Definitizability of a Class of $J$-Selfadjoint Operators with Applications

Jussi Behrndt$^1$, Friedrich Philipp$^{2,*}$, and Carsten Trunk$^2$

$^1$ Institut für Mathematik, Technische Universität Berlin, Straße des 17. Juni 136, 10623 Berlin, Germany
$^2$ Institut für Mathematik, Technische Universität Ilmenau, PF 100565, 98684 Ilmenau, Germany

We formulate an abstract result concerning the definitizability of $J$-selfadjoint operators which, roughly speaking, differ by at most finitely many dimensions from the orthogonal sum of a $J$-selfadjoint operator with finitely many negative squares and a semibounded selfadjoint operator in a Hilbert space. The general perturbation result is applied to a class of singular Sturm-Liouville operators with indefinite weight functions.

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1 Introduction

Let $(\mathcal{H},(\cdot,\cdot))$ be a Hilbert space and let $J$ be a bounded and boundedly invertible linear operator in $\mathcal{H}$ with $J = J^{-1} = J^*$. Such an operator induces an additional inner product $(\cdot,\cdot) = (J\cdot,J\cdot)$ in $\mathcal{H}$. The linear operators in $\mathcal{H}$ which are selfadjoint with respect to $[\cdot,\cdot]$ are called $J$-selfadjoint. The spectrum of $J$-selfadjoint operators is known to be symmetric with respect to the real axis and it is easy to see that the mapping $A \mapsto JA$ establishes a one-to-one correspondence between the selfadjoint operators and the $J$-selfadjoint operators in $\mathcal{H}$.

Among the $J$-selfadjoint operators is the particularly interesting class of definitizable operators. A $J$-selfadjoint operator $T$ is called definitizable if $\rho(T) \neq \emptyset$ and if there exists a polynomial $p$ such that $[p(T)f,f] \geq 0$ for all $f \in \text{dom} \, p(T)$. Such a polynomial is called definitizing for $T$. If $T$ is a definitizable $J$-selfadjoint operator, then the nonreal spectrum of $T$ consists of at most finite many eigenvalues and $T$ possesses a spectral function $E$ on $\mathbb{R}$ with singularities; cf. [5]. We say that a definitizable operator $T$ is nonnegative in a neighbourhood of $\infty$ if there exists some $c > 0$ such that $[Tf,f] \geq 0$ for all $f \in \text{dom} \, T \cap E(\mathbb{R}\setminus(-c,c))\mathcal{H}$. This is the case, if, e.g., the polynomial $p(t) = tq(t)\overline{q}(t)$, where $q$ is a monic polynomial, is definitizing for $T$, see [5].

The following result is well known, see, e.g. [3, Remark 1.3 and Proposition 1.1]. Recall that the essential spectrum $\sigma_{\text{ess}}(A)$ of a selfadjoint operator $A$ in $\mathcal{H}$ is the set of all spectral points which are no isolated eigenvalues of finite multiplicity.

Theorem 1.1 Let $A$ be a selfadjoint operator in $\mathcal{H}$ which is semibounded from below. If $\sigma_{\text{ess}}(A) \cap (-\infty, 0] = \emptyset$, then $JA$ is definitizable and nonnegative in a neighbourhood of $\infty$ with definitizing polynomial $p(t) = tq(t)\overline{q}(t)$, where $q$ is a monic polynomial.

We mention that the statement of Theorem 1.1 does not hold in general if $\min \sigma_{\text{ess}}(A) \leq 0$. Simple examples show that even $\rho(JA) = \emptyset$ may happen. The main purpose of the present note is to show that under additional assumptions on $A$ the definitizability and nonnegativity of $JA$ are preserved if $\min \sigma_{\text{ess}}(A) \leq 0$. In Section 3 we apply this result to a class of indefinite Sturm-Liouville operators which are regular at one endpoint.

2 A perturbation result

Throughout this section let $(\mathcal{H},(\cdot,\cdot))$ be a Hilbert space, let $J$ be a bounded operator in $\mathcal{H}$ with $J = J^{-1} = J^*$. Let $\mathcal{H}_\pm$ be the eigenspace of $J$ with respect to the eigenvalue $\pm 1$. Then $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$. For a closed operator $B$ in $\mathcal{H}$ denote by $\sigma_{\text{ess}}(B)$ the set of points $\lambda \in \mathbb{C}$ such that $B - \lambda$ is not Fredholm. Note that for a selfadjoint operator in $\mathcal{H}$ this coincides with the definition above. In the next theorem we show that under suitable assumptions also in the case $\min \sigma_{\text{ess}}(A) \leq 0$ the assertions in Theorem 1.1 remain true. Although the statement is essentially a consequence of the results in [4] and [5] we give a short proof for the convenience of the reader.

