Existence of Nonpositive Solutions for the Kalman-Yakubovich-Popov Inequality

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Differential-Algebraic Systems

Linear time-invariant differential-algebraic systems

$$\tfrac{\mathrm{d}}{\mathrm{d}t}Ex(t) = Ax(t) + Bu(t).$$

Assumptions: $sE - A \in \mathbb{R}[s]^{n\times n}$ regular, $B \in \mathbb{R}^{n\times m}$
Terminology:

a) behavior:

$$\mathfrak{B}_{[E,A,B]} := \left\{(x,u) \in \mathcal{L}^2_{\mathrm{loc}}(\mathbb{R},\mathbb{R}^n) \times \mathcal{L}^2_{\mathrm{loc}}(\mathbb{R},\mathbb{R}^m) : \tfrac{\mathrm{d}}{\mathrm{d}t}Ex = Ax + Bu\right\}.$$

b) system space: smallest subspace in $\mathbb{R}^{n+m}$ such that

$$\begin{pmatrix} x(t) \\ u(t) \end{pmatrix} \in \mathcal{V}_{\mathrm{sys}} \quad \forall (x,u) \in \mathfrak{B}_{[E,A,B]} \text{ and almost all } t \in \mathbb{R}.$$

c) space of consistent initial differential variables:

$$\mathcal{V}_{\mathrm{diff}} := \left\{x_0 \in \mathbb{K}^n : \exists (x,u) \in \mathfrak{B}_{[E,A,B]} \text{ with } Ex(0) = Ex_0\right\}.$$

Controllability and Stabilizability

$[E, A, B] \in \Sigma_{n,m}$ is called

a) behaviorally controllable:

$$\Leftrightarrow \ \operatorname{rank} \begin{bmatrix} \lambda E - A & B \end{bmatrix} = n \quad \forall \lambda \in \mathbb{C};$$

b) behaviorally stabilizable:

$$\Leftrightarrow \ \operatorname{rank} \begin{bmatrix} \lambda E - A & B \end{bmatrix} = n \quad \forall \lambda \in \mathbb{C}^+;$$

c) behaviorally sign-controllable:

$$\Leftrightarrow \ \operatorname{rank} \begin{bmatrix} \lambda E - A & B \end{bmatrix} = n \text{ or } \operatorname{rank} \begin{bmatrix} -\overline{\lambda} E - A & B \end{bmatrix} = n \quad \forall \lambda \in \mathbb{C}.$$

With the output equation $y = Cx + Du$ it is called behaviorally detectable:

$$\Leftrightarrow \ \operatorname{rank} \begin{bmatrix} \lambda E - A \\ C \end{bmatrix} = n \quad \forall \lambda \in \mathbb{C}^+.$$

Kalman-Yakubovich-Popov Lemma

Modified Popov function: For $Q = Q^{\mathsf{T}} \in \mathbb{R}^{n \times n}$, $S \in \mathbb{R}^{n \times m}$, $R = R^{\mathsf{T}} \in \mathbb{R}^{m \times m}$:

$$\Psi(\lambda) = \begin{bmatrix} (\lambda E - A)^{-1} B \\ I_m \end{bmatrix}^{\mathsf{H}} \begin{bmatrix} Q & S \\ S^{\mathsf{T}} & R \end{bmatrix} \begin{bmatrix} (\lambda E - A)^{-1} B \\ I_m \end{bmatrix}.$$

KYP Lemma [Reis, Rendel, V. '15]

a) If the KYP inequality

$$\begin{pmatrix} x \\ u \end{pmatrix}^{\mathsf{T}} \begin{bmatrix} A^{\mathsf{T}} P E + E^{\mathsf{T}} P A + Q & E^{\mathsf{T}} P B + S \\ B^{\mathsf{T}} P E + S^{\mathsf{T}} & R \end{bmatrix} \begin{pmatrix} x \\ u \end{pmatrix} \geq 0 \quad \forall \begin{pmatrix} x \\ u \end{pmatrix} \in \mathcal{V}_{\mathrm{sys}},$$

has a solution $P = P^{\mathsf{T}} \in \mathbb{R}^{n \times n}$, then $\Psi(\mathrm{i}\omega) \geq 0 \, \forall \mathrm{i}\omega \in \mathrm{i}\mathbb{R} \setminus \sigma(E, A)$.

