|_2^2_{L_2(\Omega)}} \leq \bar{\lambda}_K \delta^d, \quad \kappa(K) \lesssim \delta^{-2}
$$

$$
\lambda_S \delta^{d+2} \leq \frac{s(u_\Gamma, u_\Gamma)}{\|u_\Gamma\|_2^2_{L_2(\Gamma)}} \leq \bar{\lambda}_S \delta^d, \quad \kappa(S_\Gamma) \lesssim \delta^{-2}
$$

$$
\lambda_{\text{sharp}} \delta^{d+1} \leq \frac{s(u_\Gamma, u_\Gamma)}{\|u_\Gamma\|_2^2_{L_2(\Gamma)}} \leq \bar{\lambda}_{\text{sharp}} \delta^d, \quad \kappa_{\text{sharp}}(S_\Gamma) \lesssim \delta^{-1}
$$

$$
k_1 \leq \frac{\ell(u, u)}{\|u\|_2^2_{L_2(\Omega)}} \leq k_2 h^{-2}, \quad \kappa(K_{\text{local}}) \lesssim h^{-2}
$$

$$
k_3 \leq \frac{s_{\text{local}}(u, u)}{\|u\|_\Gamma} \leq k_4 h^{-1}, \quad \kappa(S_{\text{local}}) \lesssim h^{-1}
$$

In conditioning, $\delta$ somewhat plays the role of $h$ probably due to intrinsic lengthscale. Laplace operator in a nonlocal sauce.
Related publications


Minimum eigenvalue characterization

Since $|\mathbf{x}' - \mathbf{x}|^{d+2s} \leq \delta^{d+2s}$, then $\frac{a(u,u)}{\delta^{d+2s}} \leq b(u, u)$. By Corollary for the same family of kernels, we immediately have:

$$\lambda \frac{\delta^{d+2}}{\delta^{d+2s}} \|u\|^2_{L^2(\Omega)} \leq b(u, u).$$

Hence, $\lambda_{\min} \sim \delta^{2-2s} h^d$. Unlike the $a(\cdot, \cdot)$ case, $\lambda_{\min}(\delta, h, s, \Omega) = c(s) \delta^{2-2s} h^d$.

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<td>49.2809</td>
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<tr>
<td>$2^{-10}$</td>
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<td>8.9164</td>
<td>10.9082</td>
<td>19.6765</td>
<td>49.2809</td>
</tr>
</tbody>
</table>
Numerical setup

Piecewise linear FEM with homogenous Dirichlet BD, 1D domain
\( \Omega = (0, 1), \)
\( h = 2^{-n}, \) hence, the system size \( N = 2^n - 1. \)

\[
\begin{array}{cccccccc}
  h & \frac{1}{2^{10}} & \frac{1}{2^{11}} & \frac{1}{2^{12}} & \frac{1}{2^{13}} & \frac{1}{2^{14}} & \frac{1}{2^{15}} & \frac{1}{2^{16}} \\
  N & 1023 & 2047 & 4095 & 8191 & 16383 & 32767 & 65535 \\
\end{array}
\]

\( \delta = 1/\{2^8, \ldots, 2^{12}\}, \)
Shift function $c(s)$

We call $c(s)$ the shift function in $\lambda^{\min}(\delta, h, s, \Omega) = c(s)\delta^{2-2s}h^d$.

The plot is the same for both fixed mesh and fixed delta indicating that it only depends on $s$.

**Figure:** $\delta = 2^{-10}$ and $h = 2^{-12}$. 
Numerical verification of $\lambda^{\text{min}}$

Based on the theoretical result and omitting $\Omega$ dependence, we want to verify that

$$
\lambda^{\text{min}}(\delta, s, h) = c(s)\delta^{m(s)}h^{r(s)}
$$

where $m(s) = 2 - 2s$ and $r(s) = 1$.

By fixing $h$ and $s$, by using two values of $\delta$; $\delta_1$ and $\delta_2$, we can extract $c(s_0)$:

$$
\lambda_1^{\text{min}} := \lambda^{\text{min}}(\delta_1, s_0, h_0) = c(s_0)\delta_1^{m(s_0)}h_0^{r(s_0)}
$$

$$
\lambda_2^{\text{min}} := \lambda^{\text{min}}(\delta_2, s_0, h_0) = c(s_0)\delta_2^{m(s_0)}h_0^{r(s_0)}
$$

$$
m(s_0) = \frac{\log \lambda_2^{\text{min}} - \log \lambda_1^{\text{min}}}{\log \delta_2 - \log \delta_1}.
$$
Verification of $m(s) = 2 - 2s$

