Analysis and Numerical Solution of Structured Descriptor System Problems

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Motivation

Given:
Continuous-time LTI descriptor system

$$\Sigma: \quad \begin{cases} E\dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t) + Du(t), \end{cases}$$

- $E, A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{m\times n}$, $D \in \mathbb{R}^{m\times m}$,
- descriptor vector $x(t) \in \mathbb{R}^n$, input vector $u(t) \in \mathbb{R}^m$, output vector $y(t) \in \mathbb{R}^m$,
- assumption: $\lambda E - A$ is regular and stable,
- corresponding transfer function:

$$G(s) := C(sE - A)^{-1}B + D.$$

Applications:

- Modeling of electrical circuits or flexible mechanical systems,
- semi-discretization of partial differential equations playing a role in (bio-)chemical process engineering (Stokes, Navier-Stokes, ...),
- and many more.

Structural Properties

Often, the considered systems have additional structures such as

- (strict) bounded realness:
  $I_m - G(\mathrm{i}\omega)G^H(\mathrm{i}\omega) \succcurlyeq 0$ ($\succ 0$) (or the maximum singular value $\sigma_{\max}(G(\mathrm{i}\omega)) \le 1$ (< 1)) for all $\omega \in \mathbb{R}$,
- (strict) positive realness:
  $G(\mathrm{i}\omega) + G^H(\mathrm{i}\omega) \succcurlyeq 0$ ($\succ 0$) for all $\omega \in \mathbb{R}$,
- (strict) negative imaginarity:
  $\mathrm{i}\left(G(\mathrm{i}\omega) - G^H(\mathrm{i}\omega)\right) \succcurlyeq 0$ ($\succ 0$) for all $\omega > 0$,
- (strict) general frequency domain inequality:
  $U(\mathrm{i}\omega)^H \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} U(\mathrm{i}\omega) \succcurlyeq 0$ ($\succ 0$) with
  $U(\mathrm{i}\omega) = \begin{bmatrix} (\mathrm{i}\omega E - A)^{-1}B \\ I_m \end{bmatrix}$ for all $\omega \in \mathbb{R}$.

These additional structures are equivalent to the passivity of the system Σ (with an appropriate energy functional). Passivity means, that the system cannot internally generate energy.

Equivalent Characterizations:
Equivalent conditions often given in terms of

- feasibility of linear matrix inequalities,
- solvability of matrix equations (Riccati, Lur'e),
- spectral conditions of matrix pencils with even or skew-Hamiltonian/Hamiltonian structure:
  Under certain conditions, a transfer function is strictly bounded real / positive real / negative imaginary or fulfills the strict frequency domain inequality if and only if the associated skew-Hamiltonian/Hamiltonian matrix pencil has no (nonzero) finite, purely imaginary eigenvalues.

Computational Tasks:

- Checking if the above structures are satisfied (by checking if the spectral conditions of the related matrix pencils are fulfilled),
- enforcement of the above structures:
  Often, the theoretically fulfilled system structure is violated due to modeling errors $\longrightarrow$ need post-processing procedure to restore the structural properties.

Cooperations & Coworkers

- Prof. Dr. Timo Reis (University of Hamburg, Germany)
- Prof. Dr. Volker Mehrmann (TU Berlin, Germany)
- Dr. Vasile Sima (ICI Bucharest, Romania)

Skew-Hamiltonian/Hamiltonian Matrix Pencils

Definition:
Let $\mathcal{J} := \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$. We call

- a matrix $\mathcal{S} \in \mathbb{R}^{2n\times 2n}$ *skew-Hamiltonian* if $(\mathcal{S}\mathcal{J})^T = -\mathcal{S}\mathcal{J}$,
- a matrix $\mathcal{H} \in \mathbb{R}^{2n\times 2n}$ *Hamiltonian* if $(\mathcal{H}\mathcal{J})^T = \mathcal{H}\mathcal{J}$,
- a matrix pencil $\lambda\mathcal{S} - \mathcal{H} \in \mathbb{R}^{2n\times 2n}$ *skew-Hamiltonian/Hamiltonian* if $\mathcal{S}$ is skew-Hamiltonian and $\mathcal{H}$ is Hamiltonian.