b) If $\Psi(\mathrm{i}\omega) \geq 0 \, \forall \mathrm{i}\omega \in \mathrm{i}\mathbb{R} \setminus \sigma(E, A)$ and at least one of the two properties

i) $\operatorname{rank} \Psi(\mathrm{i}\omega_0) = m$ for $\mathrm{i}\omega_0 \in \mathrm{i}\mathbb{R} \setminus \sigma(E, A)$, $[E, A, B]$ is behaviorally sign-controllable

ii) $[E, A, B]$ is behaviorally controllable

is satisfied, then there exists some $P \in \mathbb{R}^{n \times n}$ that solves the KYP inequality.

Kalman-Yakubovich-Popov Lemma

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KYP Lemma [Reis, Rendel, V. '15]

a) If the KYP inequality

$$\begin{bmatrix} A^{\mathsf{T}}PE + E^{\mathsf{T}}PA + Q & E^{\mathsf{T}}PB + S \\ B^{\mathsf{T}}PE + S^{\mathsf{T}} & R \end{bmatrix} \geq_{\mathcal{V}_{\mathrm{sys}}} 0,$$

has a solution $P = P^{\mathsf{T}} \in \mathbb{R}^{n\times n}$, then $\Psi(\mathrm{i}\omega) \geq 0 \; \forall \mathrm{i}\omega \in \mathrm{i}\mathbb{R} \setminus \sigma(E, A)$.

b) If $\Psi(\mathrm{i}\omega) \geq 0 \; \forall \mathrm{i}\omega \in \mathrm{i}\mathbb{R} \setminus \sigma(E, A)$ and at least one of the two properties
   i) $\operatorname{rank} \Psi(\mathrm{i}\omega_0) = m$ for $\mathrm{i}\omega_0 \in \mathrm{i}\mathbb{R} \setminus \sigma(E, A)$, $[E, A, B]$ is behaviorally sign-controllable
   ii) $[E, A, B]$ is behaviorally controllable

   is satisfied, then there exists some $P \in \mathbb{R}^{n\times n}$ that solves the KYP inequality.

Relations to Cyclo-Dissipativity

Let $[E, A, B]$ be behaviorally controllable. Then the following are equivalent:

a) It holds that

$$\int_0^T \begin{pmatrix} x(\tau) \\ u(\tau) \end{pmatrix}^\mathsf{T} \begin{bmatrix} Q & S \\ S^\mathsf{T} & R \end{bmatrix} \begin{pmatrix} x(\tau) \\ u(\tau) \end{pmatrix} \mathrm{d}\tau \geq 0$$

for all $(x, u) \in \mathfrak{B}_{[E,A,B]}$ with $Ex(0) = Ex(T) = 0$.

b) The KYP inequality

$$\begin{bmatrix} A^\mathsf{T} PE + E^\mathsf{T} PA + Q & E^\mathsf{T} PB + S \\ B^\mathsf{T} PE + S^\mathsf{T} & R \end{bmatrix} \geq_{\mathcal{V}_{\mathrm{sys}}} 0, \quad P = P^\mathsf{T}$$

has a solution P.

c) It holds that $\Psi(\mathrm{i}\omega) \geq 0 \; \forall \mathrm{i}\omega \in \mathrm{i}\mathbb{R} \setminus \sigma(E, A)$.

