Computation of Robustness Measures for Descriptor Systems

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Differential-Algebraic Equations (DAEs)

Consider a differential-algebraic equation

$$\frac{\mathrm{d}}{\mathrm{d}t}Ex(t) = Ax(t),$$

with E, $A \in \mathbb{R}^{n \times n}$. Here we assume E to be singular, but $sE - A$ regular, i. e., $\det(sE - A) \not\equiv 0$ (possibly algebraic constraints in the dynamics). Fields of applications:

- constrained multi-body systems,
- semi-discretized PDEs in flow control,
- electrical circuits,
- many more ...

Stability of DAEs

Stability

A DAE is called stable, if for all solutions of the DAE it holds

$$\|x(t)\| \to 0 \quad \text{for} \quad t \to \infty.$$

Equivalent characterization:

$$\begin{aligned} & \text{DAE stable} \\ \Leftrightarrow \quad & \Lambda(E, A) := \{\lambda \in \mathbb{C} : \det(\lambda E - A) = 0\} \subseteq \mathbb{C}^- := \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) < 0\} \end{aligned}$$

Robust Control

Consider a DAE (a plant P) with inputs and outputs

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}Ex(t) &= Ax(t) + B_1u(t) + B_2w(t),\\ y(t) &= C_1x(t),\\ z(t) &= C_2x(t),\end{aligned}$$

with $B_i \in \mathbb{R}^{n \times m_i}$, $C_i \in \mathbb{R}^{p_i \times n}$.

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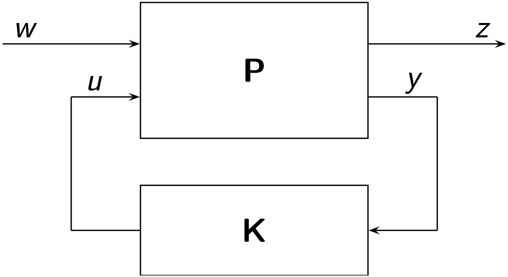
with $B_i \in \mathbb{R}^{n \times m_i}$, $C_i \in \mathbb{R}^{p_i \times n}$.
Task: design a controller $K \in \mathbb{R}^{m \times n}$ such that $u(t) = Ky(t)$.



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Closed-loop system:

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t}Ex(t) &= (A + B_1KC_1)x(t) + B_2w(t), \\ z(t) &= C_2x(t). \end{aligned}$$

Want: controller K such that the influence on stability of the closed loop system by disturbances of the form $w(t) = \Delta z(t)$ is as small as possible.

Structured Stability Radius

Need to evaluate robustness of a system with respect to structured perturbations!

Definitions

- structured complex stability radius $r_{\mathbb{C}}$:

$$r_{\mathbb{C}} = \inf \left\{ \|\Delta\|_2 : \Lambda(E, A + B\Delta C) \cap \mathrm{i}\mathbb{R} \neq \emptyset \right\},$$

- structured pseudospectrum Λ_ε:

$$\Lambda_\varepsilon = \left\{ \lambda \in \mathbb{C} : \lambda \in \Lambda(E, A + B\Delta C) \text{ for } \Delta \in \mathbb{C}^{m \times p} \text{ with } \|\Delta\|_2 < \varepsilon \right\},$$

- structured pseudospectral abscissa $\alpha(\varepsilon)$:

$$\alpha(\varepsilon) := \max \left\{ \operatorname{Re} \lambda : \lambda \in \Lambda_\varepsilon \right\}.$$

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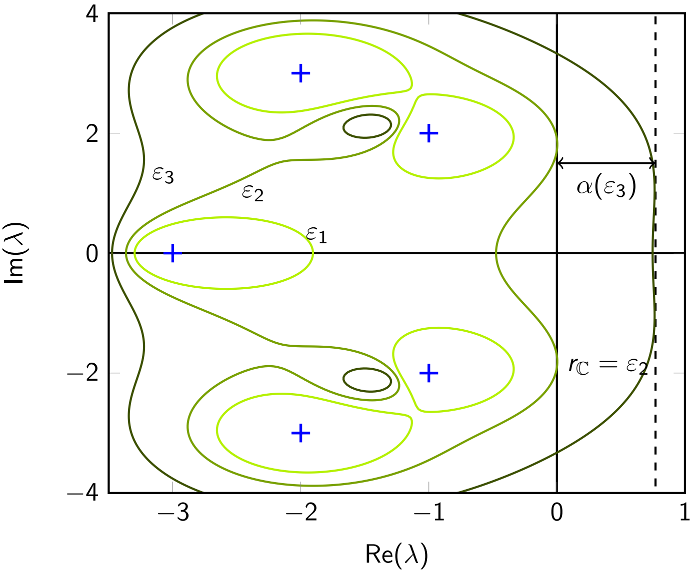
WARNING

