The Linear-Quadratic Optimal Control Problem Revisited

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Problem Description

Minimize

$$\mathcal{J}(x,u) = \int_0^\infty \begin{pmatrix} x(\tau) \\ u(\tau) \end{pmatrix}^\mathsf{T} \begin{bmatrix} Q & S \\ S^\mathsf{T} & R \end{bmatrix} \begin{pmatrix} x(\tau) \\ u(\tau) \end{pmatrix} \mathrm{d}t$$

subject to the differential-algebraic control system

$$E\dot{x}(t) = Ax(t) + Bu(t), \quad Ex(0) = Ex_0, \quad \lim_{t\to\infty} Ex(t) = 0.$$

Here:

- $sE - A \in \mathbb{R}[s]^{n\times n}$ regular, $B \in \mathbb{R}^{n\times m}$,
- state $x \in \mathcal{L}^2_{\mathrm{loc}}(\mathbb{R},\mathbb{R}^n)$ with $E\dot{x} \in \mathcal{L}^2_{\mathrm{loc}}(\mathbb{R},\mathbb{R}^n)$, control input $u \in \mathcal{L}^2_{\mathrm{loc}}(\mathbb{R},\mathbb{R}^m)$,
- $Q = Q^\mathsf{T} \in \mathbb{R}^{n\times n}$, $S \in \mathbb{R}^{n\times m}$, $R = R^\mathsf{T} \in \mathbb{R}^{m\times m}$.

Lur'e Equations

Main tool: the Lur'e equation

$$\begin{bmatrix} E^\mathsf{T}XA + A^\mathsf{T}XE + Q & E^\mathsf{T}XB + S \\ B^\mathsf{T}XE + S^\mathsf{T} & R \end{bmatrix} =_{\mathcal{V}_{\mathrm{sys}}} \begin{bmatrix} K^\mathsf{T} \\ L^\mathsf{T} \end{bmatrix} \begin{bmatrix} K & L \end{bmatrix}, \quad X = X^\mathsf{T},$$

where a solution $(X,K,L) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{q\times n} \times \mathbb{R}^{q\times m}$ fulfills

$$\operatorname{rank}_{\mathbb{R}(s)} \begin{bmatrix} -sE + A & B \\ K & L \end{bmatrix} = n + q.$$

Notation:

$$M =_\mathcal{V} N \Leftrightarrow v^\mathsf{T}Mv = v^\mathsf{T}Nv \quad \forall v \in \mathcal{V}.$$

System Space:

$\mathcal{V}_{\mathrm{sys}}$ = the smallest subspace in $\mathbb{R}^{n+m}$ in which the solution trajectories (x,u) evolve.

Stabilizing solution: (X,K,L) even fulfills

$$\operatorname{rank} \begin{bmatrix} -\lambda E + A & B \\ K & L \end{bmatrix} = n + q \quad \forall \lambda \in \mathbb{C}^+ := \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) > 0\}.$$

Feasibility Condition

Define the space of consistent initial differential variables

$$\mathcal{V}_{\mathrm{diff}} = \{x_0 \in \mathbb{R}^n : \exists \text{ a solution } (x,u) \text{ of the the DAE with } Ex(0) = Ex_0\}$$

and the optimal value function $V : E\mathcal{V}_{\mathrm{diff}} \to \mathbb{R} \cup \{-\infty\}$ with

$$V^+(Ex_0) = \inf\{\mathcal{J}(x,u) : (x,u) \text{ with } Ex(0) = Ex_0 \text{ solves the DAE}\}.$$

Feasibility Theorem:
The following statements are equivalent:

a) $V^+(Ex_0) \in \mathbb{R} \quad \forall x_0 \in \mathcal{V}_{\mathrm{diff}}$.

b) The system (E,A,B) has no uncontrollable modes on the imaginary axis and the Lur'e equation has a stabilizing solution (X^+,K^+,L^+).