Important Special Case

Let $[E, A, B]$ be behaviorally controllable with output $y = Cx + Du$, transfer function $G(s) = C(sE - A)^{-1}B + D$. Then the following are equivalent:

a) It holds that

$$\int_0^T \begin{pmatrix} x(\tau) \\ u(\tau) \end{pmatrix}^\mathsf{T} \begin{bmatrix} 0 & C^\mathsf{T} \\ C & D + D^\mathsf{T} \end{bmatrix} \begin{pmatrix} x(\tau) \\ u(\tau) \end{pmatrix} \mathrm{d}\tau \geq 0$$

for all $(x, u) \in \mathfrak{B}_{[E,A,B]}$ with $Ex(0) = 0$. ($Ex(T)$ free)

b) The KYP inequality

$$\begin{bmatrix} A^\mathsf{T}PE + E^\mathsf{T}PA & E^\mathsf{T}PB + C^\mathsf{T} \\ B^\mathsf{T}PE + C & D + D^\mathsf{T} \end{bmatrix} \geq_{\mathcal{V}_\mathrm{sys}} 0, \quad P = P^\mathsf{T}$$

has a solution with $E^\mathsf{T}PE \leq_{\mathcal{V}_\mathrm{diff}} 0$.

c) It holds that $\Psi(\lambda) = G(\lambda) + G(\lambda)^\mathsf{H} \geq 0 \; \forall \lambda \in \mathbb{C}^+ \setminus \sigma(E, A)$.

Does such a relation hold true for general Q, S, R?

Let $[E, A, B]$ be behaviorally controllable. Are the following equivalent?

a) It holds that

$$\int_0^T \begin{pmatrix} x(\tau) \\ u(\tau) \end{pmatrix}^{\mathsf{T}} \begin{bmatrix} Q & S \\ S^{\mathsf{T}} & R \end{bmatrix} \begin{pmatrix} x(\tau) \\ u(\tau) \end{pmatrix} \mathrm{d}\tau \geq 0$$

for all $(x, u) \in \mathfrak{B}_{[E,A,B]}$ with $Ex(0) = 0$.

b) The KYP inequality

$$\begin{bmatrix} A^{\mathsf{T}}PE + E^{\mathsf{T}}PA + Q & E^{\mathsf{T}}PB + S \\ B^{\mathsf{T}}PE + S^{\mathsf{T}} & R \end{bmatrix} \geq_{\mathcal{V}_{\mathrm{sys}}} 0, \quad P = P^{\mathsf{T}}$$

has a solution $E^{\mathsf{T}}PE \leq_{\mathcal{V}_{\mathrm{diff}}} 0$.

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$$\begin{bmatrix} A^\mathsf{T} PE + E^\mathsf{T} PA + Q & E^\mathsf{T} PB + S \\ B^\mathsf{T} PE + S^\mathsf{T} & R \end{bmatrix} \geq_{\mathcal{V}_{\mathrm{sys}}} 0, \quad P = P^\mathsf{T}$$

has a solution $E^\mathsf{T} PE \leq_{\mathcal{V}_{\mathrm{diff}}} 0$.

c) It holds that $\Psi(\lambda) \geq 0 \; \forall \lambda \in \mathbb{C}^+ \setminus \sigma(E, A)$.

Answer:

a) $\Leftrightarrow$ b) $\Rightarrow$ c), but in general c) $\not\Rightarrow$ a), b)

Counterexample

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ Q = \begin{bmatrix} 1 & \alpha \\ \alpha & 0 \end{bmatrix},\ S = \begin{bmatrix} 0 \\ 0 \end{bmatrix},\ R = 1,$$

see [Willems '74]

Minimal solution of KYP inequality:

$$P = \begin{bmatrix} \sqrt{2} - \alpha & 1 \\ 1 & -\sqrt{2} \end{bmatrix} \left(\leq 0 \ \Leftrightarrow \ \alpha \geq -1/\sqrt{2} \approx -0.7071 \right)$$

Counterexample

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ Q = \begin{bmatrix} 1 & \alpha \\ \alpha & 0 \end{bmatrix},\ S = \begin{bmatrix} 0 \\ 0 \end{bmatrix},\ R = 1,$$

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Modified Popov function:

$$\Psi(\lambda) = \frac{1}{|\lambda|^2} + \alpha \left(\frac{1}{|\lambda|\lambda} + \frac{1}{|\lambda|\overline{\lambda}} \right) + 1$$

