Inner-Outer Factorization for Differential-Algebraic Systems

Timo Reis[1] **Matthias Voigt[2]**

[1]Universität Hamburg
Fachbereich Mathematik

[2]Technische Universität Berlin
Institut für Mathematik

DAE Day
Berlin, Germany
December 1, 2015

Differential-Algebraic Systems

Linear time-invariant differential-algebraic systems

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}Ex(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t), \end{aligned}$$

where $sE - A \in \mathbb{R}[s]^{n \times n}$ is regular, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{p \times n}$, $D \in \mathbb{R}^{p \times m}$ and with the transfer function

$$G(s) = C(sE - A)^{-1}B + D.$$

Inner-Outer Factorizations

Definition: A rational function $G(s) \in \mathbb{R}(s)^{p \times m}$ is called

a) outer if $p = \operatorname{rank}_{\mathbb{R}(s)} G(s)$ and $G(s)$ has no zeros in $\mathbb{C}^+$;

b) inner if $G(s)$ has no poles in $\mathbb{C}^+$ and $G^{\mathsf{H}}(-\overline{s})G(s) = I_m$.

Goal of this talk

Determination of a factorization of the form

$$G(s) = G_{\mathrm{i}}(s)G_{\mathrm{o}}(s),$$

where $G_{\mathrm{i}}(s)$ is inner and $G_{\mathrm{o}}(s)$ is outer.

Terminology

a) behavior:

$$\mathfrak{B}_{[E,A,B]} := \left\{ (x,u) \in \mathcal{L}^2_{\mathrm{loc}}(\mathbb{R},\mathbb{R}^n) \times \mathcal{L}^2_{\mathrm{loc}}(\mathbb{R},\mathbb{R}^m) : \tfrac{\mathrm{d}}{\mathrm{d}t} Ex = Ax + Bu \right\}.$$

b) system space: smallest subspace in $\mathbb{R}^{n+m}$ such that

$$\begin{pmatrix} x(t) \\ u(t) \end{pmatrix} \in \mathcal{V}_{\mathrm{sys}} \quad \forall (x,u) \in \mathfrak{B}_{[E,A,B]} \text{ and almost all } t \in \mathbb{R}.$$

c) space of consistent initial differential variables:

$$\mathcal{V}_{\mathrm{diff}} := \left\{ x_0 \in \mathbb{K}^n : \exists (x,u) \in \mathfrak{B}_{[E,A,B]} \text{ with } Ex(0) = Ex_0 \right\}.$$

Lur'e Equations

For $Q = Q^{\mathsf{T}} \in \mathbb{R}^{n \times n}$, $S \in \mathbb{R}^{n \times m}$, $R = R^{\mathsf{T}} \in \mathbb{R}^{m \times m}$, the Lur'e equation is given by

$$\begin{bmatrix} A^{\mathsf{T}}XE + E^{\mathsf{T}}XA + Q & E^{\mathsf{T}}XB + S \\ B^{\mathsf{T}}XE + S^{\mathsf{T}} & R \end{bmatrix} =_{\mathcal{V}_{\mathrm{sys}}} \begin{bmatrix} K^{\mathsf{T}} \\ L^{\mathsf{T}} \end{bmatrix} \begin{bmatrix} K & L \end{bmatrix}, \quad X = X^{\mathsf{T}}.$$

A triple $(X, K, L) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{q \times n} \times \mathbb{R}^{q \times m}$ for some $q \in \mathbb{N}_0$ is called solution if

$$\operatorname{rank}_{\mathbb{R}(s)} \begin{bmatrix} -sE + A & B \\ K & L \end{bmatrix} = n + q.$$

