Signal Recovery from Thresholded Frame Measurements

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ABSTRACT

We study the exact recovery of signals from quantized frame coefficients. Here, the basis of the quantization is hard thresholding, and we present a simple algorithm for the recovery of reconstructable signals. The set of non-reconstructable signals is shown to be star-shaped and symmetric with respect to the origin. Moreover, we provide a criterion on the frame for the boundedness of this set. In this case, we also give a priori bounds.

Keywords: Frame, threshold, recovery

1. INTRODUCTION

Frames in Hilbert spaces are complete, yet redundant systems of vectors. It is this redundancy which makes frames robust against perturbations such as erasures or noise [6,10] and thus – for many tasks – superior to bases. In the recent past, frames have therefore become of remarkable interest to various fields of the applied sciences. To name a few, we refer to the following works in, e.g., wireless communication [14,17], bioimaging [8], and speech recognition [2]. Driven by this rather practical interest but also by purely theoretical considerations [5,11], frames have also gained high importance in several fields of mathematical signal processing, such as sampling theory [1], coding [3], and image processing [4,12]. For a detailed study of frames in theory and applications we refer to [7,9,13,15,16] and the references therein.

The above-mentioned stability of frames under noise or losses is obviously highly advantagous in signal processing when signals are to be sent from, e.g., a sensor to a receiver and perfect transmission cannot be guaranteed. In the present paper we have a similar but different point of view. Here, we would like to erase some of the signal's frame coefficients *on purpose* before sending the signal. In fact, we erase those coefficients which are smaller than a given threshold parameter which leads to both

— numerical stability in the reconstruction process of the signal and

— a smaller amount of transmission data (compression).

Moreover, as we utilize frames for coding and decoding, we can hope to be able to reconstruct the exact original signal after reception.

Our setting is as follows: Given a finite frame Φ and $\varepsilon > 0$ we define the *domain of reconstructability* by

$$\mathcal{R}(\varepsilon, \Phi) := \{x : \tau_\varepsilon(\langle y, \varphi\rangle) = \tau_\varepsilon(\langle x, \varphi\rangle)\ \forall \varphi \in \Phi \implies y = x\},$$

where τ_ε is the hard threshold operator with threshold parameter ε. By the definition of $\mathcal{R}(\varepsilon, \Phi)$, a signal x can be (exactly) recovered from the measurements $\tau_\varepsilon(\langle x, \varphi\rangle)$, $\varphi \in \Phi$, if and only if $x \in \mathcal{R}(\varepsilon, \Phi)$. In Section 2 we characterize this set in terms of spanning properties of the frame Φ. This characterization then leads to a simple reconstruction formula for signals $x \in \mathcal{R}(\varepsilon, \Phi)$, see Theorem 2.2.

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It is evident that signals cannot be recovered when they are too small in norm. Hence, $\mathcal{R}(\varepsilon, \Phi)$ can never be the entire space, so that the complement $\mathcal{R}(\varepsilon, \Phi)^c$ is of special interest. We describe its geometrical properties in Section 3. In particular, it turns out that it is open and both star-shaped and symmetric with respect to the origin.

As we certainly seek for frames which keep the complement $\mathcal{R}(\varepsilon, \Phi)^c$ "as small as possible," it is natural to ask for necessary and sufficient conditions for its boundedness. In Section 4 we prove that $\mathcal{R}(\varepsilon, \Phi)^c$ is bounded if and only if the frame Φ has the so-called complement property. This means that in every partition $\Phi = \Phi_1 \cup \Phi_2$ of the frame one of Φ_1 and Φ_2 is still a frame. Surprisingly, this is exactly the same characterization as for real phase retrieval (see [2]). We conclude the paper by providing a priori bounds for $\mathcal{R}(\varepsilon, \Phi)^c$.

2. PRELIMINARIES

The symbol $\mathbb{F}$ will always stand for one of the fields $\mathbb{R}$ or $\mathbb{C}$. Recall that a system of vectors $\Phi = \{\varphi_i\}_{i\in I}$ in a real or complex Hilbert space $\mathcal{H}$ is called a *frame for* $\mathcal{H}$ if there exist positive constants A and B such that

$$A\|x\|^2 \le \sum_{i\in I} |\langle x, \varphi_i\rangle|^2 \le B\|x\|^2 \tag{1}$$

holds for all $x \in \mathcal{H}$. The constants A and B are called *lower and upper frame bounds* of Φ, respectively. The largest possible lower frame bound (smallest possible upper frame bound) in (1) will be denoted by $A(\Phi)$ ($B(\Phi)$, respectively). The values $A(\Phi)$ and $B(\Phi)$ are called the *optimal frame bounds*.