With $\lambda = \sigma + \mathrm{i}\omega$:

$$\begin{aligned} \Psi(\lambda) \geq 0 \quad \forall \lambda \in \mathbb{C}^+ \quad & \Leftrightarrow \quad (\sigma^2 + \omega^2)^2 + 2\alpha\sigma + 1 \geq 0 \quad \forall (\sigma, \omega) \in \mathbb{R}^+ \times \mathbb{R} \\ & \Leftrightarrow \quad \alpha \geq -2/\left(3 \cdot \sqrt[4]{3}\right) \approx -0.8774. \end{aligned}$$

Sufficient Conditions for c) $\Rightarrow$ b)

Theorem [ODEs: Willems '74, Reis, V. '15]

Assume that $\Psi(\lambda) \geq 0 \ \forall \lambda \in \mathbb{C}^+ \setminus \sigma(E, A)$ and at least one of the two properties

i) $\operatorname{rank} \Psi(\mathrm{i}\omega_0) = m$ for $\mathrm{i}\omega_0 \in \mathrm{i}\mathbb{R} \setminus \sigma(E, A)$, $[E, A, B]$ is behaviorally stabilizable

ii) $[E, A, B]$ is behaviorally controllable

is satisfied. Let $C_1 \in \mathbb{R}^{m \times n}$, $C_2 \in \mathbb{R}^{p_2 \times n}$, $D_1 \in \mathbb{R}^{m \times m}$, $D_2 \in \mathbb{R}^{p_2 \times m}$ be such that

$$\begin{bmatrix} Q & S \\ S^\mathsf{T} & R \end{bmatrix} =_{\mathcal{V}_{\mathrm{sys}}} \begin{bmatrix} C_1^\mathsf{T} C_1 & C_1^\mathsf{T} D_1 \\ D_1^\mathsf{T} C_1 & D_1^\mathsf{T} D_1 \end{bmatrix} - \begin{bmatrix} C_2^\mathsf{T} C_2 & C_2^\mathsf{T} D_2 \\ D_2^\mathsf{T} C_2 & D_2^\mathsf{T} D_2 \end{bmatrix}$$

and

$$G_1(s) := C_1(sE - A)^{-1} B + D_1$$

is invertible. Then the following holds true:

Sufficient Conditions for c) $\Rightarrow$ b)

Theorem [ODEs: Willems '74, Reis, V. '15]

a) There exists a solution of

$$\begin{bmatrix} A^\mathsf{T}PE + E^\mathsf{T}PA + Q & E^\mathsf{T}PB + S \\ B^\mathsf{T}PE + S^\mathsf{T} & R \end{bmatrix} \geq_{\mathcal{V}_{\rm sys}} 0, \quad P = P^\mathsf{T}$$

with $E^\mathsf{T}PE \leq_{\mathcal{V}_{\rm diff}} 0$.

b) If, furthermore,

$$\operatorname{rank} \begin{bmatrix} -\lambda E + A & B \\ C_1 & D_1 \\ C_2 & D_2 \end{bmatrix} = n + m \quad \forall \lambda \in \mathbb{C}^+,$$

then all solutions of the KYP inequality fulfill $E^\mathsf{T}PE \leq_{\mathcal{V}_{\rm diff}} 0$.

Proof

Special Case: $C_1 = 0,\ D_1 = I_m$, i. e.,

$$\begin{bmatrix} Q & S \\ S^{\mathsf{T}} & R \end{bmatrix} =_{\mathcal{V}_{\rm sys}} \begin{bmatrix} 0 & 0 \\ 0 & I_m \end{bmatrix} - \begin{bmatrix} C_2^{\mathsf{T}} C_2 & C_2^{\mathsf{T}} D_2 \\ D_2^{\mathsf{T}} C_2 & D_2^{\mathsf{T}} D_2 \end{bmatrix}.$$

- $\Psi(\lambda) = I_m - G_2(\lambda)^{\mathsf{H}} G_2(\lambda) \geq 0\ \forall \lambda \in \mathbb{C}^+ \setminus \sigma(E, A) \Rightarrow G_2 \in \mathcal{RH}_\infty^{p_2 \times m}$.

