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Linear-Quadratic Optimal Control of Differential-Algebraic Equations

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1. Short Intro to DAEs
2. Problem Formulation
3. Descriptor Lur'e Equations and Even Matrix Pencils
4. Feasibility and Solution
5. Conclusions and Outlook

Differential-Algebraic Equations

Model Equations

$$\Sigma : \quad E\dot{x}(t) = Ax(t) + Bu(t), \quad Ex(0) = Ex_0$$

with

- $E,\ A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, E is singular!
- descriptor vector $x(t) \in \mathbb{R}^n$,
- input vector $u(t) \in \mathbb{R}^m$.

Example

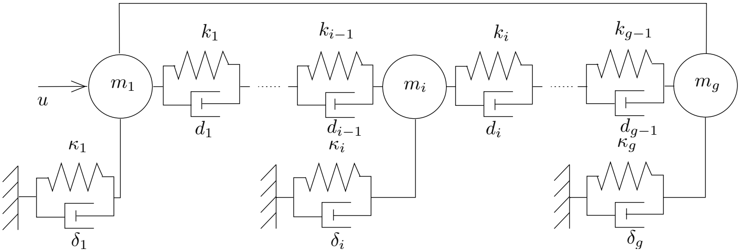

Figure: Constrained mass-spring-damper system

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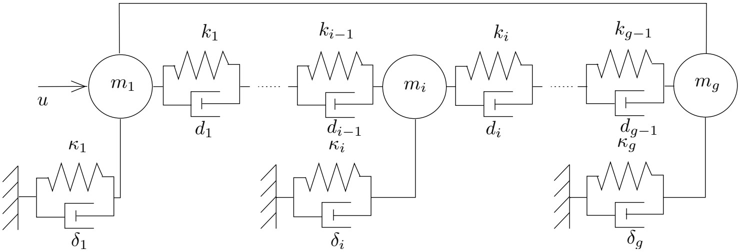

Figure: Constrained mass-spring-damper system

$$
\begin{aligned}
M\ddot{p}(t) &= D\dot{p}(t) + Kp(t) - G^T\lambda(t) + B_2u(t), \\
0 &= Gp(t).
\end{aligned}
$$

Example

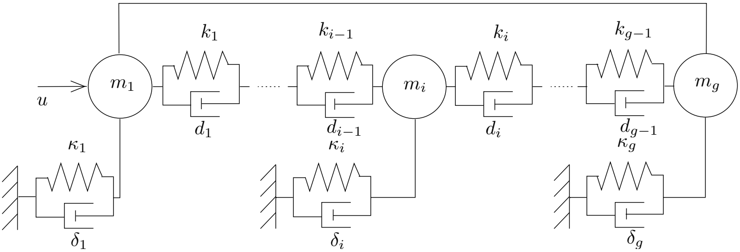

Figure: Constrained mass-spring-damper system

$$\begin{bmatrix} I_g & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} \dot{p}(t) \\ \dot{v}(t) \\ \dot{\lambda}(t) \end{pmatrix} = \begin{bmatrix} 0 & I_g & 0 \\ K & D & -G^T \\ G & 0 & 0 \end{bmatrix} \begin{pmatrix} p(t) \\ v(t) \\ \lambda(t) \end{pmatrix} + \begin{bmatrix} 0 \\ B_2 \\ 0 \end{bmatrix} u(t).$$

Matrix Pencils

Definition

A linear matrix polynomial of the form $sE - A$ is called a matrix pencil.

Terminology:

- eigenvalues: roots of the characteristic polynomial $\det(sE - A)$,
- regular pencil: if $\det(sE - A) \not\equiv 0$, otherwise singular pencil.

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Examples

- $s \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, $\det(sE - A) = (s-1)(3s-2)$, $\Lambda(E, A) = \{1, \frac{2}{3}\}$;
- $s \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, $\det(sE - A) = -2(s-1)$, $\Lambda(E, A) = \{1, \infty\}$;
- $s \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $\det(sE - A) \equiv 0$ (singular pencil).

Weierstraß Canonical Form

Reminder: Jordan canonical form

For every matrix $A \in \mathbb{C}^{n \times n}$ there exists a nonsingular $X \in \mathbb{C}^{n \times n}$ such that

$$X^{-1}AX = \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_r \end{bmatrix}, \quad J_k = \begin{bmatrix} \lambda_k & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_k \end{bmatrix}, \quad k = 1, \ldots, r.$$

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Weierstraß canonical form

For every regular pencil $sE - A$ there exist nonsingular $W, T \in \mathbb{C}^{n \times n}$ such that

$$W(sE - A)T = s \begin{bmatrix} I_{n_f} & 0 \\ 0 & N \end{bmatrix} - \begin{bmatrix} J & 0 \\ 0 & I_{n_\infty} \end{bmatrix},$$

where J and N are in Jordan canonical form, $sI_{n_f} - J$ has only finite eigenvalues and $sN - I_{n_\infty}$ has only infinite eigenvalues.