If the above are satisfied then it holds

$$V^+(Ex_0) = x_0^\mathsf{T}E^\mathsf{T}X^+Ex_0 \quad \forall x_0 \in \mathcal{V}_{\mathrm{diff}}.$$

Optimal Controls

Feasibility condition suggests the existence of a sequence of solution trajectories $\{(x_k,u_k)\}_{k=1}^\infty$ with $Ex_k(0) = Ex_0$ and $\lim_{t\to\infty} Ex_k(t) = 0$ such that

$$V^+(Ex_0) = \lim_{k\to\infty} \mathcal{J}(x_k,u_k).$$

Questions:

a) Does there exist an optimal control, i.e., a solution (x,u) with $Ex(0) = Ex_0$, $\lim_{t\to\infty} Ex(t) = 0$, and $V^+(Ex_0) = \mathcal{J}(x,u)$?

b) Is it unique?

Answers: With the stabilizing solution (X^+,K^+,L^+) of the Lur'e equation and let

$$\mathcal{R}(s) := \begin{bmatrix} -\lambda\Pi E + A & B \\ K^+ & L^+ \end{bmatrix},$$

where Π is a certain projector with $\operatorname{im}\Pi = E\mathcal{V}_{\mathrm{diff}}$. Then it holds:

a) There exists an optimal control (x,u) if and only if

$$\operatorname{rank}\mathcal{R}(\lambda) = n + q \quad \forall \lambda \in \overline{\mathbb{C}^+} \text{ and the index of } \mathcal{R}(s) \text{ is at most one.}$$

b) It is even unique if and only if

$$\operatorname{rank}\mathcal{R}(\lambda) = n + m \quad \forall \lambda \in \overline{\mathbb{C}^+} \text{ and the index of } \mathcal{R}(s) \text{ is at most one.}$$

Even Matrix Pencils

Consider the even matrix pencil

$$s\mathcal{E} - \mathcal{A} := \begin{bmatrix} 0 & -s\Pi E + A & B \\ sE^\mathsf{T}\Pi^\mathsf{T} + A^\mathsf{T} & Q & S \\ B^\mathsf{T} & S^\mathsf{T} & R \end{bmatrix} \in \mathbb{R}[s]^{2n+m\times 2n+m},$$

where Π is a certain projector with $\operatorname{im}\Pi = E\mathcal{V}_{\mathrm{diff}}$.

Construction of solution of the Lur'e equation: Use deflating subspaces, i.e., determine accordingly partitioned matrices

$$Y = \begin{bmatrix} Y_\mu \\ Y_x \\ Y_u \end{bmatrix} \in \mathbb{R}^{(2n+m)\times(n+m)}, \quad Z = \begin{bmatrix} Z_\mu \\ Z_x \\ Z_u \end{bmatrix} \in \mathbb{R}^{(2n+m)\times(n+q)},$$

such that

a) the space $\operatorname{im} Y$ is $n+m$-dimensional and $y_1^\mathsf{T}\mathcal{E}y_2 = 0$ for all $y_1, y_2 \in \operatorname{im} Y$;

b) $\mathcal{V}_{\mathrm{sys}} \subseteq \operatorname{im}\begin{bmatrix} Y_x \\ Y_u \end{bmatrix}$;

c) $\operatorname{rank}\Pi EY_x = \operatorname{rank}\Pi E$;

d) there exist $\widetilde{\mathcal{E}}, \widetilde{\mathcal{A}} \in \mathbb{R}^{n+q\times n+m}$ with $\operatorname{rank}_{\mathbb{R}(s)}(s\widetilde{\mathcal{E}} - \widetilde{\mathcal{A}}) = n + q$, such that

$$(s\mathcal{E} - \mathcal{A})Y = Z(s\widetilde{\mathcal{E}} - \widetilde{\mathcal{A}}).$$

Remarks:

- With $\begin{bmatrix} Y_x^- & Y_u^- \end{bmatrix} := \begin{bmatrix} Y_x \\ Y_u \end{bmatrix}^{-1}$ it holds

$$E^\mathsf{T}\Pi^\mathsf{T}X\Pi E = E^\mathsf{T}\Pi^\mathsf{T}Y_\mu Y_x^- \quad \Longrightarrow \quad \text{Reconstruction of } X.$$

- The matrices Y and Z can be chosen such that

$$s\widetilde{\mathcal{E}} - \widetilde{\mathcal{A}} = \begin{bmatrix} -s\Pi E + A & B \\ K & L \end{bmatrix} \quad \Longrightarrow \quad \text{Reconstruction of } K, L.$$

- Obtain stabilizing solutions by choosing semi-stable deflating subspaces of $s\mathcal{E} - \mathcal{A}$.