Special care needs be taken of perturbations changing the regularity or index of the pencil!

Graphical Interpretation



Relation to the $\mathcal{H}_\infty$-Norm

Consider the transfer function of the descriptor system given

$$G(s) = C(sE - A)^{-1}B$$

Definitions: the space $\mathcal{RH}_\infty^{p\times m}$ and the $\mathcal{H}_\infty$-norm

$\mathcal{RH}_\infty^{p\times m}$: Hardy space of $p \times m$ functions of the form $G(s) = C(sE - A)^{-1}B$ which are analytic and bounded in the open right half-plane, i. e., they are

- well-defined ($sE - A$ regular);
- stable (all poles in open left half-plane);
- proper (bounded at infinity).

$\mathcal{H}_\infty$-norm: $\|G\|_{\mathcal{H}_\infty} := \sup_{s\in\mathbb{C}^+} \sigma_{\max}(G(s)) = \sup_{\omega\in\mathbb{R}} \sigma_{\max}(G(\mathrm{i}\omega))$.

Connection to the structured stability radius

$$r_{\mathbb{C}} = 1/\|G\|_{\mathcal{H}_\infty}$$

Basic Results

Theorem [Boyd, Balakrishnan, Kabamba '89], [Bruinsma, Steinbuch '90], [Benner, Sima, V. '12]

Assume that $sE - A$ is regular and stable, $G \in \mathcal{RH}_\infty^{p \times m}$, $\gamma > 0$ is not a singular value of D, and $\omega_0 \in \mathbb{R}$. Then, γ is a singular value of $G(\mathrm{i}\omega_0)$ if and only if

$$H_\gamma(s) := \begin{bmatrix} sE - A & 0 \\ 0 & sE^\mathsf{T} + A^\mathsf{T} \end{bmatrix} - \begin{bmatrix} B & 0 \\ 0 & -C^\mathsf{T} \end{bmatrix} \begin{bmatrix} -D & \gamma I_p \\ \gamma I_m & -D^\mathsf{T} \end{bmatrix}^{-1} \begin{bmatrix} C & 0 \\ 0 & B^\mathsf{T} \end{bmatrix}$$

has the eigenvalue $\mathrm{i}\omega_0$.

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has the eigenvalue $\mathrm{i}\omega_0$.

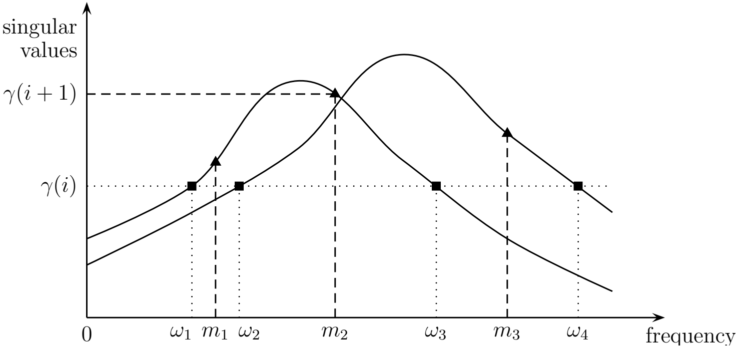
Corollary

Assume that $sE - A$ is regular and stable, $G \in \mathcal{RH}_\infty^{p \times m}$ and let $\gamma > \inf_{\omega \in \mathbb{R}} \sigma_{\max}(G(\mathrm{i}\omega))$ be not a singular value of D. Then $\|G\|_{\mathcal{H}_\infty} \geq \gamma$ if and only if $H_\gamma(s)$ has purely imaginary eigenvalues.