A solution is called

a) stabilizing, if

$$\operatorname{rank} \begin{bmatrix} -\lambda E + A & B \\ K & L \end{bmatrix} = n + q \quad \forall\, \lambda \in \mathbb{C}^{+};$$

b) nonnegative, if

$$E^{\mathsf{T}}XE \geq_{\mathcal{V}_{\mathrm{diff}}} 0.$$

Theorem

Let $[E, A, B] \in \Sigma_{n,m}(\mathbb{K})$ be behaviorally stabilizable. If

$$\begin{bmatrix} A^{\mathsf{T}}PE + E^{\mathsf{T}}PA + Q & E^{\mathsf{T}}PB + S \\ B^{\mathsf{T}}PE + S^{\mathsf{T}} & R \end{bmatrix} \geq_{\mathcal{V}_{\mathrm{sys}}} 0,$$

is solvable, then the Lur'e equation has a stabilizing solution $(X, K, L) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{q\times n} \times \mathbb{R}^{q\times m}$ with the properties:

a) **Maximality:** $E^{\mathsf{T}}XE \geq_{\mathcal{V}_{\mathrm{diff}}} E^{\mathsf{T}}PE$ for all $P \in \mathbb{R}^{n\times n}$ fulfilling the LMI.

b) **Spectral factorization:** Let

$$\Phi(s) := \begin{bmatrix} (-\overline{s}E - A)^{-1}B \\ I_m \end{bmatrix}^{\mathsf{H}} \begin{bmatrix} Q & S \\ S^{\mathsf{T}} & R \end{bmatrix} \begin{bmatrix} (sE - A)^{-1}B \\ I_m \end{bmatrix} \in \mathbb{R}(s)^{m\times m}.$$

Then $\Phi(s) = W^{\mathsf{H}}(-\overline{s})W(s)$ for outer

$$W(s) = K(sE - A)^{-1}B + L \in \mathbb{R}(s)^{q\times m}.$$

c) **Rank:** $q = \operatorname{rank}_{\mathbb{R}(s)} \Phi(s)$.

Construction of the Factors

Theorem

Let $[E, A, B, C, D]$ be behaviorally stabilizable with transfer function $G(s) \in \mathbb{R}(s)^{p \times m}$. Let $q = \operatorname{rank}_{\mathbb{R}(s)} G(s)$ and $Z \in \mathbb{R}^{m \times q}$ be a matrix with $\operatorname{rank}_{\mathbb{R}(s)} G(s)Z = q$. Let (X, K, L) be a stabilizing solution of the Lur'e equation

$$\begin{bmatrix} A^{\mathsf{T}}XE + E^{\mathsf{T}}XA + C^{\mathsf{T}}C & E^{\mathsf{T}}XB + C^{\mathsf{T}}D \\ B^{\mathsf{T}}XE + D^{\mathsf{T}}C & D^{\mathsf{T}}D \end{bmatrix} =_{\mathcal{V}_{\mathrm{sys}}} \begin{bmatrix} K^{\mathsf{T}} \\ L^{\mathsf{T}} \end{bmatrix} \begin{bmatrix} K & L \end{bmatrix}, \quad X = X^{\mathsf{T}}.$$

Then an $G_{\mathrm{i}}(s) \in \mathbb{R}(s)^{p \times q}$ is the transfer function of

$$[E_{\mathrm{i}}, A_{\mathrm{i}}, B_{\mathrm{i}}, C_{\mathrm{i}}, D_{\mathrm{i}}] := \left[\begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A & BZ \\ K & LZ \end{bmatrix}, \begin{bmatrix} 0 \\ -I_q \end{bmatrix}, \begin{bmatrix} C & DZ \end{bmatrix}, 0_{p \times q} \right]$$

and $G_o(s) \in \mathbb{R}(s)^{q \times m}$ is the transfer function of

$$[E_{\mathrm{o}}, A_{\mathrm{o}}, B_{\mathrm{o}}, C_{\mathrm{o}}, D_{\mathrm{o}}] := [E, A, B, K, L].$$

Proof Idea

Step 1: $G_{\mathrm{o}}(s) = K(sE - A)^{-1}B + L$ is outer:
Follows from the fact, that (X, K, L) is a stabilizing solution.