Decoupling of the System

Change of variables

$$E \quad \dot{x}(t) = \quad A \quad x(t) + \quad Bu(t)$$

Decoupling of the System

Change of variables

$$WETT^{-1}\dot{x}(t) = WATT^{-1}x(t) + WBu(t)$$

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$$\begin{bmatrix} I_{n_f} & 0 \\ 0 & N \end{bmatrix} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} J & 0 \\ 0 & I_{n_\infty} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(t)$$

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Decomposition into slow/fast subsystems

- slow: $\dot{x}_1(t) = Jx_1(t) + B_1 u(t)$,

  solution: $x_1(t) = e^{Jt}x_1(0) + \int_0^t e^{J(t-\tau)} B_1 u(\tau)\mathrm{d}\tau$,
- fast: $N\dot{x}_2(t) = x_2(t) + B_2 u(t)$,

  solution: $x_2(t) = -\sum_{i=0}^{\nu-1} N^i B_2 u^{(i)}(t)$.

The Fast Subsystem

$$x_2(t) = -\sum_{i=0}^{\nu-1} N^i B_2 u^{(i)}(t)$$

Features

- need the first $\nu - 1$ derivatives of the input signal $\Longrightarrow$ inputs must be sufficiently smooth,
- $\nu =$ index of nilpotency of N ($=$ smallest k such that $N^k = 0$), ν is called algebraic index of the system,
- There are restrictions to the initial conditions (consistency):

$$x_2(0) = -\sum_{i=0}^{\nu-1} N^i B_2 u^{(i)}(0).$$

Behavior and Controllability

Definition

The behavior $\mathfrak{B}$ of the system Σ is the set of all its solution trajectories $(x(\cdot), u(\cdot))$.

Definitions

The system Σ is called

- finite dynamics controllable if for all consistent initial conditions and all $x_1 \in \mathbb{R}^n$ there exist $T > 0$ and a trajectory $(x, u) \in \mathfrak{B}$ such that $Ex(T) = Ex_1$;
- impulse controllable if for all $x_0 \in \mathbb{R}^n$ there exists a trajectory $(x, u) \in \mathfrak{B}$ with $Ex(0) = Ex_0$.

Equivalent Conditions

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Equivalent conditions

The system Σ is

- finite dynamics controllable $\iff$ rank $\begin{bmatrix} \lambda E - A & B \end{bmatrix} = n$ for all $\lambda \in \mathbb{C}$;
- impulse controllable $\iff$ rank $\begin{bmatrix} E & AS_\infty & B \end{bmatrix} = n$ with $\operatorname{im} S_\infty = \ker E$.

Problem Formulation

Minimize

$$\mathcal{J}(x_0, u) = \int_0^\infty \begin{pmatrix} x(t) \\ u(t) \end{pmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{pmatrix} x(t) \\ u(t) \end{pmatrix} \mathrm{d}t$$

subject to

$$E\dot{x}(t) = Ax(t) + Bu(t), \quad Ex(0) = Ex_0, \quad \lim_{t\to\infty} Ex(t) = 0$$

where

- E, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$,
- $Q = Q^T \in \mathbb{R}^{n\times n}$, $S \in \mathbb{R}^{n\times m}$, $R = R^T \in \mathbb{R}^{m\times m}$.

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- $Q = Q^T \in \mathbb{R}^{n \times n},\ S \in \mathbb{R}^{n \times m},\ R = R^T \in \mathbb{R}^{m \times m}$.

Definitions

Minimizer: A function $\hat{u} \in \mathcal{L}^2_{\mathrm{loc}}(\mathbb{R}^+, \mathbb{R}^m)$ such that

$$\mathcal{J}(x_0, \hat{u}) = \inf \left\{ \mathcal{J}(x_0, u) : u \in \mathcal{L}^2_{\mathrm{loc}}(\mathbb{R}^+, \mathbb{R}^m) \right\}.$$

Optimal value:

$$V(Ex_0) = \mathcal{J}(x_0, \hat{u}).$$

Regular ODE Theory

Questions

1. Feasibility: Does a minimizer exist for all initial values x_0?
2. Construction: If yes, how can we construct such a minimizer?