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Special Case: $C_1 = 0,\ D_1 = I_m$, i. e.,

$$\begin{bmatrix} Q & S \\ S^{\mathsf{T}} & R \end{bmatrix} =_{\mathcal{V}_{\mathrm{sys}}} \begin{bmatrix} 0 & 0 \\ 0 & I_m \end{bmatrix} - \begin{bmatrix} C_2^{\mathsf{T}} C_2 & C_2^{\mathsf{T}} D_2 \\ D_2^{\mathsf{T}} C_2 & D_2^{\mathsf{T}} D_2 \end{bmatrix}.$$

- $\Psi(\lambda) = I_m - G_2(\lambda)^{\mathsf{H}} G_2(\lambda) \geq 0\ \forall \lambda \in \mathbb{C}^+ \setminus \sigma(E, A) \Rightarrow G_2 \in \mathcal{RH}_\infty^{p_2 \times m}$.

Statement b):

- **Lemma:** If $[E, A, B, C, D]$ is beh. stabilizable and beh. detectable and $G \in \mathcal{RH}_\infty^{p \times m}$, then $\sigma(E, A) \subset \mathbb{C}^-$.
- We have

$$\operatorname{rank} \begin{bmatrix} -\lambda E + A & B \\ C_1 & D_1 \\ C_2 & D_2 \end{bmatrix} = n + m \quad \forall \lambda \in \mathbb{C}^+$$

$$\Rightarrow \operatorname{rank} \begin{bmatrix} \lambda E - A \\ C_2 \end{bmatrix} = n \quad \forall \lambda \in \mathbb{C}^+$$

$$\Rightarrow \sigma(E, A) \subset \mathbb{C}^- \text{ (follows from Lemma).}$$

Proof

- transform to system equivalence form with invertible W, T:

$$W\begin{bmatrix} sE - A & B \end{bmatrix}\begin{bmatrix} T & 0 \\ 0 & I_m \end{bmatrix} = \begin{bmatrix} sI_{n_1} - A_{11} & 0 & sE_{13} - A_{13} & B_1 \\ 0 & sE_{22} - I_{n_2} & sE_{23} - A_{23} & B_2 \\ 0 & 0 & sE_{33} - I_{n_3} & 0 \end{bmatrix},$$

where E_{22}, E_{33} nilpotent + some controllability condition.

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- transform to system equivalence form with invertible W, T:

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  where E_{22}, E_{33} nilpotent + some controllability condition.
- by KYP inequality, $\sigma(A_{11}) \subset \mathbb{C}^-$, and

$$P = \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{12}^\mathsf{T} & P_{22} & P_{23} \\ P_{13}^\mathsf{T} & P_{23}^\mathsf{T} & P_{33} \end{bmatrix}, \quad C_2 = \begin{bmatrix} C_{21} & C_{22} & C_{23} \end{bmatrix},$$

  we have $0 \leq A_{11}^\mathsf{T} P_{11} + P_{11}A_{11} - C_{21}^\mathsf{T} C_{21} \Rightarrow P_{11} \leq 0$.
- further show $P_{12}E_{22} = 0$, $E_{22}^\mathsf{T} P_{22} E_{22} = 0$ (complicated),
- conclude result:

$$\begin{bmatrix} I_{n_1} & 0 & 0 \\ 0 & E_{22}^\mathsf{T} & 0 \\ E_{13}^\mathsf{T} & E_{23}^\mathsf{T} & E_{33}^\mathsf{T} \end{bmatrix}\begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{12}^\mathsf{T} & P_{22} & P_{23} \\ P_{13}^\mathsf{T} & P_{23}^\mathsf{T} & P_{33} \end{bmatrix}\begin{bmatrix} I_{n_1} & 0 & E_{13} \\ 0 & E_{22} & E_{23} \\ 0 & 0 & E_{33} \end{bmatrix} =_{\mathcal{V}_{\mathrm{diff}}} \begin{bmatrix} P_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\leq_{\mathcal{V}_{\mathrm{diff}}} 0.$$

Proof

Statement a):

- Kalman decomposition

$$sE - A = \begin{bmatrix} sE_{11} - A_{11} & 0 \\ sE_{21} - A_{21} & sE_{22} - A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad C_2 = \begin{bmatrix} C_{21} & 0 \end{bmatrix},$$

  where $[E_{11}, A_{11}, B_1, C_{21}, 0]$ behaviorally detectable.