Throughout this paper, let $\Phi = \{\varphi_i\}_{i=1}^M$ be a frame for $\mathbb{F}^N$. For $\varepsilon > 0$ the (hard) threshold function $\tau_\varepsilon : \mathbb{F} \to \mathbb{F}$ is defined as follows:

$$\tau_\varepsilon(z) := \begin{cases} 0 & \text{if } |z| < \varepsilon \\ z & \text{if } |z| \ge \varepsilon, \end{cases} \qquad z \in \mathbb{F}.$$

Moreover, let the (non-linear) operator $M(\varepsilon, \Phi) : \mathbb{F}^N \to \mathbb{F}^M$ be defined by

$$M(\varepsilon, \Phi)x := \Big(\tau_\varepsilon\big(\langle x, \varphi_i\rangle\big)\Big)_{i=1}^M, \qquad x \in \mathbb{F}^N.$$

Clearly, if for a pair $x, y \in \mathbb{F}^N$ of signals we have that $M(\varepsilon, \Phi)x = M(\varepsilon, \Phi)y$, we cannot recover, e.g., x from the thresholded measurements $M(\varepsilon, \Phi)x$. Therefore, we say that a signal $x \in \mathbb{F}^N$ is *reconstructable* if $M(\varepsilon, \Phi)x \ne M(\varepsilon, \Phi)y$ holds for all signals $y \in \mathbb{F}^N \setminus \{x\}$, and we define the *domain of reconstructability* with respect to ε and Φ by

$$\mathcal{R}(\varepsilon, \Phi) := \left\{x \in \mathbb{F}^N : \nexists y \in \mathbb{F}^N \setminus \{x\} \text{ such that } M(\varepsilon, \Phi)y = M(\varepsilon, \Phi)x\right\}.$$

In particular, the restriction of $M(\varepsilon, \Phi)$ to $\mathcal{R}(\varepsilon, \Phi)$ is injective and has therefore an inverse on its image which can be utilized as a reconstruction operator. For $x \in \mathbb{F}^N$ we define the subset $\Phi(\varepsilon, x)$ of Φ by

$$\Phi(\varepsilon, x) := \{\varphi \in \Phi : |\langle x, \varphi\rangle| \ge \varepsilon\}.$$

The following lemma relates the domain of reconstructability to spanning properties of the frame Φ.

LEMMA 2.1. *We have*

$$\mathcal{R}(\varepsilon, \Phi) = \{x \in \mathbb{F}^N : \operatorname{span} \Phi(\varepsilon, x) = \mathbb{F}^N\}. \tag{2}$$

Proof. "$\supset$": Let $x \in \mathbb{F}^N$ be contained in the set on the right hand side of (2), i.e. we have $\operatorname{span} \Phi(\varepsilon, x) = \mathbb{F}^N$. Suppose that there exists some $y \in \mathbb{F}^N$, $y \ne x$, such that $M(\varepsilon, \Phi)y = M(\varepsilon, \Phi)x$. By the definition of the operator M this implies that $\langle y, \varphi\rangle = \langle x, \varphi\rangle$ for all $\varphi \in \Phi(\varepsilon, x)$ and hence $y - x \in (\operatorname{span} \Phi(\varepsilon, x))^\perp = \{0\}$. Consequently, we have $y = x$, a contradiction.

"$\subset$": Let $x \in \mathcal{R}(\varepsilon, \Phi)$ and suppose that $\operatorname{span}\Phi(\varepsilon, x) \neq \mathbb{F}^N$. We can then choose some $u \in (\operatorname{span}\Phi(\varepsilon, x))^\perp$, $u \neq 0$, such that

$$\|u\| < \min\left\{\frac{\varepsilon - |\langle x, \varphi\rangle|}{\|\varphi\|} : \varphi \in \Phi \setminus \Phi(\varepsilon, x)\right\},$$

and put $y := x + u$. Since $u \in (\operatorname{span}\Phi(\varepsilon, x))^\perp$, we have

$$\langle y, \varphi\rangle = \langle x, \varphi\rangle \quad \text{for } \varphi \in \Phi(\varepsilon, x).$$

And for $\varphi \in \Phi \setminus \Phi(\varepsilon, x)$,

$$|\langle y, \varphi\rangle| \le |\langle x, \varphi\rangle| + \|u\|\|\varphi\| < \varepsilon.$$

This proves $M(\varepsilon, \Phi)y = M(\varepsilon, \Phi)x$. But $x \in \mathcal{R}(\varepsilon, \Phi)$ which implies $x + u = y = x$ and thus $u = 0$, which is a contradiction. □

Lemma 2.1 leads to a conceivably simple reconstruction formula for signals in the domain of reconstructability.