Theorem 2.1 Let $J$ and $\mathcal{H}_\pm$ be as above and let $A$ be a selfadjoint operator in $\mathcal{H}$ which is semibounded from below with $\mu := \min \sigma_{\text{ess}}(A) \leq 0$. Suppose there are linear manifolds $\mathcal{D}_+$ and $\mathcal{D}_0$ in dom $A$ with $\mathcal{D}_+ \subset \mathcal{H}_+ + \mathcal{D}_0 = \mathcal{D}_+^\perp$ such that $\dim \,(\text{dom} \, A/(\mathcal{D}_+ \oplus \mathcal{D}_0)) < \infty$, $AD_0 \subset \overline{\mathcal{D}_0}$ and $\sigma_{\text{ess}}(A|\mathcal{D_+}) = \emptyset$. Then the operator $JA$ is definitizable and nonnegative in a neighbourhood of $\infty$ with definitizing polynomial $p(t) = (t - \mu)q(t)\overline{q}(t)$, where $q$ is a monic polynomial.

Proof. It is no restriction to assume that $\mathcal{D}_+$ and $\mathcal{D}_0$ are closed with respect to the graph norm of $A$. From the selfadjointness of $A$ we conclude $AD_+ \subset \overline{\mathcal{D}_+}$. Set $K_+ := \overline{\mathcal{D}_+}$, $K_0 := \overline{\mathcal{D}_0} = \overline{\mathcal{D}_+}^\perp$, $S_+ := A|\mathcal{D}_+$ and $S_0 := A|\mathcal{D}_0$. Then $S_0$ and $S_+$ are densely defined closed symmetric operators in $K_0$ and $K_+$, respectively. Since every $\lambda < \min \sigma(A)$ is a point of regular type of both $S_+$ and $S_0$, these operators admit selfadjoint extensions $A_+$ and $A_0$ in the Hilbert spaces $K_+$ and $K_0$, respectively.

* Corresponding author E-mail: friedrich.philipp@tu-ilmenau.de, Phone: +49 3677 69 3255 Fax: +49 3677 69 3270

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Then, by assumption, $A_+ \oplus A_0$ and $A$ both are finite dimensional extensions of $S_+ \oplus S_0$. Hence $\sigma_{\text{ess}}(A_+) \cap (-\infty, \mu) = \emptyset$ and $\sigma_{\text{ess}}(A_0) = \sigma_{\text{ess}}(S_0) = \emptyset$. Therefore there exists a monic polynomial $q_+ \in \mathbb{C}[t]$ such that $(A_+ - \mu)q_+(A_+)\overline{q}(A_+)_{\text{ess}}$ is a nonnegative operator in the Hilbert space $K_+$. From Theorem 1.1, applied to $A_0$, we see that $J_0 A_0$ is definitizable and since $\sigma_{\text{ess}}(A_0) = \emptyset$ it can be shown that $p_0(\ell) := (t - \mu)q_0(t)\overline{q_0}(t)$ is a definitizing polynomial for $J_0 A_0$, where $q_0$ is some monic polynomial. We have $J(A_+ \oplus A_0) = A_+ \oplus J_0 A_0$. With $p(\ell) := (t - \mu)q(t)\overline{q}(t)\overline{q}(t)$ and $f = f_+ + f_0$, we obtain 

$$
[p(A_+ \oplus J_0 A_0) f, f] = [(A_+ - \mu)q_+(A_+)\overline{q}(A_+)]q_0(A_+)f_+ + q_0(A_+)f_+ + [p_0(J_0 A_0)q_+(J_0 A_0)]f_0 + q_0(J_0 A_0)f_0 \geq 0.
$$

Hence, $J(A_+ \oplus A_0)$ is definitizable. From [2, Theorem 2.2], [1, Theorem 3.1] and the proof of [4, Theorem 1] it follows that $J A$ is definitizable with a definitizing polynomial of the desired form, hence $J A$ is nonnegative in a neighbourhood of $\infty$. \hfill $\square$