Properties:

- Block structure:

$$\lambda\mathcal{S} - \mathcal{H} = \lambda \begin{bmatrix} F & G \\ H & F^T \end{bmatrix} - \begin{bmatrix} R & S \\ T & -R^T \end{bmatrix}$$

with skew-symmetric G, H, symmetric S, T,

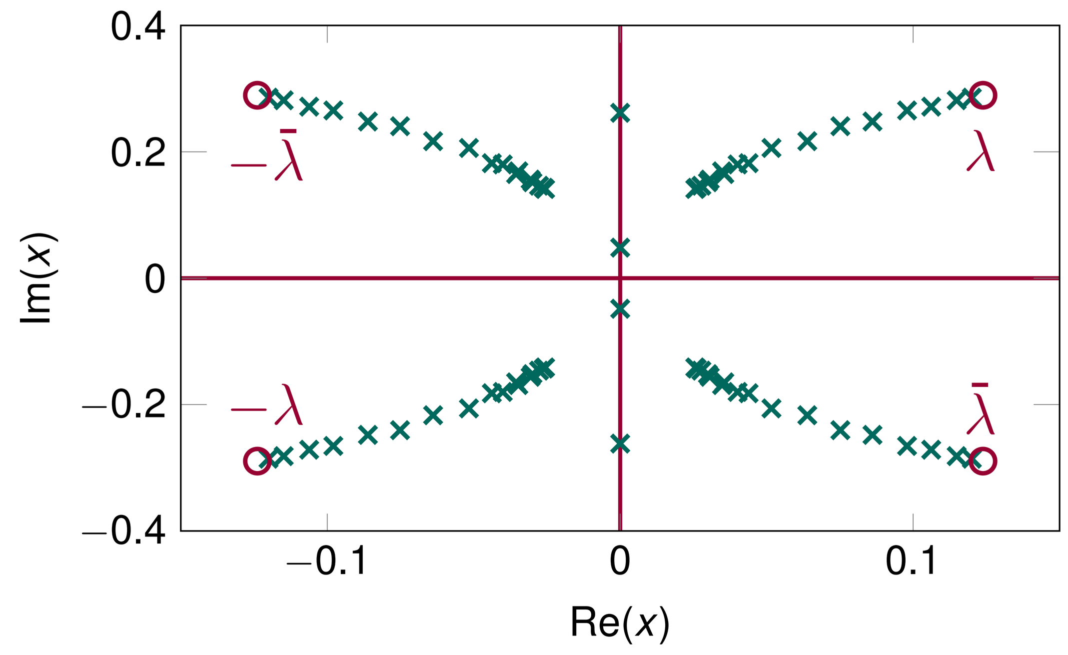

Figure 1: Typical Hamiltonian spectrum

- Hamiltonian eigensymmetry (symmetry with respect to real and imaginary axis),
- $\mathcal{J}$-congruence transformations

$$\lambda\tilde{\mathcal{S}} - \tilde{\mathcal{H}} := \mathcal{J}\mathcal{P}^T\mathcal{J}^T(\lambda\mathcal{S} - \mathcal{H})\mathcal{P}$$

preserve the skew-Hamiltonian/Hamiltonian structure,

- structured Schur form:

$$\mathcal{J}\mathcal{Q}^T\mathcal{J}^T(\lambda\mathcal{S} - \mathcal{H})\mathcal{Q} = \lambda\begin{bmatrix} S_1 & S_2 \\ 0 & S_1^T \end{bmatrix} - \begin{bmatrix} H_1 & H_2 \\ 0 & -H_1^T \end{bmatrix},$$

with orthogonal $\mathcal{Q}$, upper triangular S_1, and upper quasi-triangular H_1; however *existence cannot be guaranteed*, doubling of the pencil can solve this issue.

Structure-Preserving Algorithms

The algorithms rely on the embedding

$$\lambda\mathcal{B}_{\mathcal{S}} - \mathcal{B}_{\mathcal{H}} := \lambda\begin{bmatrix} \mathcal{S} & 0 \\ 0 & \mathcal{S} \end{bmatrix} - \begin{bmatrix} \mathcal{H} & 0 \\ 0 & -\mathcal{H} \end{bmatrix},$$

and restoring the structure by the orthogonal transformation

$$\lambda\tilde{\mathcal{B}}_{\mathcal{N}} - \tilde{\mathcal{B}}_{\mathcal{H}} := \mathcal{X}^T(\lambda\mathcal{B}_{\mathcal{N}} - \mathcal{B}_{\mathcal{H}})\mathcal{X}.$$

By using the generalized symplectic URV decomposition we get

$$\mathcal{J}\mathcal{Q}^T\mathcal{J}^T(\lambda\tilde{\mathcal{B}}_{\mathcal{N}} - \tilde{\mathcal{B}}_{\mathcal{H}})\mathcal{Q} = \lambda\left[\begin{array}{cc|cc} S_1 & 0 & S_2 & 0 \\ 0 & T_1 & 0 & T_2 \\ \hline 0 & 0 & S_1^T & 0 \\ 0 & 0 & 0 & T_1^T \end{array}\right] - \left[\begin{array}{cc|cc} 0 & H_{11} & 0 & H_{12} \\ -H_{22}^T & 0 & H_{12}^T & 0 \\ \hline 0 & 0 & 0 & H_{22} \\ 0 & 0 & -H_{11}^T & 0 \end{array}\right]$$

with orthogonal $\mathcal{Q}$, upper triangular S_1, T_1, H_{11}, and upper quasi-triangular H_{22}^T.