THEOREM 2.2. *Let $\Phi = \{\varphi_i\}_{i=1}^M$ be a frame for $\mathbb{F}^N$. Then for each $x \in \mathcal{R}(\varepsilon, \Phi)$ we have*

$$x = \sum_{i \in J(x)} \langle x, \varphi_i\rangle \psi_i = \sum_{i \in J(x)} \big(M(\varepsilon, \Phi)x\big)_i \psi_i,$$

where $J(x) = \{i \in \{1, \ldots, M\} : \varphi_i \in \Phi(\varepsilon, x)\}$ and $(\psi_i)_{i \in J(x)}$ is some dual frame of $\Phi(\varepsilon, x)$.

Indeed, if $x \in \mathcal{R}(\varepsilon, \Phi)$ and the data $M(\varepsilon, \Phi)x \in \mathbb{F}^M$ is given, then $\Phi(\varepsilon, x)$ is known and is, in addition, a frame for $\mathbb{F}^N$ by Lemma 2.1. Hence, in contrast to other reconstruction tasks (e.g. phase retrieval, see for example [2]), the reconstruction step does not need to be considered further. Instead, it is important to investigate the properties of the domain of reconstructability or, equivalently, of its complement

$$\mathcal{R}(\varepsilon, \Phi)^c := \mathbb{F}^N \setminus \mathcal{R}(\varepsilon, \Phi).$$

This will be the content of the following two sections.

3. GEOMETRICAL FEATURES OF $\mathcal{R}(\varepsilon, \Phi)^C$

In this section we study the geometrical properties of the complement of the domain of reconstructability. The next lemma shows in particular that this set is star-shaped. Recall that a set $S \subset \mathbb{F}^N$ is called *star-shaped* with respect to $x_0 \in S$ if with each $x \in S$ also the line segment $\{tx + (1-t)x_0 : t \in (0, 1)\}$ is contained in S.

LEMMA 3.1. *The complement $\mathcal{R}(\varepsilon, \Phi)^c$ is open in $\mathbb{F}^N$ and both star-shaped and symmetric with respect to the origin. Moreover, for $t > 0$ we have*

$$\mathcal{R}(t\varepsilon, \Phi)^c = t\mathcal{R}(\varepsilon, \Phi)^c.$$

In particular, $\mathcal{R}(\varepsilon, \Phi)^c = \varepsilon\mathcal{R}(1, \Phi)^c$.

Proof. Let $x \in \mathcal{R}(\varepsilon, \Phi)^c$. Then $\Phi(\varepsilon, x)$ does not span $\mathbb{F}^N$ by Lemma 2.1. Set

$$\delta := \min\left\{\frac{\varepsilon - |\langle x, \varphi\rangle|}{\|\varphi\|} : \varphi \in \Phi \setminus \Phi(\varepsilon, x)\right\}.$$

Then for all $u \in \mathbb{F}^N$ with $\|u - x\| < \delta$ and all $\varphi \in \Phi \setminus \Phi(\varepsilon, x)$ we have

$$|\langle u, \varphi\rangle| \le \|u - x\|\|\varphi\| + |\langle x, \varphi\rangle| < \varepsilon.$$

Hence, $\Phi(\varepsilon, u) \subset \Phi(\varepsilon, x)$. And as $\Phi(\varepsilon, x)$ does not span $\mathbb{F}^N$, $\Phi(\varepsilon, u)$ doesn't either. Again from Lemma 2.1 we conclude that $u \in \mathcal{R}(\varepsilon, \Phi)^c$. This proves that $\mathcal{R}(\varepsilon, \Phi)^c$ is an open set.

For the star-shape and symmetry of $\mathcal{R}(\varepsilon, \Phi)^c$ with respect to the origin we only mention that

$$\Phi(\varepsilon, -x) = \Phi(\varepsilon, x) \quad \text{and} \quad \Phi(\varepsilon, tx) \subset \Phi(\varepsilon, x) \ \text{ for } t \in (0, 1).$$

The statement then follows directly from Lemma 2.1.

