Meeting of the GAMM Activity Group
"Dynamics and Control Theory"
University of Stuttgart
March 09-10, 2012

Kalman-Yakubovich-Popov Lemma for Differential-Algebraic Equations

Timo Reis[1] **Matthias Voigt[2]**

[1]Department of Mathematics
University of Hamburg
Hamburg, Germany

[2]Computational Methods in Systems and Control Theory
Max Planck Institute for Dynamics of Complex Technical Systems
Magdeburg, Germany

MAX PLANCK INSTITUTE FOR DYNAMICS OF COMPLEX TECHNICAL SYSTEMS MAGDEBURG

Introduction & Preliminaries

Differential-algebraic equations

Consider the differential-algebraic equation

$$E\dot{x}(t) = Ax(t) + Bu(t), \quad Ex(0) = Ex_0$$

with $E, A \in \mathbb{K}^{n\times n}$, $B \in \mathbb{K}^{n\times m}$, descriptor vector $x(t) \in \mathbb{K}^n$, and input vector $u(t) \in \mathbb{K}^m$.

Introduction & Preliminaries

Differential-algebraic equations

Consider the differential-algebraic equation

$$E\dot{x}(t) = Ax(t) + Bu(t), \quad Ex(0) = Ex_0$$

with $E, A \in \mathbb{K}^{n \times n}$, $B \in \mathbb{K}^{n \times m}$, descriptor vector $x(t) \in \mathbb{K}^n$, and input vector $u(t) \in \mathbb{K}^m$.

Spectral density functions

Define the spectral density function by

$$\Phi(s) = \begin{bmatrix} (-\overline{s}E - A)^{-1}B \\ I_m \end{bmatrix}^* \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{bmatrix} (sE - A)^{-1}B \\ I_m \end{bmatrix},$$

with $Q = Q^* \in \mathbb{K}^{n \times n}$, $S \in \mathbb{K}^{n \times m}$, $R = R^* \in \mathbb{K}^{m \times m}$.

Applications

Question in systems and control theory

Is $\Phi(\mathrm{i}\omega) \geq 0 \ \forall \ \mathrm{i}\omega \notin \Lambda(E, A)$?

Answer can be given by analyzing solvability of linear matrix inequalities (LMIs) of the form

$$\begin{bmatrix} A^*X + X^*A + Q & X^*B + S \\ B^*X + S^* & R \end{bmatrix} \geq 0, \quad E^*X = X^*E.$$

Applications

- Characterization of system properties such as dissipativity,
- structure-preserving model order reduction.

LMIs are also useful for linear-quadratic optimal control.

Overview over Existing Theories

Reminder: LMI

$$\begin{bmatrix} A^*X + X^*A + Q & X^*B + S \\ B^*X + S^* & R \end{bmatrix} \geq 0, \quad E^*X = X^*E.$$

Overview over Existing Theories

Reminder: LMI

$$\begin{bmatrix} A^*X + X^*A + Q & X^*B + S \\ B^*X + S^* & R \end{bmatrix} \geq 0, \quad E^*X = X^*E.$$

R nonsingular, E (possibly) singular

If R is nonsingular the solvability of the LMI is equivalent to the solvability of the generalized algebraic Riccati equation

$$A^*X + X^*A - (X^*B + S)R^{-1}(B^*X + S^*) + Q = 0,$$
$$E^*X = X^*E.$$

Overview over Existing Theories

Reminder: LMI

$$\begin{bmatrix} A^*X + X^*A + Q & X^*B + S \\ B^*X + S^* & R \end{bmatrix} \geq 0, \quad E^*X = X^*E.$$

R nonsingular, E (possibly) singular

If R is nonsingular the solvability of the LMI is equivalent to the solvability of the generalized algebraic Riccati equation

$$A^*X + X^*A - (X^*B + S)R^{-1}(B^*X + S^*) + Q = 0, \\ E^*X = X^*E.$$

Solvability criteria and solution can be given in terms of the eigenstructure of the skew-Hamiltonian/Hamiltonian pencil

$$s\mathcal{E}_H - \mathcal{A}_H = s\begin{bmatrix} E & 0 \\ 0 & E^* \end{bmatrix} - \begin{bmatrix} A - BR^{-1}S & -BR^{-1}B^* \\ S^*R^{-1}S - Q & -\left(A - BR^{-1}S\right)^* \end{bmatrix}.$$

Overview over Existing Theories

Reminder: LMI

$$\begin{bmatrix} A^*X + X^*A + Q & X^*B + S \\ B^*X + S^* & R \end{bmatrix} \geq 0, \quad E^*X = X^*E.$$

R singular, E nonsingular (set $E = I_n$)

If R is singular the generalized algebraic Riccati equation cannot be formulated!! Then, the solvability of the LMI is equivalent to the solvability of

$$\begin{bmatrix} A^*X + XA + Q & XB + S \\ B^*X + S^* & R \end{bmatrix} \geq 0, \quad X = X^*.$$