Purely Imaginary Eigenvalues:
are associated to the 1×1 diagonal blocks of the formal product $-S_1^{-1}H_{11}T_1^{-1}H_{22}^T$.

Corresponding Eigenvectors:
can be extracted from S_1, T_1, H_{11}, H_{22}^T and the orthogonal matrix $\mathcal{Q}$.

Enforcement of Bounded Realness

Given:
Descriptor system $(\lambda E - A, B, C, D)$ with non-bounded real transfer function $G(s)$.

Desired:
Descriptor system $(\lambda\tilde{E} - \tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ with bounded real transfer function $\tilde{G}(s)$ and the error $\|\tilde{G} - G\|$ is "small".

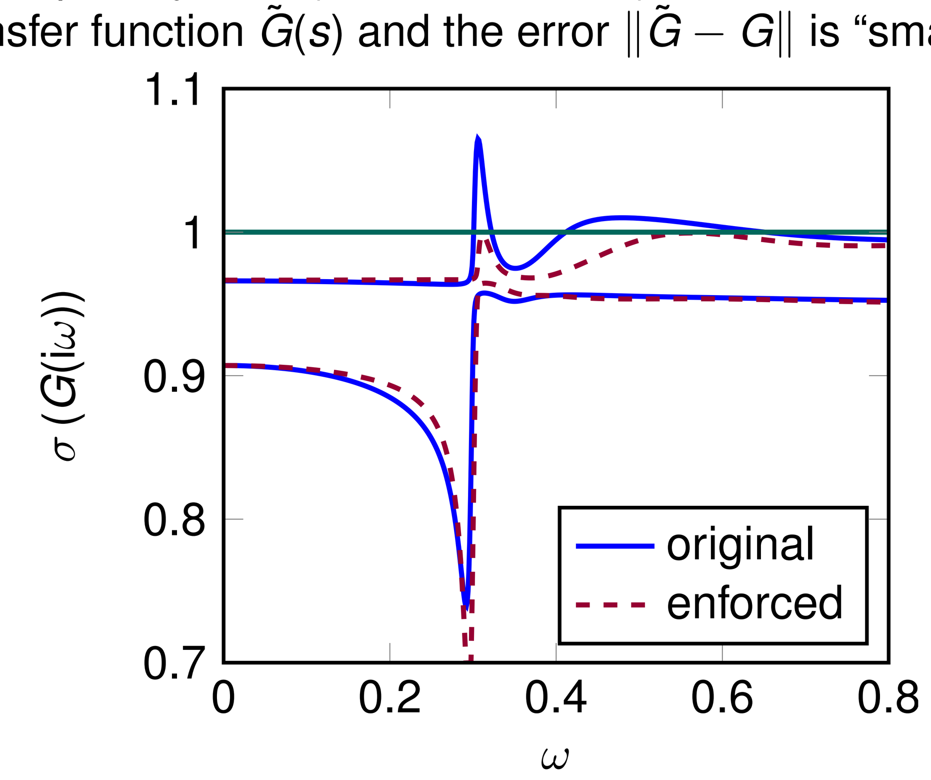

Figure 2: Non-bounded real (original) and bounded-real (enforced) transfer functions

- Violation of bounded realness in $[0.3009, 0.3223] \cup [0.4117, 0.6512]$,
- associated skew-Hamiltonian/Hamiltonian matrix pencil $\lambda\mathcal{S} - \mathcal{H}$ has purely imaginary eigenvalues $0.3009\mathrm{i}, 0.3223\mathrm{i}, 0.4117\mathrm{i}, 0.6512\mathrm{i}$,
- **main idea:** perturb these eigenvalues to decrease the level of bounded realness violations $\Longrightarrow$ results in certain *structured* perturbations of $\lambda\mathcal{S} - \mathcal{H}$, i.e., $\lambda\mathcal{S} - \left(\mathcal{H} + \hat{\mathcal{H}}\right)$,
- first order perturbation theory yields

$$\tilde{\omega} - \omega = \frac{v^H\mathcal{J}\hat{\mathcal{H}}v}{\mathrm{i}v^H\mathcal{J}\mathcal{S}v},$$

where $\mathrm{i}\omega$ is eigenvalue of $\lambda\mathcal{S} - \mathcal{H}$ with eigenvector v, and $\mathrm{i}\tilde{\omega}$ is the perturbed eigenvalue,

- it is possible to find the perturbation which results in the least system error measured in the $\mathcal{H}_2$-norm.

Future Topics

- Detailed perturbation theory for purely imaginary eigenvalues of skew-Hamiltonian/Hamiltonian pencils, in particular multiple ones,
- structure-preserving balancing as pre-processing for the skew-Hamiltonian/Hamiltonian eigenvalue problem to improve numerical accuracy,
- adapt theory for continuous-time to discrete-time systems (deal with palindromic matrix pencils),
- develop algorithms for large systems.

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