The claim in the moreover-part is a direct consequence of

$$\Phi(t\varepsilon, x) = \{\varphi \in \Phi : |\langle x, \varphi\rangle| \geq t\varepsilon\} = \left\{\varphi \in \Phi : \left|\left\langle \frac{x}{t}, \varphi\right\rangle\right| \geq \varepsilon\right\} = \Phi\left(\varepsilon, \frac{x}{t}\right)$$

and Lemma 2.1. □

Figure 1 below illustrates the properties of $\mathcal{R}(\varepsilon, \Phi)^c$ in Lemma 3.1 in the special case of the so-called Mercedes-Benz Frame for $\mathbb{R}^2$, given by

$$\Phi = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1/2 \\ \sqrt{3}/2 \end{pmatrix}, \begin{pmatrix} -1/2 \\ -\sqrt{3}/2 \end{pmatrix} \right\}.$$

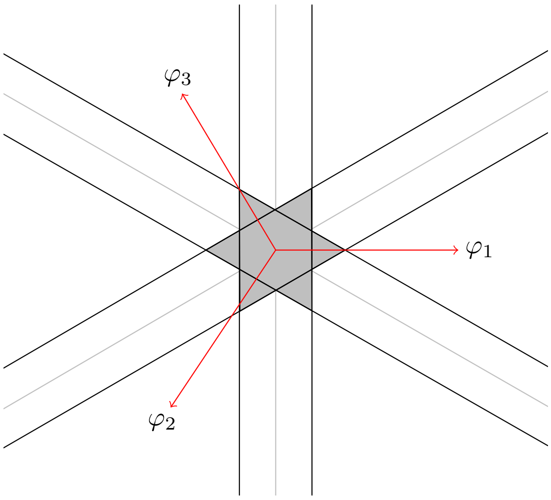

Figure 1. The complement of the domain of reconstructability (gray) of the unit-norm Mercedes Benz Frame (red) with respect to the threshold $\varepsilon \approx 0.2$.

Next, it is our aim to obtain an explicit representation of the complement $R(\varepsilon, \Phi)^c$. To this end, for a vector $\varphi \in \mathbb{F}^N$ we define the set*

$$S_\varepsilon(\varphi) := \{x \in \mathbb{F}^N : |\langle x, \varphi\rangle| < \varepsilon\},$$

and for $\Psi \subset \mathbb{F}^N$,

$$S_\varepsilon(\Psi) := \bigcap_{\varphi \in \Psi} S_\varepsilon(\varphi).$$

Lemma 3.2. *We have*

$$\mathcal{R}(\varepsilon, \Phi)^c = \bigcup_{\substack{\Psi \subset \Phi \\ \dim \operatorname{span} \Psi = N-1}} S_\varepsilon(\Phi \setminus \Psi).$$

In particular, if each subset of Φ *with* N *elements forms a basis of* $\mathbb{F}^N$*, then*

$$\mathcal{R}(\varepsilon, \Phi)^c = \bigcup_{\substack{\Psi \subset \Phi \\ |\Psi| = N-1}} S_\varepsilon(\Phi \setminus \Psi).$$

Proof. First of all, we prove that

$$\mathcal{R}(\varepsilon, \Phi)^c = \bigcup_{\substack{\Psi \subset \Phi \\ \operatorname{span} \Psi \neq \mathbb{F}^N}} S_\varepsilon(\Phi \setminus \Psi). \tag{3}$$

*The "S" in S_ε shall indicate the "slice-like" form of the set in the case of one vector.

"$\subset$": Let $x \in \mathcal{R}(\varepsilon,\Phi)^c$. Then $\Phi(\varepsilon,x)$ does not span F^N by Lemma 2.1. And as $x \in S_\varepsilon(\Phi \setminus \Phi(\varepsilon,x))$, we can choose $\Psi = \Phi(\varepsilon,x)$, and the inclusion "$\subset$" is proved.

"$\supset$": For the opposite inclusion, let $x \in \mathbb{F}^N$ be contained in the set on the right hand side of (3). Then there exists a subset $\Psi \subset \Phi$ which is not spanning $\mathbb{F}^N$ and such that $|\langle x,\varphi\rangle| < \varepsilon$ for all $\varphi \in \Phi \setminus \Psi$. Then we have $\Phi(\varepsilon,x) \subset \Psi$ which implies that $\Phi(\varepsilon,x)$ is not spanning $\mathbb{F}^N$. The claim now follows from Lemma 2.1.