Outline of the Algorithm

Sketch of the algorithm

1. Choose initial value of γ.
2. Check $H_\gamma(s)$ for imaginary eigenvalues.
3. If imaginary eigenvalues exist, then increase γ, else $\|G\|_{\mathcal{H}_\infty}$ is found.



Properties and Numerical Results

Properties of the algorithm

- global convergence with quadratic rate,
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Numerical example:

iteration i	iterate $\gamma(i)$	rel. error
1	0.15756820450368172	0.008983922316548
2	0.15899264646186639	$2.497728868261070 \cdot 10^{-5}$
3	0.15899661773662260	$1.865781701059065 \cdot 10^{-10}$
4	0.15899661776628779	$6.982683281081451 \cdot 10^{-16}$

The Eigenvalue Problems

Reminder: original matrix pencil

$$
\begin{aligned}
H_\gamma(s) &:= \begin{bmatrix} sE - A & 0 \\ 0 & sE^\mathsf{T} + A^\mathsf{T} \end{bmatrix} - \begin{bmatrix} B & 0 \\ 0 & -C^\mathsf{T} \end{bmatrix} \begin{bmatrix} -D & \gamma I_p \\ \gamma I_m & -D^\mathsf{T} \end{bmatrix}^{-1} \begin{bmatrix} C & 0 \\ 0 & B^\mathsf{T} \end{bmatrix} \\
&= \begin{bmatrix} sE - A + BR_\gamma^{-1} D^\mathsf{T} C & \gamma B R_\gamma^{-1} B^\mathsf{T} \\ -\gamma C^\mathsf{T} S_\gamma^{-1} C & sE^\mathsf{T} + A^\mathsf{T} - C^\mathsf{T} D R_\gamma^{-1} B^\mathsf{T} \end{bmatrix}
\end{aligned}
$$

with $R_\gamma := DD^\mathsf{T} - \gamma^2 I_p$ and $S_\gamma := D^\mathsf{T} D - \gamma^2 I_m$.

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Remarks

- If γ is close to a singular value of D, then R_γ, S_γ are ill-conditioned.

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- If γ is close to a singular value of D, then R_γ, S_γ are ill-conditioned.
- Forming "matrix-times-its-transpose" products numerically unstable
  $\Longrightarrow$ explicitly forming $H_\gamma(s)$ should be avoided if possible!

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- If γ is close to a singular value of D, then R_γ, S_γ are ill-conditioned.
- Forming "matrix-times-its-transpose" products numerically unstable
  $\Longrightarrow$ explicitly forming $H_\gamma(s)$ should be avoided if possible!
- $H_\gamma(s)$ is a skew-Hamiltonian/Hamiltonian pencil
  $\Longrightarrow$ structure-preserving algorithms?

Matrix Structures

Even pencils

A real $n \times n$ matrix pencil $s\mathcal{S} - \mathcal{H}$ is called even, if $\mathcal{S}$ is skew-symmetric and $\mathcal{H}$ is symmetric.

Skew-Hamiltonian/Hamiltonian pencils

Let $\mathcal{J} = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$. A real $2n \times 2n$ matrix pencil $s\mathcal{S} - \mathcal{H}$ is called skew-Hamiltonian/Hamiltonian if $\mathcal{S}$ is skew-Hamiltonian ($(\mathcal{S}\mathcal{J})^{\mathsf{T}} = -\mathcal{S}\mathcal{J}$) and $\mathcal{H}$ is Hamiltonian ($(\mathcal{H}\mathcal{J})^{\mathsf{T}} = \mathcal{H}\mathcal{J}$).

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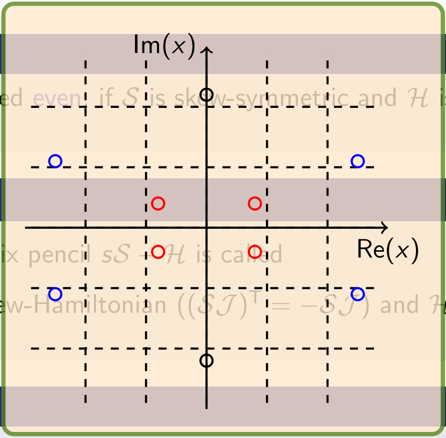

Properties

- Hamiltonian spectral symmetry,
- easy to switch between the two types.