Proof Idea

Step 1: $G_o(s) = K(sE - A)^{-1}B + L$ is outer:
Follows from the fact, that (X, K, L) is a stabilizing solution.
Step 2: $\ker_{\mathbb{R}(s)} G_o(s) = \ker_{\mathbb{R}(s)} G(s)$:
Follows from

$$\Phi(\mathrm{i}\omega) = G(\mathrm{i}\omega)^{\mathsf{H}} G(\mathrm{i}\omega) = G_o(\mathrm{i}\omega)^{\mathsf{H}} G_o(\mathrm{i}\omega) \quad \forall \mathrm{i}\omega \notin \Lambda(E, A)$$

and

$$\|G(\mathrm{i}\omega)v(\mathrm{i}\omega)\|_2 = 0 \quad \Leftrightarrow \quad \|G_o(\mathrm{i}\omega)v(\mathrm{i}\omega)\|_2 = 0 \quad \forall \mathrm{i}\omega \notin \Lambda(E, A) \cup \mathfrak{P}(v).$$

Proof Idea

Step 1: $G_{\mathrm{o}}(s) = K(sE - A)^{-1}B + L$ is outer:
Follows from the fact, that (X, K, L) is a stabilizing solution.
Step 2: $\ker_{\mathbb{R}(s)} G_{\mathrm{o}}(s) = \ker_{\mathbb{R}(s)} G(s)$:
Follows from

$$\Phi(\mathrm{i}\omega) = G(\mathrm{i}\omega)^{\mathsf{H}} G(\mathrm{i}\omega) = G_{\mathrm{o}}(\mathrm{i}\omega)^{\mathsf{H}} G_{\mathrm{o}}(\mathrm{i}\omega) \quad \forall \mathrm{i}\omega \notin \Lambda(E, A)$$

and

$$\|G(\mathrm{i}\omega)v(\mathrm{i}\omega)\|_2 = 0 \quad \Leftrightarrow \quad \|G_{\mathrm{o}}(\mathrm{i}\omega)v(\mathrm{i}\omega)\|_2 = 0 \quad \forall \mathrm{i}\omega \notin \Lambda(E, A) \cup \mathfrak{P}(v).$$

Step 3: $G_{\mathrm{o}}(s)Z$ is invertible:

- Step 2 $\quad \Rightarrow \quad \operatorname{rank}_{\mathbb{R}(s)} G_{\mathrm{o}}(s) = \operatorname{rank}_{\mathbb{R}(s)} G(s) = q$,
- $G_{\mathrm{o}}(s)$ outer $\quad \Rightarrow \quad G_{\mathrm{o}}(s) \in \mathbb{R}(s)^{q \times m}$,
- $(G(\mathrm{i}\omega)Z)^{\mathsf{H}}(G(\mathrm{i}\omega)Z) = (G_{\mathrm{o}}(\mathrm{i}\omega)Z)^{\mathsf{H}}(G_{\mathrm{o}}(\mathrm{i}\omega)Z) \quad \forall \mathrm{i}\omega \notin \Lambda(E, A)$.

Then the statement follows from $\operatorname{rank}_{\mathbb{R}(s)} G(s)Z = q$.

Proof Idea

Step 4: $G_{\mathrm{i}}(s)G_{\mathrm{o}}(s) = G(s)$:

$$G_{\mathrm{i}}(s) = \begin{bmatrix} C & DZ \end{bmatrix} \begin{bmatrix} sE - A & -BZ \\ -K & -LZ \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ -I_q \end{bmatrix} = G(s)Z(G_{\mathrm{o}}(s)Z)^{-1}.$$

Further, $Z(G_{\mathrm{o}}(s)Z)^{-1}G_{\mathrm{o}}(s)$ is projector along $\ker_{\mathbb{R}(s)} G(s)$ and thus

$$\begin{aligned} G_{\mathrm{i}}(s) \cdot G_{\mathrm{o}}(s) &= G(s)Z(G_{\mathrm{o}}(s)Z)^{-1} \cdot G_{\mathrm{o}}(s) \\ &= G(s) - \underbrace{G(s)(I_m - Z(G_{\mathrm{o}}(s)Z)^{-1}G_{\mathrm{o}}(s))}_{=0} = G(s). \end{aligned}$$