- From statement b):

$$\begin{bmatrix} A_{11}^{\mathsf{T}} P_{11} E_{11} + E_{11}^{\mathsf{T}} P_{11} A_{11} - C_{21}^{\mathsf{T}} C_{21} & E_{11}^{\mathsf{T}} P_{11} B_1 - C_{21}^{\mathsf{T}} D_2 \\ B_1^{\mathsf{T}} P_{11} E_{11} - D_2^{\mathsf{T}} C_{21} & I_m - D_2^{\mathsf{T}} D_2. \end{bmatrix} \geq_{\mathcal{V}_{\mathrm{sys},1}} 0,$$

$$E_{11}^{\mathsf{T}} P_{11} E_{11} \leq_{\mathcal{V}_{\mathrm{diff},1}} 0.$$

- $P = \begin{bmatrix} P_{11} & 0 \\ 0 & 0 \end{bmatrix}$ is nonpositive solution of overall system.

Proof

General Case: general C_1, D_1:
Statement a):

- $\Psi(\lambda) = G_1(\lambda)^{\mathsf{H}} G_1(\lambda) - G_2(\lambda)^{\mathsf{H}} G_2(\lambda) \geq 0$,
- G_1 invertible $\Rightarrow I_m - G_{\mathrm{e}}(\lambda)^{\mathsf{H}} G_{\mathrm{e}}(\lambda) \geq 0$ with

$$G_{\mathrm{e}}(s) = G_2(s) G_1(s)^{-1}$$

which has the beh. stabilizable realization

$$[E_{\mathrm{e}}, A_{\mathrm{e}}, B_{\mathrm{e}}, C_{\mathrm{e}}, D_{\mathrm{e}}] = \left[\begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A & B \\ C_1 & D_1 \end{bmatrix}, \begin{bmatrix} 0 \\ -I_m \end{bmatrix}, \begin{bmatrix} C_2 & D_2 \end{bmatrix}, 0 \right].$$

Proof

General Case: general C_1, D_1:

Statement a):

- $\Psi(\lambda) = G_1(\lambda)^{\mathsf{H}} G_1(\lambda) - G_2(\lambda)^{\mathsf{H}} G_2(\lambda) \geq 0$,
- G_1 invertible $\Rightarrow I_m - G_{\mathrm{e}}(\lambda)^{\mathsf{H}} G_{\mathrm{e}}(\lambda) \geq 0$ with

$$G_{\mathrm{e}}(s) = G_2(s) G_1(s)^{-1}$$

  which has the beh. stabilizable realization

$$[E_{\mathrm{e}}, A_{\mathrm{e}}, B_{\mathrm{e}}, C_{\mathrm{e}}, D_{\mathrm{e}}] = \left[\begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A & B \\ C_1 & D_1 \end{bmatrix}, \begin{bmatrix} 0 \\ -I_m \end{bmatrix}, \begin{bmatrix} C_2 & D_2 \end{bmatrix}, 0 \right].$$

- With

$$Q_{\mathrm{e}} = -\begin{bmatrix} C_2^{\mathsf{T}} C_2 & C_2^{\mathsf{T}} D_2 \\ D_2^{\mathsf{T}} C_2 & D_2^{\mathsf{T}} D_2 \end{bmatrix}, \quad S_{\mathrm{e}} = 0, \quad R_{\mathrm{e}} = I_m$$

  we obtain the modified Popov function $\Psi_{\mathrm{e}}(\lambda) = I_m - G_{\mathrm{e}}(\lambda)^{\mathsf{H}} G_{\mathrm{e}}(\lambda)$.