Transformation of $H_\gamma(s)$

Reminder: original matrix pencil

$$H_\gamma(s) := \begin{bmatrix} sE - A & 0 \\ 0 & sE^\mathsf{T} + A^\mathsf{T} \end{bmatrix} - \begin{bmatrix} B & 0 \\ 0 & -C^\mathsf{T} \end{bmatrix} \begin{bmatrix} -D & \gamma I_p \\ \gamma I_m & -D^\mathsf{T} \end{bmatrix}^{-1} \begin{bmatrix} C & 0 \\ 0 & B^\mathsf{T} \end{bmatrix}$$

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Exploit Schur complement structure and get the extended matrix pencil

$$\mathcal{H}_\gamma^{(1)}(s) = \left[\begin{array}{cc|cc} sE - A & 0 & -B & 0 \\ 0 & sE^\mathsf{T} + A^\mathsf{T} & 0 & C^\mathsf{T} \\ \hline -C & 0 & -D & \gamma I_p \\ 0 & -B^\mathsf{T} & \gamma I_m & -D^\mathsf{T} \end{array}\right],$$

which has the same eigenvalues as $H_\gamma(s)$ with additional infinite eigenvalues.

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which has the same eigenvalues as $H_\gamma(s)$ with additional infinite eigenvalues.
Block permutations yield the even pencil

$$\mathcal{H}_\gamma^{(2)}(s) = \left[\begin{array}{cc|cc} 0 & sE - A & 0 & -B \\ -sE^\mathsf{T} - A^\mathsf{T} & 0 & -C^\mathsf{T} & 0 \\ \hline 0 & -C & \gamma I_p & -D \\ -B^\mathsf{T} & 0 & -D^\mathsf{T} & \gamma I_m \end{array}\right].$$

Numerical Solution of Even Eigenvalue Problems

Transform pencil to a skew-Hamiltonian/Hamiltonian (sH/H) pencil of the form and solve this problem instead. Idea: Use a structured Schur-like form!

Generalized real Schur form

Let $sE - A$ be regular. Then there exist orthogonal matrices Q and Z such that

$$Q^{\mathsf{T}}(sE - A)Z = \begin{bmatrix} sE_{11} - A_{11} & \dots & sE_{1k} - A_{1k} \\ & \ddots & \vdots \\ & & sE_{kk} - A_{kk} \end{bmatrix},$$

where $sE_{ii} - A_{ii}$ are either 1×1 (real/infinite eigenvalues) or 2×2 (complex conjugate eigenvalues).

Numerical Solution of Even Eigenvalue Problems

- Block-structure of sH/H pencils:

$$s\mathcal{S} - \mathcal{H} = s \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{11}^{\mathsf{T}} \end{bmatrix} - \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & -H_{11}^{\mathsf{T}} \end{bmatrix},$$

S_{12}, S_{21} skew-symmetric, H_{12}, H_{21} symmetric,

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- $\mathcal{J}$-congruence transformations preserve structure:

$s\widetilde{\mathcal{S}} - \widetilde{\mathcal{H}} = \mathcal{J}\mathcal{X}^{\mathsf{T}}\mathcal{J}^{\mathsf{T}}(s\mathcal{S} - \mathcal{H})\mathcal{X}$ with nonsingular $\mathcal{X}$ is again sH/H,

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- Hope: find an orthogonal transformation $\mathcal{Q}$ such that

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where $sS_{11} - H_{11}$ is in generalized Schur form!

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  where $sS_{11} - H_{11}$ is in generalized Schur form!
- But: This form does not exist in general!