Proof Idea

Step 4: $G_{\mathrm{i}}(s)G_{\mathrm{o}}(s) = G(s)$:

$$G_{\mathrm{i}}(s) = \begin{bmatrix} C & DZ \end{bmatrix} \begin{bmatrix} sE - A & -BZ \\ -K & -LZ \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ -I_q \end{bmatrix} = G(s)Z(G_{\mathrm{o}}(s)Z)^{-1}.$$

Further, $Z(G_{\mathrm{o}}(s)Z)^{-1}G_{\mathrm{o}}(s)$ is projector along $\ker_{\mathbb{R}(s)} G(s)$ and thus

$$\begin{aligned} G_{\mathrm{i}}(s) \cdot G_{\mathrm{o}}(s) &= G(s)Z(G_{\mathrm{o}}(s)Z)^{-1} \cdot G_{\mathrm{o}}(s) \\ &= G(s) - \underbrace{G(s)(I_m - Z(G_{\mathrm{o}}(s)Z)^{-1}G_{\mathrm{o}}(s))}_{=0} = G(s). \end{aligned}$$

Step 5: $G_{\mathrm{i}}(s)$ is inner:
Show that

$$\begin{bmatrix} A_{\mathrm{i}}^{\mathsf{T}} P_{\mathrm{i}} E_{\mathrm{i}} + E_{\mathrm{i}}^{\mathsf{T}} P_{\mathrm{i}} A_{\mathrm{i}} + C_{\mathrm{i}}^{\mathsf{T}} C_{\mathrm{i}} & E_{\mathrm{i}}^{\mathsf{T}} P_{\mathrm{i}} B_{\mathrm{i}} \\ B_{\mathrm{i}}^{\mathsf{T}} P_{\mathrm{i}} E_{\mathrm{i}} & -I_q \end{bmatrix} =_{\mathcal{V}_{\mathrm{sys,i}}} 0, \quad E_{\mathrm{i}}^{\mathsf{T}} P_{\mathrm{i}} E_{\mathrm{i}} \geq_{\mathcal{V}_{\mathrm{diff,i}}} 0,$$

simple calculation, leads to the Lur'e equation.

Example

Let the behaviorally stabilizable system

$$sE - A = \begin{bmatrix} -1 & s \\ 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} -1 & -1 \\ -1 & 0 \end{bmatrix}, \quad D = 0_{2\times 1}$$

be given. The transfer function is $G(s) = \begin{bmatrix} s+1 \\ s \end{bmatrix} \in \mathbb{R}(s)^{2\times 1}$.

Example

Let the behaviorally stabilizable system

$$sE - A = \begin{bmatrix} -1 & s \\ 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} -1 & -1 \\ -1 & 0 \end{bmatrix}, \quad D = 0_{2\times 1}$$

be given. The transfer function is $G(s) = \begin{bmatrix} s+1 \\ s \end{bmatrix} \in \mathbb{R}(s)^{2\times 1}$.

A stabilizing solution of the Lur'e equation

$$\begin{bmatrix} A^\mathsf{T} XE + E^\mathsf{T} XA + C^\mathsf{T} C & E^\mathsf{T} XB + C^\mathsf{T} D \\ B^\mathsf{T} XE + D^\mathsf{T} C & D^\mathsf{T} D \end{bmatrix} =_{\mathcal{V}_{\mathrm{sys}}} \begin{bmatrix} K^\mathsf{T} \\ L^\mathsf{T} \end{bmatrix} \begin{bmatrix} K & L \end{bmatrix}, \quad X = X^\mathsf{T}$$

is

$$(X, K, L) = \left(\begin{bmatrix} \sqrt{2} - 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} -\sqrt{2} & -1 \end{bmatrix}, 0 \right).$$