- Results for special case give

$$\begin{bmatrix} A_{\mathrm{e}}^{\mathsf{T}} P_{\mathrm{e}} E_{\mathrm{e}} + E_{\mathrm{e}}^{\mathsf{T}} P_{\mathrm{e}} A_{\mathrm{e}} + Q_{\mathrm{e}} & E_{\mathrm{e}}^{\mathsf{T}} P_{\mathrm{e}} B_{\mathrm{e}} + S_{\mathrm{e}} \\ B_{\mathrm{e}}^{\mathsf{T}} P_{\mathrm{e}} E_{\mathrm{e}} + S_{\mathrm{e}}^{\mathsf{T}} & R_{\mathrm{e}} \end{bmatrix} \geq_{\mathcal{V}_{\mathrm{sys,e}}} 0,$$

$$E_{\mathrm{e}}^{\mathsf{T}} P_{\mathrm{e}} E_{\mathrm{e}} \leq_{\mathcal{V}_{\mathrm{diff,e}}} 0.$$

- Let $P_{\mathrm{e}} = \begin{bmatrix} P & P_{\mathrm{e},12} \\ P_{\mathrm{e},21} & P_{\mathrm{e},22} \end{bmatrix}$. Simple calculation shows

$$\begin{bmatrix} A^{\mathsf{T}} PE + E^{\mathsf{T}} PA + Q & E^{\mathsf{T}} PB + S \\ B^{\mathsf{T}} PE + S^{\mathsf{T}} & R \end{bmatrix} \geq_{\mathcal{V}_{\mathrm{sys}}} 0, \quad E^{\mathsf{T}} PE \leq_{\mathcal{V}_{\mathrm{diff}}} 0.$$

- Results for special case give

$$\begin{bmatrix} A_{\mathrm{e}}^{\mathsf{T}} P_{\mathrm{e}} E_{\mathrm{e}} + E_{\mathrm{e}}^{\mathsf{T}} P_{\mathrm{e}} A_{\mathrm{e}} + Q_{\mathrm{e}} & E_{\mathrm{e}}^{\mathsf{T}} P_{\mathrm{e}} B_{\mathrm{e}} + S_{\mathrm{e}} \\ B_{\mathrm{e}}^{\mathsf{T}} P_{\mathrm{e}} E_{\mathrm{e}} + S_{\mathrm{e}}^{\mathsf{T}} & R_{\mathrm{e}} \end{bmatrix} \geq_{\mathcal{V}_{\mathrm{sys,e}}} 0,$$
$$E_{\mathrm{e}}^{\mathsf{T}} P_{\mathrm{e}} E_{\mathrm{e}} \leq_{\mathcal{V}_{\mathrm{diff,e}}} 0.$$

- Let $P_{\mathrm{e}} = \begin{bmatrix} P & P_{\mathrm{e},12} \\ P_{\mathrm{e},21} & P_{\mathrm{e},22} \end{bmatrix}$. Simple calculation shows

$$\begin{bmatrix} A^{\mathsf{T}} PE + E^{\mathsf{T}} PA + Q & E^{\mathsf{T}} PB + S \\ B^{\mathsf{T}} PE + S^{\mathsf{T}} & R \end{bmatrix} \geq_{\mathcal{V}_{\mathrm{sys}}} 0, \quad E^{\mathsf{T}} PE \leq_{\mathcal{V}_{\mathrm{diff}}} 0.$$

Statement b):

- P solves KYP inequality $\Rightarrow P_{\mathrm{e}} = \begin{bmatrix} P & 0 \\ 0 & 0 \end{bmatrix}$ solves "extended" KYP inequality.
- We have

$$\operatorname{rank} \begin{bmatrix} -\lambda E_{\mathrm{e}} + A_{\mathrm{e}} & B_{\mathrm{e}} \\ C_{\mathrm{e}} & 0 \\ 0 & I_m \end{bmatrix} = \operatorname{rank} \begin{bmatrix} -\lambda E + A & B \\ C_1 & D_1 \\ C_2 & D_2 \end{bmatrix} + m = n + 2m \quad \forall \lambda \in \overline{\mathbb{C}^+}.$$

- $[E_{\mathrm{e}}, A_{\mathrm{e}}, B_{\mathrm{e}}, C_{\mathrm{e}}, D_{\mathrm{e}}]$ is beh. detectable $\Rightarrow$ Results on special case show $E^{\mathsf{T}} PE \leq_{\mathcal{V}_{\mathrm{diff}}} 0$.