**Figure:** Plot of $m(s) = 2 - 2s$
Characterization of the shift function $c(s)$

To find an expression for $c(s_0)$, connect through $m(s_0)$:

$$
(m(s_0)) = \frac{\log \lambda_2^{\text{min}} - \log(c(s_0)h_0^{r(s_0)})}{\log \delta_2} = \frac{\log \lambda_1^{\text{min}} - \log(c(s_0)h_0^{r(s_0)})}{\log \delta_1}
$$

$$
c(s_0) = e^{\frac{\log \delta_2 \log \lambda_1^{\text{min}} - \log \delta_1 \log \lambda_2^{\text{min}}}{\log \delta_2 - \log \delta_1}} / h_0^{r(s_0)}
$$

Since we have $c(s)$ at hand, we can identify $\delta$- and $h$-quantifications of $\lambda^{\text{min}}$. Define

$$
c^{\text{min}}(h, s) := \frac{\lambda^{\text{min}}(\delta, s, h)}{c(s)\delta^{2-2s}} \bigg|_{\delta=\delta_0}
$$

$$
c^{\text{min}}(\delta, s) := \frac{\lambda^{\text{min}}(\delta, s, h)}{c(s)h^d} \bigg|_{h=h_0}
$$
Plot of $c^\text{min}(\delta, s) := \frac{\lambda^\text{min}(\delta, h, s)}{c(s)h^d}$

Figure: $c^\text{min}(\delta, s)$ as a function of $\delta$ for each $s$.

Figure: $c^\text{min}(\delta, s)$ as a function of $s$ for each $\delta$. 
Plot of $c_{shifted}(\delta, s) = c(s)c_{\text{min}}(\delta, s)$

**Figure:** $c_{\text{min}}(\delta, s)$ as a function of $\delta$ for each $s$.  

**Figure:** $c_{\text{min}}(\delta, s)$ as a function of $s$ for each $\delta$. 
3D log scale plot of $c_{\min}(\delta, s) := \frac{\lambda_{\min}(\delta, h_0, s)}{c(s) h_0^d}$, $h_0 = 2^{-12}$.

Figure: (Light blue) $c_{\min}(\delta, s) := \frac{\lambda_{\min}(\delta, h_0, s)}{c(s) h_0^d}$, (dark blue) $\text{guess}(\delta, s) = \delta^{2-2s}$.
$h$- and $s$-quantification of $c^{\min}(h, s), \delta_0 = 2^{-8}$
Norm equivalence between $b(u, u)$ and $\|u\|_{H^s(\Omega)}^2$

It is easy to prove that

$$|u|_{H^s(\Omega)}^2 \leq b(u, u) + 4|\Omega|\delta^{-(d+2s)}\|u\|_{L^2(\Omega)}^2.$$

Du-Lehoucq-Gunzburger-Zhou 2012 gave a nonlocal Poincare inequality for $b(u, u)$.

$$c_{Pncr}\|u\|_{L^2(\Omega)}^2 \leq b(u, u).$$

Hence,

$$\|u\|_{H^s(\Omega)}^2 \leq c(\delta, s, \Omega)b(u, u).$$

On the other hand, we trivially have

$$b(u, u) \leq 1\|u\|_{H^s(\Omega)}^2.$$

Consequently, we have the norm equivalence

$$b(u, u) \sim \|u\|_{H^s(\Omega)}^2, \quad c = c(\delta, s, \Omega).$$
Our strategy of finding $\delta$- and $h$-quantification of $\lambda^{\text{max}}$ is based on the norm equivalence $b(u, u) \sim \|u\|_{H^{s}(\Omega)}^2$.

\[
b(u, u) \leq c_{1}(\delta, s, \Omega)\|u\|_{H^{s}(\Omega)}^2 \\
\leq c_{2}(s, \kappa, k)c_{1}(\delta, s, \Omega)h^{-2s}\|u\|_{L^2(\Omega)}^2 \\
\leq c_{3}(\kappa)c_{2}(s, \kappa, k)c_{1}(\delta, s, \Omega)h^{d-2s} u^t u.
\]

Since $\kappa, k = 1$, and $d = 1$, $\Omega = (0, 1)$ are fixed, only concentrate on

\[
\lambda^{\text{max}}(\delta, h, s) = c(\delta, s)h^{d-2s}.
\]

This quantification is compatible with the condition number estimate given by Du-Lehoucq-Gunzburger-Zhou 2012; $\kappa(K) \lesssim h^{-2s}$. 