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Answers

If $R > 0$ and some further assumptions hold, then:

1. The optimal control problem is feasible if and only if the algebraic Riccati equation

$$A^T X + XA + Q - (XB + S)R^{-1}(B^T X + S^T) = 0, \quad X = X^T$$

has a solution X.

Regular ODE Theory

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1. Feasibility: Does a minimizer exist for all initial values x_0?
2. Construction: If yes, how can we construct such a minimizer?

Answers

If $R > 0$ and some further assumptions hold, then:

2. The optimal solution is constructed by the unique maximal solution X^+ of the ARE, i.e.,

$$\hat{u}(t) = -R^{-1}(B^T X^+ + S^T)x(t) \quad \text{and} \quad V(x_0) = x_0^T X^+ x_0.$$

Properties of the solution:

a) **Maximality:** $X \leq X^+$ for all other solutions X.
b) **Stabilization:** Closed-loop matrix $A - BR^{-1}(B^T X^+ + S^T)$ has only eigenvalues in the closed left half-plane.

Regular ODE Theory

Construction of the solution [Lancaster, Rodman '95]

Let

$$\begin{bmatrix} A - BR^{-1}S & -BR^{-1}B^T \\ S^T R^{-1} S - Q & -\left(A - BR^{-1}S\right)^T \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} T,$$

with $Y := \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} \in \mathbb{R}^{2n \times n}$ and $T \in \mathbb{R}^{n \times n}$. If Y spans a semi-stable Lagrangian invariant subspace, then

1. $X^+ = Y_2 Y_1^{-1}$;
2. $T = A - BR^{-1}(B^T X^+ + S^T)$ with $\Lambda(T) \subset \mathbb{C}^- \cup \mathrm{i}\mathbb{R}$.

Generalizations

Generalizations of LQR theory

- for ODEs ($E = I_n$) and $R > 0$: classical theory with algebraic Riccati equations

$$A^T X + XA + Q - \big(XB + S\big) R^{-1} \big(B^T X + S^T\big) = 0, \quad X = X^T$$

[Lancaster, Rodman '95]

Generalizations

Generalizations of LQR theory

- for ODEs ($E = I_n$) and $R > 0$: classical theory with algebraic Riccati equations [Lancaster, Rodman '95]
- generalization to DAEs: generalized algebraic Riccati equations

$$A^T XE + E^T XA + Q - (E^T XB + S) R^{-1} (B^T XE + S^T) = 0, \quad X = X^T,$$

[Mehrmann '91]

$$A^T X + X^T A + Q - (X^T B + S) R^{-1} (B^T X + S^T) = 0, \quad E^T X = X^T E,$$

however need artificial side conditions to be solvable [Kawamoto, Takaba, Katayama '99]

Generalizations

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- for ODEs ($E = I_n$) and $R > 0$: classical theory with algebraic Riccati equations [Lancaster, Rodman '95]
- generalization to DAEs: generalized algebraic Riccati equations [Mehrmann '91], [Kawamoto, Takaba, Katayama '99]
- generalization to singular control, i.e., $R \geq 0$: Lur'e equations

$$\begin{aligned} A^T X + XA + Q &= K^T K, \\ XB + S &= K^T L, \quad X = X^T, \\ R &= L^T L \end{aligned}$$

[Reis '11]

Generalizations

Generalizations of LQR theory

- for ODEs ($E = I_n$) and $R > 0$: classical theory with algebraic Riccati equations [Lancaster, Rodman '95]
- generalization to DAEs: generalized algebraic Riccati equations [Mehrmann '91], [Kawamoto, Takaba, Katayama '99]
- generalization to singular control, i.e., $R \geq 0$: Lur'e equations [Reis '11]
- Here: singular control for DAEs: solution theory of descriptor Lur'e equations [Reis, V. '14]

Descriptor Lur'e Equations

$$\begin{aligned} A^T X + X^T A + Q &= K^T K + V_\infty^T \Sigma V_\infty, \\ X^T B + S &= K^T L + V_\infty^T \Sigma W_\infty, \quad E^T X = X^T E \\ R &= L^T L + W_\infty^T \Sigma W_\infty, \end{aligned}$$

has a solution

$$(X, K, L, V_\infty, W_\infty, \Sigma) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{p \times n} \times \mathbb{R}^{p \times m} \times \mathbb{R}^{n-r \times n} \times \mathbb{R}^{n-r \times m} \times \mathbb{R}^{n-r \times n-r}$$

where

- $r = \operatorname{rank} E$,
- p is minimal,
- Σ is a signature matrix,
- $\ker \begin{bmatrix} V_\infty & W_\infty \end{bmatrix} = \mathcal{V} := \left\{ \begin{pmatrix} x \\ u \end{pmatrix} \in \mathbb{R}^{n+m} : Ax + Bu \in \operatorname{im} E \right\}$.