Positive Real Lemma

Positive realness: For $G(s) = C(sE - A)^{-1}B + D$:

$$G(s) \text{ positive real } :\Leftrightarrow G(\lambda) + G(\lambda)^{\mathsf{H}} \geq 0 \quad \forall \lambda \in \mathbb{C}^{+}.$$

Positive real lemma [ODEs: Anderson, Vongpanitlerd '73, Reis, V. '15]

a) If there exists a solution the KYP inequality

$$\begin{bmatrix} A^{\mathsf{T}}PE + E^{\mathsf{T}}PA & E^{\mathsf{T}}PB + C^{\mathsf{T}} \\ B^{\mathsf{T}}PE + C & D + D^{\mathsf{T}} \end{bmatrix} \geq_{\mathcal{V}_{\mathrm{sys}}} 0, \quad P = P^{\mathsf{T}}$$

with $E^{\mathsf{T}}PE \leq_{\mathcal{V}_{\mathrm{diff}}} 0$, then $G(s)$ is positive real.

b) Assume $G(s)$ is positive real. Let at least one of the two properties
   i) $G(\mathrm{i}\omega_0) + G(\mathrm{i}\omega_0)^{\mathsf{H}} > 0$ for $\mathrm{i}\omega_0 \in \mathrm{i}\mathbb{R} \setminus \sigma(E, A)$, $[E, A, B]$ is beh. stabilizable
   ii) $[E, A, B]$ is behaviorally controllable

   be satisfied. Then there exists a solution with $E^{\mathsf{T}}PE \leq_{\mathcal{V}_{\mathrm{diff}}} 0$.

c) If further $[E, A, B, C, D]$ is behaviorally detectable, then all solutions fulfill $E^{\mathsf{T}}PE \leq_{\mathcal{V}_{\mathrm{diff}}} 0$.

Bounded Real Lemma

Positive realness: For $G(s) = C(sE - A)^{-1}B + D$:

$$G(s) \text{ bounded real } :\Leftrightarrow I_m - G(\lambda)^{\mathsf{H}} G(\lambda) \geq 0 \quad \forall \lambda \in \mathbb{C}^+.$$

Bounded real lemma [ODEs: Anderson, Vongpanitlerd '73, Reis, V. '15]

a) If there exists a solution the KYP inequality

$$\begin{bmatrix} A^{\mathsf{T}}PE + E^{\mathsf{T}}PA - C^{\mathsf{T}}C & E^{\mathsf{T}}PB - C^{\mathsf{T}}D \\ B^{\mathsf{T}}PE - D^{\mathsf{T}}C & I_m - D^{\mathsf{T}}D \end{bmatrix} \geq_{\mathcal{V}_{\mathrm{sys}}} 0, \quad P = P^{\mathsf{T}}$$

with $E^{\mathsf{T}}PE \leq_{\mathcal{V}_{\mathrm{diff}}} 0$, then $G(s)$ is bounded real.

b) Assume $G(s)$ is bounded real. Let at least one of the two properties
   i) $I_m - G(\mathrm{i}\omega_0)^{\mathsf{H}} G(\mathrm{i}\omega_0) > 0$ for $\mathrm{i}\omega_0 \in \mathrm{i}\mathbb{R} \setminus \sigma(E, A)$, $[E, A, B]$ is beh. stabilizable
   ii) $[E, A, B]$ is behaviorally controllable

   be satisfied. Then there exists a solution with $E^{\mathsf{T}}PE \leq_{\mathcal{V}_{\mathrm{diff}}} 0$.

c) If further $[E, A, B, C, D]$ is behaviorally detectable, then all solutions fulfill $E^{\mathsf{T}}PE \leq_{\mathcal{V}_{\mathrm{diff}}} 0$.

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Thanks for the Attention!

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