Example

Then with $Z = 1$

$$G_{\mathrm{o}}(s) = \begin{bmatrix} -\sqrt{2} & -1 \end{bmatrix} \begin{bmatrix} -1 & s \\ 0 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \sqrt{2}s + 1 \in \mathbb{R}(s),$$

$$G_{\mathrm{i}}(s) = \begin{bmatrix} -1 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} -1 & s & 0 \\ 0 & -1 & -1 \\ \sqrt{2} & 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{s+1}{1+\sqrt{2}s} \\ \frac{s}{1+\sqrt{2}s} \end{bmatrix} \in \mathbb{R}(s)^{2\times 1}.$$

We have $G(s) = G_{\mathrm{i}}(s)G_{\mathrm{o}}(s) = \begin{bmatrix} s+1 \\ s \end{bmatrix} \in \mathbb{R}(s)^{2\times 1}$. Further,

- $G_{\mathrm{o}}(s)$ is outer, since it has full row rank and the only zero $\lambda_0 = -\frac{1}{\sqrt{2}} \notin \mathbb{C}^+$.
- $G_{\mathrm{i}}(s)$ is inner, since the only pole $\lambda_0 = -\frac{1}{\sqrt{2}} \notin \mathbb{C}^+$ and

$$G_{\mathrm{i}}^{\mathsf{H}}(-\overline{s})G_{\mathrm{i}}(s) = \begin{bmatrix} \frac{-s+1}{1-\sqrt{2}s} & \frac{-s}{1-\sqrt{2}s} \end{bmatrix} \begin{bmatrix} \frac{s+1}{1+\sqrt{2}s} \\ \frac{s}{1+\sqrt{2}s} \end{bmatrix} = 1.$$

Conclusions

Have shown the construction of inner-outer factorization for arbitrary transfer functions given by behaviorally stabilizable differential-algebraic equations. The construction based on simple formulas using Lur'e equations.

Final remarks:

- the formulas do not need previous successive reductions of $[E, A, B, C, D]$ such as in [Oară, Varga '00],
- no restrictive assumptions such as properness, stability, or right invertibility [Yeh, Wei '90], [Xin, Mita '98],
- algorithms in [Varga '98], [Oară, Varga '00].

Conclusions

Have shown the construction of inner-outer factorization for arbitrary transfer functions given by behaviorally stabilizable differential-algebraic equations. The construction based on simple formulas using Lur'e equations.

Final remarks:

- the formulas do not need previous successive reductions of $[E, A, B, C, D]$ such as in [Oară, Varga '00],
- no restrictive assumptions such as properness, stability, or right invertibility [Yeh, Wei '90], [Xin, Mita '98],
- algorithms in [Varga '98], [Oară, Varga '00].

Thanks for the Attention!

References

- F.-B. Yeh and L.-F. Wei. Inner-outer factorizations of right-invertible real-rational matrices. *Systems Control Lett.*, 14(1):31–36, 1990.
- X. Xin and T. Mita. Inner-outer factorization for non-square proper functions with infinite and finite $j\omega$-axis zeros. *Internat. J. Control*, 71(1):145–161, 1998.
- A. Varga. Computation of inner-outer factorizations of rational matrices. *IEEE Trans. Automat. Control.*, 43(5):684–688, 1998.
- C. Oară and A. Varga. Computation of general inner-outer and spectral factorizations. *IEEE Trans. Automat. Control*, 45(12):2307–2325, 2000.
- M. Voigt. *On Linear-Quadratic Optimal Control and Robustness of Differential-Algebraic Systems*. Logos-Verlag, Berlin, 2015. Also as Dissertation, Otto-von-Guericke-Universität Magdeburg, Fakultät für Mathematik, 2015.
- T. Reis and M. Voigt. Inner-outer factorizations for differential-algebraic systems, Hamburger Beiträge zur angewandten Mathematik 2015-31, 2015. Submitted for publication.