Concerning the actual assertion of the lemma, it is evident that

$$\bigcup_{\substack{\Psi\subset\Phi \\ \dim \operatorname{span}\Psi = N-1}} S_\varepsilon(\Phi\setminus\Psi) \subset \bigcup_{\substack{\Psi\subset\Phi \\ \operatorname{span}\Psi \neq \mathbb{F}^N}} S_\varepsilon(\Phi\setminus\Psi).$$

Now, observe that $\Psi_1 \subset \Psi_2 \subset \Phi$ implies

$$S_\varepsilon(\Phi\setminus\Psi_1) \subset S_\varepsilon(\Phi\setminus\Psi_2).$$

This proves the lemma. □

4. BOUNDEDNESS OF $\mathcal{R}(\varepsilon,\Phi)^C$

Clearly, our aim is to keep the complementary set $\mathcal{R}(\varepsilon,\Phi)^c$ "as small as possible". A minimal requirement is obviously its boundedness. In Theorem 4.3 below we provide a necessary and sufficient condition for the boundedness of $\mathcal{R}(\varepsilon,\Phi)^c$. For this, we need the following definition.

Definition 4.1. *A frame $\Phi = \{\varphi_i\}_{i=1}^M$ for $\mathbb{F}^N$ is said to have the* complement property *if for each partition*

$$\Phi = \Phi_1 \cup \Phi_2$$

of Φ one of the sets Φ_1 and Φ_2 spans $\mathbb{F}^N$.

Remark 4.2. *We say that the frame Φ for $\mathbb{R}^N$ allows for phaseless reconstruction if for each pair $(x,y) \in \mathbb{R}^N \times \mathbb{R}^N$ we have*

$$|\langle x,\varphi\rangle| = |\langle y,\varphi\rangle| \ \text{for all } \varphi\in\Phi \quad \Longrightarrow \quad y = x \ \text{ or } \ y = -x.$$

In [2, Theorem 2.8] *it was shown that a frame Φ for $\mathbb{R}^N$ allows for phaseless reconstruction if and only if Φ possesses the complement property.*

We can now state and prove our characterization for the boundedness of the complement of the domain of reconstructability.

Theorem 4.3. *The following two statements are equivalent:*

(i) *The complement $\mathcal{R}(\varepsilon,\Phi)^c$ is bounded.*

(ii) *Φ possesses the complement property.*

Proof. (i)$\Rightarrow$(ii). Assume that Φ does not have the complement property. Then there exists a partition $\Phi = \Phi_1 \cup \Phi_2$ of Φ such that neither Φ_1 nor Φ_2 is spanning $\mathbb{F}^N$. But then $\mathcal{R}(\varepsilon,\Phi)^c$ is unbounded since

$$(\operatorname{span}\Phi_1)^\perp \subset \mathcal{R}(\varepsilon,\Phi)^c.$$

To see this, let $x \in (\operatorname{span}\Phi_1)^\perp$. Then $|\langle x,\varphi\rangle| \geq \varepsilon$, $\varphi\in\Phi$, can only be true for $\varphi \in \Phi_2$, hence $\Phi(\varepsilon,x) \subset \Phi_2$. But as Φ_2 is not spanning $\mathbb{F}^N$, Lemma 2.1 implies that $x \in \mathcal{R}(\varepsilon,\Phi)^c$.

(ii)$\Rightarrow$(i). Conversely, assume that $\mathcal{R}(\varepsilon,\Phi)^c$ is unbounded. Then there exists a sequence

$$(x_n) \subset \mathcal{R}(\varepsilon,\Phi)^c \qquad \text{with} \qquad \|x_n\| \to \infty \ \text{ as } \ n\to\infty.$$

Since Φ is a finite frame, for each $n \in \mathbb{N}$ there is only a finite number of possibilities for $\Phi(\varepsilon,x_n)$. Therefore, after possibly passing to a subsequence, we can assume that $\Phi(\varepsilon,x_n)$ is the same set $\Psi \subset \Phi$ for each $n\in\mathbb{N}$. Note that $\Psi = \Phi(\varepsilon, x_n)$ is not spanning $\mathbb{F}^N$ due to $x_n \in \mathcal{R}(\varepsilon, \Phi)^c$ for every $n \in \mathbb{N}$ (cf. Lemma 2.1). Now, define the sequence (u_n) by $u_n := \frac{x_n}{\|x_n\|}$, $n \in \mathbb{N}$. As $\{u_n : n \in \mathbb{N}\}$ is compact, we can – again, after possibly passing to a subsequence – furthermore assume without loss of generality that (u_n) converges to some $u \in \mathbb{F}^N$ with $\|u\| = 1$. For $\varphi \in \Phi \setminus \Psi$ we have

$$|\langle u_n, \varphi\rangle| = \frac{|\langle x_n, \varphi\rangle|}{\|x_n\|} < \frac{\varepsilon}{\|x_n\|} \longrightarrow 0 \qquad \text{as } n \to \infty.$$