Even Matrix Pencils

Descriptor Lur'e equations have a close relationship to matrix pencils

$$s\mathcal{E} - \mathcal{A} = \begin{bmatrix} 0 & -sE + A & B \\ sE^T + A^T & Q & S \\ B^T & S^T & R \end{bmatrix}.$$

The pencil $s\mathcal{E} - \mathcal{A}$ is called even, since $\mathcal{E} = -\mathcal{E}^T$ and $\mathcal{A} = \mathcal{A}^T$.

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Deflating subspaces

- A subspace $\mathcal{Y} := \operatorname{im} Y$ is called deflating subspace of $s\mathcal{E} - \mathcal{A}$ if there exist Z, $\tilde{\mathcal{E}}$, and $\tilde{\mathcal{A}}$ of appropriate dimensions such that

$$(s\mathcal{E} - \mathcal{A})Y = Z(s\tilde{\mathcal{E}} - \tilde{\mathcal{A}}).$$

- A subspace $\mathcal{Y}$ is called $\mathcal{E}$-neutral if $x^T\mathcal{E}y = 0$ for all $x, y \in \mathcal{Y}$.

Even Kronecker Canonical Form

Definition: even Kronecker canonical form [Thompson '76]

Let $s\mathcal{E} - \mathcal{A}$ be an even pencil. Then there exists a nonsingular $U \in \mathbb{C}^{n \times n}$ such that $U^H(s\mathcal{E} - \mathcal{A})U = \operatorname{diag}(\mathcal{D}_1(s), \ldots, \mathcal{D}_k(s))$ where each $\mathcal{D}_j(s)$, $j = 1, \ldots, k$ is of one of the following structures:

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Type 1 (finite non-imaginary eigenvalues):

$$\left[\begin{array}{cccc|cccc} & & & & -s+\mu_j & -1 & & \\ & & & & & \ddots & \ddots & \\ & & & & & & \ddots & -1 \\ & & & & & & & -s+\mu_j \\ \hline s+\bar{\mu}_j & & & & & & & \\ -1 & \ddots & & & & & & \\ & \ddots & \ddots & & & & & \\ & & -1 & s+\bar{\mu}_j & & & & \end{array}\right] \in \mathbb{C}[s]^{2\ell_j \times 2\ell_j},$$

with finite non-imaginary eigenvalues $\mu_j \in \mathbb{C}^+$, $-\bar{\mu}_j \in \mathbb{C}^-$.

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Type 2 (finite imaginary eigenvalues):

$$\varepsilon_j \begin{bmatrix} & & 1 & -s\mathrm{i} - \mu_j \\ & \iddots & \iddots & \\ 1 & \iddots & & \\ -s\mathrm{i} - \mu_j & & & \end{bmatrix} \in \mathbb{C}[s]^{\ell_j \times \ell_j},$$

with finite imaginary eigenvalues $\mathrm{i}\mu_j \in \mathrm{i}\mathbb{R}$ and block signature $\varepsilon_j \in \{-1, 1\}$.

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Type 3 (infinite eigenvalues):

$$\varepsilon_j \begin{bmatrix} & & si & 1 \\ & \iddots & \iddots & \\ si & \iddots & & \\ 1 & & & \end{bmatrix} \in \mathbb{C}[s]^{\ell_j \times \ell_j},$$

with block signature $\varepsilon_j \in \{-1, 1\}$.

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Type 4 (singular structure):

$$\left[\begin{array}{cccc|cccc} & & & & 1 & -s & & \\ & & & & & \ddots & \ddots & \\ & & & & & & 1 & -s \\ \hline 1 & & & & & & & \\ s & \ddots & & & & & & \\ & \ddots & 1 & & & & & \\ & & s & & & & & \end{array}\right] \in \mathbb{C}[s]^{(2\ell_j+1)\times(2\ell_j+1)}.$$

Feasibility Conditions

Theorem

Under some stabilizability conditions, the following statements are equivalent:

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3. There exists a p such that in the EKCF of $s\mathcal{E} - \mathcal{A}$, the blocks have the following structure:
   (a) All blocks of Type 2 (imaginary evs) have even size and negative signature.

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   (c) There exist p blocks of Type 3 with positive signature.

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   (a) All blocks of Type 2 (imaginary evs) have even size and negative signature.
   (b) There exist exactly $2(n-r)+p$ blocks of Type 3 (infinite evs).
   (c) There exist p blocks of Type 3 with positive signature.
   (d) The remaining $2(n-r)$ blocks of Type 3 are either of even size; or the number of odd-sized blocks with positive and negative signature is equal.
   (e) There exist exactly $m-p$ blocks of Type 4 (singular structure).