Power of $h$-quantification of $\lambda_{\text{max}}(\delta_0, h, s)$

Lack of resolution in $h$ value causes some loss of accuracy for the power of $h$. It still scales like $r(s) \sim 1 - 2s$. As $h$ resolution increases scaling behavior becomes more accurate.

**Figure:** $\delta = 2^{-8}$ thin lines, $\delta = 2^{-10}$ thick lines.
$c_{\text{max}}(h,s)$ vs $s$, fixed delta size

$h = -15$
$h = -14$
$h = -13$
$h = -12$
$h = -11$
$h = -10$

$h = 1 - 2s$

$c_{\text{max}}(h,s)$ and $h^{1-2s}$, fixed delta size
By fixing $\delta$ and $s$, we report

\[ c_{\text{shifted}}^{\max}(\delta, s) = \frac{\lambda^{\max}(\delta, h, s)}{h^{d-2s}} \bigg|_{h=h_0}. \]

We have no prediction for the structure of $c_{\text{shifted}}^{\max}(\delta, s)$. For instance, a power function prediction $\delta^m(s)$ may become totally useless. If there is additional factor $c(s)$, without knowing it, we cannot plot $c(s)\delta^m(s)$.

Since we do not know if $\delta$-quantification is a power function, we cannot identify a shift function. Dependences related to $s$ and $\delta$ are already absorbed in $c_{\text{shifted}}^{\max}(\delta, s)$.
Plot of $c_{\text{shifted}}^{\text{max}}(\delta, s)$

Figure: $c^{\text{max}}(\delta, s)$ as a function of $\delta$ for each $s$.

Figure: $c^{\text{max}}(\delta, s)$ as a function of $s$ for each $\delta$. $h = 2^{-12}$ thin lines, $h = 2^{-16}$ thick lines.
3D plot of $c_{\text{shifted}}^{\max}(\delta, s)$

Figure: Fixing $h = 2^{-12}$, $c_{\text{shifted}}^{\max}(\delta, s) := \frac{\lambda_{\text{max}}(\delta, h, s)}{h^{d-2s}}$, $\text{guess}(\delta, s) = \text{unknown}$. 
Lemma

Let $\Omega$ be a bounded set in $\mathbb{R}^d$ of class $C^{0,1}$ and $u \in H^1(\bar{\Omega})$. Then,

$$b(u, u) \leq c(\Omega)\delta^{2(1-s)}\|u\|^2_{H^1(\bar{\Omega})}.$$ 

Leads to:

$$\lambda_{\text{max}} \sim \delta^{2-2s} h^{d-2}.$$
Summary of results

Utilizing a kernel function leading to uniformly bounded $\lambda^{\text{max}}$ and using 1D Fourier analysis, Du-Zhou 2010 report:

$$\kappa(K) \leq c \min\{h^{-2s} \delta^{2s-2}, h^{-2}\}.$$  

By spectral equivalence, Du-Lehoucq-Gunzburger-Zhou 2012 report:

$$\kappa(K) \lesssim h^{-2s}.$$  

We report:

$$c(s) \delta^{2-2s} h^d \leq \frac{b^t b}{u^t u} \leq c(\delta, s) h^{d-2s}.$$  

With the nonsharp theoretical result for $\lambda^{\text{max}}$:

$$c(s) \delta^{2-2s} h^d \leq \frac{b^t b}{u^t u} \leq c \delta^{2-2s} h^{d-2}.$$
\[ \kappa(\delta, s) \text{ vs } s, \text{ fixed mesh size} \]

\[ \kappa(\delta, s) \text{ vs } \delta, \text{ fixed mesh size} \]

\[ \kappa(\delta, s), \text{ fixed mesh size} \]
Conclusion

1. The estimate \( \lambda_{\text{min}}(\delta, h, s) = c(s)\delta^{2-2s}h \) is numerically sharp.

2. \( \lambda_{\text{max}}(\delta, h, s) = c(\delta, s)h^{1-2s} \) and further investigation is required to determine the \( \delta \)-quantification of \( c(\delta, s) \).

3. The theoretical bound for \( \lambda_{\text{max}}(\delta, h, s) \leq c\delta^{2-2s}h^{-2} \) is not sharp in the \( h \)-quantification.

4. As \( h \to 0 \), the condition number reaches its max near \( s = 1 \), so \( \kappa(K) \lesssim h^{-2s} \) is numerically sharp.

5. As singularity degree increases \( s \to 1 \), the condition number gets larger.

6. As \( \delta \to 0 \) (closer to a local model), the condition number gets larger and reaches to its max near \( s = 1 \).

7. Police report: Fully identified the suspect for \( \lambda_{\text{min}} \) and provided a sketch of the suspect for \( \lambda_{\text{max}} \).