Numerical Solution of Even Eigenvalue Problems

Generalized symplectic URV decomposition

There exist orthogonal matrices $\mathcal{Q}_1$, $\mathcal{Q}_2$ such that

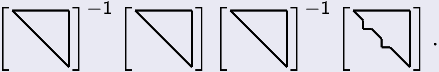
$$\mathcal{Q}_1^{\mathsf{T}} \mathcal{S} \mathcal{J} \mathcal{Q}_1 \mathcal{J}^{\mathsf{T}} = \begin{bmatrix} S_{11} & S_{12} \\ 0 & S_{11}^{\mathsf{T}} \end{bmatrix}, \quad \text{(skew-Hamiltonian)}$$

$$\mathcal{J} \mathcal{Q}_2^{\mathsf{T}} \mathcal{J}^{\mathsf{T}} \mathcal{S} \mathcal{Q}_2 = \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{11}^{\mathsf{T}} \end{bmatrix}, \quad \text{(skew-Hamiltonian)}$$

$$\mathcal{Q}_1^{\mathsf{T}} \mathcal{H} \mathcal{Q}_2 = \begin{bmatrix} H_{11} & H_{12} \\ 0 & H_{22} \end{bmatrix},$$

where $S_{11}^{-1} H_{11} T_{11}^{-1} H_{22}^{\mathsf{T}}$ is in real periodic Schur form, i. e.,

$$S_{11}^{-1} H_{11} T_{11}^{-1} H_{22}^{\mathsf{T}} =$$



Numerical Solution of Even Eigenvalue Problems

- determine a structured embedding of $s\mathcal{S} - \mathcal{H}$ into a sH/H pencil of double dimension

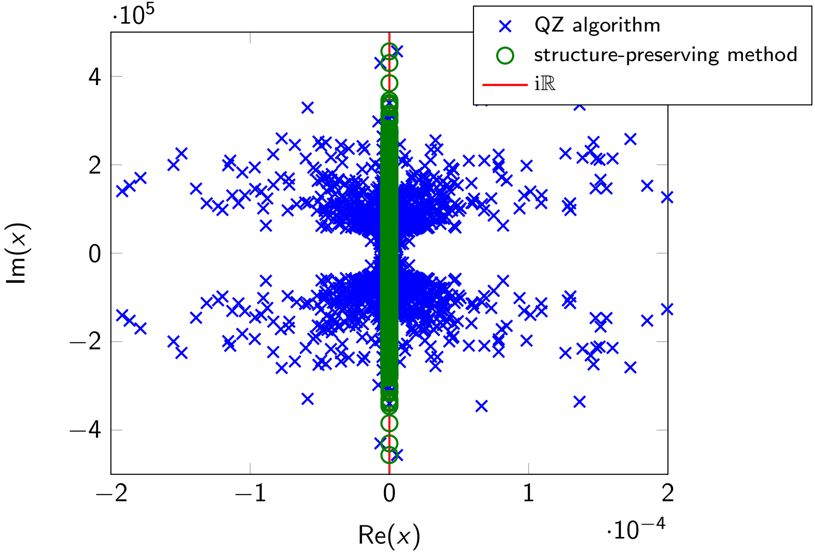
$$s\mathcal{B}_{\mathcal{S}} - \mathcal{B}_{\mathcal{H}} = \mathcal{Y}^{\mathsf{T}} \left(s \begin{bmatrix} \mathcal{S} & 0 \\ 0 & \mathcal{S} \end{bmatrix} - \begin{bmatrix} \mathcal{H} & 0 \\ 0 & -\mathcal{H} \end{bmatrix} \right) \mathcal{Y},$$

- using the symplectic URV decomposition we find

$$\begin{aligned} &\mathcal{J} \mathcal{Q}^{\mathsf{T}} \mathcal{J}^{\mathsf{T}} (s\mathcal{B}_{\mathcal{S}} - \mathcal{B}_{\mathcal{H}}) \mathcal{Q} \\ &= s \left[\begin{array}{cc|cc} S_{11} & 0 & S_{12} & 0 \\ 0 & T_{11} & 0 & T_{12} \\ \hline 0 & 0 & S_{11}^{\mathsf{T}} & 0 \\ 0 & 0 & 0 & T_{11}^{\mathsf{T}} \end{array} \right] - \left[\begin{array}{cc|cc} 0 & H_{11} & 0 & H_{12} \\ -H_{22}^{\mathsf{T}} & 0 & H_{12}^{\mathsf{T}} & 0 \\ \hline 0 & 0 & 0 & H_{22} \\ 0 & 0 & -H_{11}^{\mathsf{T}} & 0 \end{array} \right], \end{aligned}$$

- **eigenvalues:** $\pm \mathrm{i}\sqrt{\lambda_j}$ where the λ_j are the eigenvalues of $S_{11}^{-1} H_{11} T_{11}^{-1} H_{22}^{\mathsf{T}}$.