Therefore, we have $\langle u, \varphi\rangle = 0$ for $\varphi \in \Phi \setminus \Psi$ which implies that also $\Phi \setminus \Psi$ is not spanning $\mathbb{F}^N$. Hence, Φ does not possess the complement property. □

The next corollary follows from the simple fact that a frame for $\mathbb{F}^N$ having the complement property must have at least $2N - 1$ elements.

Corollary 4.4. *For the boundedness of $\mathcal{R}(\varepsilon, \Phi)^c$ it is necessary that $M \ge 2N - 1$.*

Remark 4.5. *Note that Theorem 4.3 shows that the boundedness of $\mathcal{R}(\varepsilon, \Phi)^c$ does not depend on ε but only on Φ. This also follows from Lemma 3.1.*

The open ball in $\mathbb{F}^N$ with center $x_0 \in \mathbb{F}^N$ and radius $r > 0$ will be denoted by $B_r(x_0)$. In the following theorem we provide radii r and R such that

$$B_r(0) \subset \mathcal{R}(\varepsilon, \Phi)^c \subset B_R(0)$$

(the second inclusion if $\mathcal{R}(\varepsilon, \Phi)^c$ is bounded). Recall that $A(\Psi)$ and $B(\Psi)$ denote the optimal frame bounds of a frame Ψ. Furthermore, if S is a set, then $|S|$ denotes the number of elements in S.

Theorem 4.6. *The following statements hold:*

(a) *If $r := \varepsilon\sqrt{N/B(\Phi)}$, then we always have*

$$B_r(0) \subset \mathcal{R}(\varepsilon, \Phi)^c.$$

(b) *If Φ has the complement property, then*

$$\mathcal{R}(\varepsilon, \Phi)^c \subset B_R(0),$$

where

$$R = \varepsilon \cdot \max\left\{\sqrt{\frac{|\Phi \setminus \Psi|}{A(\Phi \setminus \Psi)}} : \Psi \subset \Phi,\ \dim\operatorname{span}\Psi = N - 1\right\}.$$

Proof. (a). Let $x \in B_r(0)$. Then

$$|\Phi(\varepsilon, x)|\varepsilon^2 \le \sum_{\varphi\in\Phi(\varepsilon,x)} |\langle x, \varphi\rangle|^2 \le \sum_{\varphi\in\Phi} |\langle x, \varphi\rangle|^2 \le B(\Phi)\|x\|^2 < N\varepsilon^2.$$

Hence, $|\Phi(\varepsilon, x)| < N$ which implies that $\Phi(\varepsilon, x)$ cannot be spanning $\mathbb{F}^N$. From Lemma 2.1 we obtain $x \in \mathcal{R}(\varepsilon, \Phi)^c$.

(b). Assume that Φ possesses the complement property, and let $x \in \mathcal{R}(\varepsilon, \Phi)^c$. By Lemma 3.2, there exists $\Psi \subset \Phi$ with $\dim\operatorname{span}\Psi = N - 1$ such that $|\langle x, \varphi\rangle| < \varepsilon$ for all $\varphi \in \Phi \setminus \Psi$. As $\Phi \setminus \Psi$ must be spanning $\mathbb{F}^N$, we have $A(\Phi \setminus \Psi) > 0$ and

$$A(\Phi \setminus \Psi)\|x\|^2 \le \sum_{\varphi\in\Phi\setminus\Psi} |\langle x, \varphi\rangle|^2 < |\Phi \setminus \Psi|\varepsilon^2,$$

which shows

$$\|x\| < \varepsilon \cdot \sqrt{\frac{|\Phi \setminus \Psi|}{A(\Phi \setminus \Psi)}} \le R.$$

The theorem is proved. □

Corollary 4.7. *If each set $\Psi \subset \Phi$ with N elements is a basis of $\mathbb{F}^N$ and $M \ge 2N - 1$, then $\mathcal{R}(\varepsilon, \Phi)^c \subset B_R(0)$, where*

$$R = \varepsilon\sqrt{M - N + 1} \cdot \max\left\{A(\Psi)^{-1/2} : \Psi \subset \Phi,\ |\Psi| = M - N + 1\right\}.$$

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