3 An application to Sturm-Liouville operators with indefinite weight functions

We apply Theorem 2.1 to indefinite Sturm-Liouville operators associated to differential expressions of the form

$$
\ell = \frac{1}{w} \left( - \frac{d}{dx} p \frac{d}{dx} + q \right)
$$

on an interval $(a, b)$, $-\infty < a < b \leq \infty$, with real coefficients $w, p^{-1}, q$ integrable over $(a, c)$ for every $c \in (a, b)$ such that $p > 0$ and $w \neq 0$ a.e. on $(a, b)$. Note that the differential expression $\ell$ is thus assumed to be regular at the endpoint $a$ whereas $b$ is in general a singular endpoint. The definite counterpart $\tau = |w|^{-1}(\frac{d}{dx} p \frac{d}{dx} + q)$ of $\ell$ generates selfadjoint operators in the weighted $L^2$-space $L^2((a, b), |w|)$ which are one or two dimensional restrictions of the maximal differential operator

$$
A_{\max}(a, b) f = \tau f, \quad f \in D_{\max}(a, b) := \{ h \in L^2((a, b), |w|) : h, ph' \text{ locally absolutely continuous, } \tau h \in L^2((a, b), |w|) \},
$$

associated to $\tau$. These selfadjoint realizations of $\tau$ in $L^2((a, b), |w|)$ can also be viewed as finite dimensional extensions of the minimal operator $A_{\min}(a, b) = A_{\max}(a, b)^*$, where $^*$ denotes the adjoint with respect to the scalar product in $L^2((a, b), |w|)$. It is well known that $A_{\min}(a, b)$ is a densely defined closed symmetric operator in $L^2((a, b), |w|)$ with equal deficiency indices $(1, 1)$ (if $\tau$ is in the limit point case at $b$) or $(2, 2)$ (if $\tau$ is in the limit circle case at $b$), see, e.g. [6].

The multiplication operator $J = \text{sign}(w)$ satisfies $J = J^{-1} = J^*$ and connects the definite and indefinite Sturm-Liouville expressions $\tau$ and $\ell$, i.e., $\ell = J \tau$, or, more precisely, the $J$-selfadjoint realizations $T$ of $\ell$ in $L^2((a, b), |w|)$ are in one-to-one correspondence with the selfadjoint realizations $A$ of $\tau$ via $A \mapsto T = J A$. We note that the indefinite inner product induced by $J$ is $[f, g] = (J f, g) = \int_a^b Jg \overline{w} \, dx$, where $(\cdot, \cdot)$ is the scalar product in $L^2((a, b), |w|)$.

**Theorem 3.1** Suppose that the weight function $w$ is positive near the endpoint $b$ and that $A_{\min}(a, b)$ is semibounded from below. Then every $J$-selfadjoint realization of $\ell$ in $L^2((a, b), |w|)$ is definitizable and nonnegative in a neighbourhood of $\infty$.

**Proof.** Let $A$ be a selfadjoint realization of $\tau$ in $L^2((a, b), |w|)$. Since $A_{\min}(a, b)$ has finite deficiency indices the extension $A$ is semibounded from below. In the case $\min \sigma_{\text{ess}}(A) > 0$ the assertions of Theorem 3.1 follow from Theorem 1.1. Hence, let $\min \sigma_{\text{ess}}(A) \leq 0$. According to the assumption on $w$ there exists $c \in (a, b)$ such that $w > 0$ a.e. on $(c, b)$. Denote by $A_{\min}(a, c)$ and $A_{\min}(c, b)$ the minimal operators associated to $\tau$ in $L^2((a, c), |w|)$ and $L^2((c, b), |w|)$, respectively, and let

$$
D_0 := \text{dom} \ A_{\min}(a, c) \quad \text{and} \quad D_+ := \text{dom} \ A_{\min}(c, b).
$$

Then we have $D_0 = L^2((a, c), |w|)$ and $D_+ = L^2((c, b), |w|)$. Thus $D_0 = D_+^*$, where $D_0$ and $D_+$ are canonically embedded in $L^2((a, b), |w|)$. As $J \upharpoonright D_+ = 1$ it follows that $D_+$ is a subspace of $H_+ := \ker (J - 1)$. Moreover, $\sigma_{\text{ess}}(A \upharpoonright D_0) = \sigma_{\text{ess}}(A_{\min}(a, c)) = \emptyset$ follows from the fact that $\tau$ is regular at $a$ and $c$, and all conditions of Theorem 2.1 are satisfied. \hfill $\square$

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**References**