Previous Work

a) **Mehrmann '91:**

- considers the generalized algebraic Riccati equation:

$$E^\mathsf{T}XA + A^\mathsf{T}XE + Q - (E^\mathsf{T}XB + S)R^{-1}(B^\mathsf{T}XE + S^\mathsf{T}) = 0, \quad X = X^\mathsf{T},$$

- assumptions: $\begin{bmatrix} Q & S \\ S^\mathsf{T} & R \end{bmatrix} \geq 0$, $R > 0$, (E,A,B) impulse controllable,
- solution of optimal control problems using the even boundary value problem

$$\begin{bmatrix} 0 & E & 0 \\ -E^\mathsf{T} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} \dot{\mu}(t) \\ \dot{x}(t) \\ \dot{u}(t) \end{pmatrix} = \begin{bmatrix} 0 & A & B \\ A^\mathsf{T} & Q & S \\ B^\mathsf{T} & S^\mathsf{T} & R \end{bmatrix} \begin{pmatrix} \mu(t) \\ x(t) \\ u(t) \end{pmatrix}, \quad Ex(0) = Ex_0, \quad \lim_{t\to\infty} E^\mathsf{T}\lambda(t) = 0.$$

b) **Kurina '93:**

- considers the generalized algebraic Riccati equation:

$$XA + A^\mathsf{T}X + Q - (XB + S)R^{-1}(B^\mathsf{T}X + S^\mathsf{T}) = 0, \quad E^\mathsf{T}X = X^\mathsf{T}E,$$

- assumptions: $\begin{bmatrix} Q & S \\ S^\mathsf{T} & R \end{bmatrix} \geq 0$, $R > 0$, (E,A,B) impulse controllable,
- solution analysis: Katayama/Minamino '92 and Katayama/Kawamoto/Takaba '99, in particular solvability of an "algebraic quadratic matrix equation" necessary for existence of stabilizing solutions.

c) **Geerts '89–'94:**

- consider the linear matrix inequality

$$\begin{bmatrix} E^\mathsf{T}PA + A^\mathsf{T}PE + Q & E^\mathsf{T}PB + S \\ B^\mathsf{T}PE + S^\mathsf{T} & R \end{bmatrix} \geq 0, \quad P = P^\mathsf{T},$$

- assumptions: $\begin{bmatrix} Q & S \\ S^\mathsf{T} & R \end{bmatrix} = \begin{bmatrix} C^\mathsf{T} \\ D^\mathsf{T} \end{bmatrix} \begin{bmatrix} C & D \end{bmatrix} \geq 0$, however $sE - A \in \mathbb{R}[s]^{k\times n}$ may be singular,
- (E,A,B) is impulse controllable $\Rightarrow \exists$ maximal solution of the LMI which determines the solution the optimal control problem.

d) **Brüll '11:**

- considers the linear matrix inequality

$$\begin{bmatrix} P_1^\mathsf{T}A + A^\mathsf{T}P_1 + Q & P_1^\mathsf{T}B + A^\mathsf{T}P_2 + S \\ P_2^\mathsf{T}A + B^\mathsf{T}P_1 + S^\mathsf{T} & P_2^\mathsf{T}B + B^\mathsf{T}P_2 + R \end{bmatrix} \geq 0, \quad E^\mathsf{T}P_1 = P_1^\mathsf{T}E, \quad E^\mathsf{T}P_2 = 0,$$

- (E,A,B) is completely controllable $\Rightarrow \exists$ solution (P_1,P_2) which determines the optimal value,
- generalization to higher-order and behavioral systems possible,
- also considers the relation to the even boundary value problems.

References

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2. Voigt '15: On Linear-Quadratic Optimal Control and Robustness of Differential-Algebraic Systems. Dissertation, Fakultät für Mathematik, Otto-von-Guericke-Universität Magdeburg, 2015. Submitted.

Conclusions

We have solved the linear-quadratic optimal control problem for DAEs under very general assumptions. In particular, we do not need

- $\begin{bmatrix} Q & S \\ S^\mathsf{T} & R \end{bmatrix} \geq 0$,
- invertibility of R,
- impulse controllability.