Construction of Solutions

Theorem

Under some weak condition, the following statements are equivalent:

1. There exists a solution $(X, K, L, V_\infty, W_\infty, \Sigma)$ to the descriptor Lur'e equation.
2. There exist $Y = \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix} \in \mathbb{R}^{2n+m \times n+m}$, $\quad Z = \begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \end{bmatrix} \in \mathbb{R}^{2n+m \times n+p}$

   such that

   - the space $\mathcal{Y} = \operatorname{im} Y$ is $\mathcal{E}$-neutral and of dimension $n+m$;
   - $\mathcal{V} \subset \operatorname{im} \begin{bmatrix} Y_2 \\ Y_3 \end{bmatrix}$;
   - $\operatorname{rank} EY_2 = r$;
   - there exist $\tilde{\mathcal{E}}, \tilde{\mathcal{A}} \in \mathbb{R}^{n+p \times n+m}$

$$\begin{bmatrix} 0 & -sE + A & B \\ sE^T + A^T & Q & S \\ B^T & S^T & R \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix} = \begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \end{bmatrix} \left(s\tilde{\mathcal{E}} - \tilde{\mathcal{A}} \right).$$

Construction of the Solution

Remaining Questions

- How to ensure that rank $EY_2 = r$?
- How to find a solution that solves the optimal control problem?

Construction of the Solution

Remaining Questions

- How to ensure that rank $EY_2 = r$?
- How to find a solution that solves the optimal control problem?

Theorem

Let

$$\begin{bmatrix} 0 & -sE + A & B \\ sE^T + A^T & Q & S \\ B^T & S^T & R \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix} = \begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \end{bmatrix} \left(s\tilde{\mathcal{E}} - \tilde{\mathcal{A}}\right),$$

where Y is $\mathcal{E}$-neutral, of dimension $n+m$, and

$$\mathcal{V} := \left\{ \begin{pmatrix} x \\ u \end{pmatrix} \in \mathbb{R}^{n+m} : Ax + Bu \in \operatorname{im} E \right\} \subset \operatorname{im} \begin{bmatrix} Y_2 \\ Y_3 \end{bmatrix}.$$

If for every eigenvalue λ of $s\tilde{\mathcal{E}} - \tilde{\mathcal{A}}$, $-\bar{\lambda}$ is not an uncontrollable mode of (E, A, B), then rank $EY_2 = r$.

Construction of the Solution

Remaining Questions

- How to ensure that rank $EY_2 = r$?
- How to find a solution that solves the optimal control problem?

Theorem

If some conditions hold and the optimal control problem is feasible then

- there exists a solution $(X^+, K^+, L^+, V_\infty^+, W_\infty^+, \Sigma^+)$ of the descriptor Lur'e equation, and
- $(s\mathcal{E} - \mathcal{A})Y = Z(s\tilde{\mathcal{E}} - \tilde{\mathcal{A}})$,

such that the following properties hold:

- **Stabilization**: $s\tilde{\mathcal{E}} - \tilde{\mathcal{A}} = \begin{bmatrix} -sE + A & B \\ K^+ & L^+ \end{bmatrix}$ has only eigenvalues in the closed left half-plane.
- **Maximalility**: For $X^+ = Y_1 Y_2^-$ and every other solution X it holds $E^T X \leq E^T X^+$.

Optimal Control Signal

Optimal control signal

The optimal control signal $\hat{u}(\cdot)$ fulfills the DAE

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + B\hat{u}(t), \quad Ex(0) = Ex_0 \\ 0 &= K^{+}x(t) + L^{+}\hat{u}(t). \end{aligned}$$

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Optimal costs

$$V(Ex_0) = x_0^T E^T X^{+} x_0.$$

Conclusions

Presented in this talk

1. Singular optimal control problems for differential-algebraic equations,
2. characterization of the optimal control via the maximal solution of a generalized Lur'e equation,
3. solvability criteria and construction of solution using even matrix pencils.

Conclusions

Presented in this talk

1. Singular optimal control problems for differential-algebraic equations,
2. characterization of the optimal control via the maximal solution of a generalized Lur'e equation,
3. solvability criteria and construction of solution using even matrix pencils.

Not presented in this talk

1. Frequency domain analysis (Popov functions),
2. applications:
   (a) characterization of (lossless) (cyclo-)dissipative systems,
   (b) factorization of rational matrices (spectral factorization/normalized coprime factorizations/inner-outer factorization).

Thanks for your Attention!

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