Overview over Existing Theories

Reminder: LMI

$$\begin{bmatrix} A^*X + X^*A + Q & X^*B + S \\ B^*X + S^* & R \end{bmatrix} \geq 0, \quad E^*X = X^*E.$$

R singular, E nonsingular (set $E = I_n$)

If R is singular the generalized algebraic Riccati equation cannot be formulated!! Then, the solvability of the LMI is equivalent to the solvability of the so-called Lur'e equation

$$\begin{bmatrix} A^*X + XA + Q & XB + S \\ B^*X + S^* & R \end{bmatrix} = \begin{bmatrix} K^* \\ L^* \end{bmatrix} \begin{bmatrix} K & L \end{bmatrix}, \quad X = X^*.$$

Overview over Existing Theories

Reminder: LMI

$$\begin{bmatrix} A^*X + X^*A + Q & X^*B + S \\ B^*X + S^* & R \end{bmatrix} \geq 0, \quad E^*X = X^*E.$$

R singular, E nonsingular (set $E = I_n$)

If R is singular the generalized algebraic Riccati equation cannot be formulated!! Then, the solvability of the LMI is equivalent to the solvability of the so-called Lur'e equation

$$\begin{aligned} A^*X + XA + Q &= K^*K, \\ XB + S &= K^*L, \\ R &= L^*L, \\ X &= X^* \end{aligned}$$

for $(X, K, L) \in \mathbb{K}^{n \times n} \times \mathbb{K}^{p \times n} \times \mathbb{K}^{p \times m}$. [Reis '11]

Overview over Existing Theories

Reminder: LMI

$$\begin{bmatrix} A^*X + X^*A + Q & X^*B + S \\ B^*X + S^* & R \end{bmatrix} \geq 0, \quad E^*X = X^*E.$$

R singular, E nonsingular (set $E = I_n$)

Solvability criteria and solution can be given by analyzing the even pencil

$$s\mathcal{E} - \mathcal{A} = \begin{bmatrix} 0 & -sI_n + A & B \\ sI_n + A^* & Q & S \\ B^* & S^* & R \end{bmatrix}.$$

[Reis '11]

Overview over Existing Theories

Reminder: LMI

$$\begin{bmatrix} A^*X + X^*A + Q & X^*B + S \\ B^*X + S^* & R \end{bmatrix} \geq 0, \quad E^*X = X^*E.$$

R singular, E singular

What happens, if R and E are simultaneously singular?

Controllability

Controllability of DAEs [Rosenbrock '74]

Let $[E, A, B]$ be a given matrix triple. Further, let $r = \operatorname{rank}(E)$ and $S_\infty \in \mathbb{K}^{n,n-r}$ be a matrix with $\operatorname{im} S_\infty = \ker E$. Then, $[E, A, B]$ is called

(a) R-controllable, if $\operatorname{rank} \begin{bmatrix} sE - A & B \end{bmatrix} = n$ for all $s \in \mathbb{C}$;

(b) impulse controllable, if $\operatorname{rank} \begin{bmatrix} E & AS_\infty & B \end{bmatrix} = n$;

(c) sign-controllable, if $\operatorname{rank} \begin{bmatrix} sE - A & B \end{bmatrix} = n$ or $\operatorname{rank} \begin{bmatrix} -sE - A & B \end{bmatrix} = n$ for all $s \in \mathbb{C}$.

Sign-controllability implies the absence of uncontrollable modes on the imaginary axis!

Kalman-Yakubovich-Popov Lemma

Kalman-Yakubovich-Popov lemma

Let $E, A, Q \in \mathbb{K}^{n,n}$, $B, S \in \mathbb{K}^{n,m}$ and $R \in \mathbb{K}^{m,m}$ be given such that $sE - A$ is regular, $Q = Q^*$ and $R = R^*$. Furthermore, assume that at least one of the following two assumptions holds true:

a) $[E, A, B]$ is sign-controllable and impulse controllable. Furthermore, the spectral density function

$$\Phi(s) = \begin{bmatrix} (-\overline{s}E - A)^{-1}B \\ I_m \end{bmatrix}^* \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{bmatrix} (sE - A)^{-1}B \\ I_m \end{bmatrix}$$

satisfies

$$\operatorname{rank}_{\mathbb{K}(s)} \Phi(s) = m.$$

b) $[E, A, B]$ is R-controllable and impulse controllable.

Kalman-Yakubovich-Popov Lemma

Kalman-Yakubovich-Popov lemma

Furthermore, let

$$\mathcal{V} = \left\{ \begin{pmatrix} x \\ u \end{pmatrix} \in \mathbb{K}^{n+m} \ : \ Ax + Bu \in \operatorname{im} E \right\}.$$

Then the following two statements are equivalent:

(i) For all $\omega \in \mathbb{R}$ such that $\det(\mathrm{i}\omega E - A) \neq 0$, there holds $\Phi(i\omega) \geq 0$.