Features



$\cdot 10^5$
4
2
0
−2
−4
Im(x)
−2
−1
0
1
2
Re(x)
$\cdot 10^{-4}$
QZ algorithm
structure-preserving method
$i\mathbb{R}$

Features

- structure-preserving computations,
- faster than the QZ algorithm (less computations due to structure exploitation),
- new software for these kind of problems: `SHHEIG`, integrated in the `SLICOT` package. [Benner, Sima, V. '16]

Extensions and Other Approaches

Extensions for the $\mathcal{H}_\infty$-norm:

- extension to time-delay systems, [Michiels, Gumussoy '10]
- extension to large-scale systems (iterative eigensolvers), [Lowe, V. '13]
- computation via optimization over structured pseudospectra, [Guglielmi, Gürbüzbalaban, Overton '13], [Benner, V. '14], [Mitchell, Overton '15]
- implicit determinant method [Freitag, Spence '14]
- subspace projection approach, also works for $G(s) = C(s)D(s)^{-1}B(s)$. [Mengi, Yildrim, Kiliç '14], [Aliyev, Benner, Mengi, V. '16]

Extensions and Other Approaches

Extensions for the $\mathcal{H}_\infty$-norm:

- extension to time-delay systems, [Michiels, Gumussoy '10]
- extension to large-scale systems (iterative eigensolvers), [Lowe, V. '13]
- computation via optimization over structured pseudospectra, [Guglielmi, Gürbüzbalaban, Overton '13], [Benner, V. '14], [Mitchell, Overton '15]
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Other distances:

- real stability radii, [Qiu, Bernhardsson, Rantzer, Davison, Young, Doyle '95], [Du, Linh, Mehrmann '13]
- structured distances for Hamiltonian/symplectic systems, [Guglielmi, Kressner, Lubich '14–'15]
- structured stability radii for port-Hamiltonian systems, [Mehl, Mehrmann, Sharma '16]
- distance to singularity, [Byers, He, Mehrmann '98], [Mehl, Mehrmann, Wojtylak '15]
- many more ...

Take Home Message

- The $\mathcal{H}_\infty$-norm and other distances can be efficiently and robustly computed. This is the basis for optimizing your systems with respect to these measures!
- Many structured matrices and pencils pop up in the computations. Exploit this in your algorithms!

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Thank you for your Attention!

Our Publications

- P. Benner, V. Sima, and M. Voigt. $\mathcal{L}_\infty$-norm computation for continuous-time descriptor systems using structured matrix pencils. *IEEE Trans. Automat. Control*, 57(1):233–238, 2012.
- R. Lowe and M. Voigt. $\mathcal{L}_\infty$-norm computation for large-scale descriptor systems using structured iterative eigensolvers. Technical Report MPIMD/13-20, Max Planck Institute Magdeburg, October 2013. Available from `http://www.mpi-magdeburg.mpg.de/preprints/2013/20/`.
- P. Benner and M. Voigt. A structured pseudospectral method for $\mathcal{H}_\infty$-norm computation of large-scale descriptor systems. *Math. Control Signals Systems*, 26(2):303–338, 2014.
- M. Voigt. *On Linear-Quadratic Optimal Control and Robustness of Differential-Algebraic Systems*. Logos-Verlag, Berlin, 2015. Also as Dissertation, Otto-von-Guericke-Universität Magdeburg, Fakultät für Mathematik, 2015.
- P. Benner, V. Sima, and M. Voigt. Algorithm 961 – Fortran 77 subroutines for the solution of skew-Hamiltonian/Hamiltonian eigenproblems. *ACM Trans. Math. Software*, 42(3):Article 24, 2016.
- N. Aliyev, P. Benner, E. Mengi, and M. Voigt. Large-scale computation of $\mathcal{H}_\infty$ norms by means of a subspace method, 2016. In preparation.