(ii) There exists some $X \in \mathbb{K}^{n \times n}$ such that

$$\begin{pmatrix} x \\ u \end{pmatrix}^* \begin{bmatrix} A^*X + X^*A + Q & X^*B + S \\ B^*X + S^* & R \end{bmatrix} \begin{pmatrix} x \\ u \end{pmatrix} \geq 0, \quad E^*X = X^*E.$$

for all $\begin{pmatrix} x \\ u \end{pmatrix} \in \mathcal{V}$.

Kalman-Yakubovich-Popov Lemma

Kalman-Yakubovich-Popov lemma

Furthermore, let

$$\mathcal{V} = \left\{ \begin{pmatrix} x \\ u \end{pmatrix} \in \mathbb{K}^{n+m} \ : \ Ax + Bu \in \operatorname{im} E \right\}.$$

Then the following two statements are equivalent:

(i) For all $\omega \in \mathbb{R}$ such that $\det(\mathrm{i}\omega E - A) \neq 0$, there holds $\Phi(i\omega) \geq 0$.

(ii) There exists some $X \in \mathbb{K}^{n \times n}$ such that

$$\begin{bmatrix} A^*X + X^*A + Q & X^*B + S \\ B^*X + S^* & R \end{bmatrix} \geq_{\mathcal{V}} 0, \quad E^*X = X^*E.$$

Remarks

Some remarks

- The subspace $\mathcal{V}$ is called the hidden manifold of the DAE and contains all points located on all possible solution trajectories $(x(\cdot), u(\cdot))$.
- In contrast to standard systems we additionally need impulse controllability to be able to regularize the matrix pencil $\lambda E - A$ to index one via state feedback transformations.
- Solvability of the LMI on the subspace $\mathcal{V}$ is equivalent to the solvability of the generalized Lur'e equation

$$\begin{bmatrix} A^*X + X^*A + Q & X^*B + S \\ B^*X + S^* & R \end{bmatrix} = \begin{bmatrix} K^* \\ L^* \end{bmatrix} \begin{bmatrix} K & L \end{bmatrix} + M_{\mathcal{V}^\perp},$$
$$E^*X = X^*E,$$

where $\operatorname{im} M_{\mathcal{V}^\perp} \subset \mathcal{V}^\perp$.

Spectral Density Function and Even Matrix Pencils

Relation to even matrix pencils

We consider the matrix pencil

$$s\mathcal{E} - \mathcal{A} = \begin{bmatrix} 0 & -sE + A & B \\ sE^* + A^* & Q & S \\ B^* & S^* & R \end{bmatrix}.$$

The matrix pencil is *even*, since $\mathcal{E} = -\mathcal{E}^*$, $\mathcal{A} = \mathcal{A}^*$.
Spectral Property: Hamiltonian eigensymmetry, i.e.,

$$\lambda \in \Lambda(\mathcal{E}, \mathcal{A}) \Longrightarrow -\overline{\lambda} \in \Lambda(\mathcal{E}, \mathcal{A}).$$

Questions

- Relation between spectral density function $\Phi(s)$ and the even pencil $s\mathcal{E} - \mathcal{A}$,
- characterization of positive semidefiniteness of $\Phi(\mathrm{i}\omega)$ by properties of $s\mathcal{E} - \mathcal{A}$.

Kronecker Canonical Form

Definition: Kronecker canonical form

Let $sE - A \in \mathbb{K}[s]^{m \times n}$. Then there exist nonsingular $P \in \mathbb{C}^{m \times m}$ and $Q \in \mathbb{C}^{n \times n}$ such that $P(sE - A)Q = \operatorname{diag}(\mathcal{C}_1, \ldots, \mathcal{C}_k)$ where each $\mathcal{C}_j$, $j = 1, \ldots, k$ is of one of the following structures:

Kronecker Canonical Form

Definition: Kronecker canonical form

Let $sE - A \in \mathbb{K}[s]^{m \times n}$. Then there exist nonsingular $P \in \mathbb{C}^{m \times m}$ and $Q \in \mathbb{C}^{n \times n}$ such that $P(sE - A)Q = \operatorname{diag}(\mathcal{C}_1, \ldots, \mathcal{C}_k)$ where each $\mathcal{C}_j$, $j = 1, \ldots, k$ is of one of the following structures:

Type 1 (finite eigenvalues):

$$s \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} - \begin{bmatrix} \lambda_j & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_j \end{bmatrix} \in \mathbb{C}^{\ell_j \times \ell_j}$$

with finite eigenvalues $\lambda_j \in \mathbb{C}$.

Kronecker Canonical Form

Definition: Kronecker canonical form

Let $sE - A \in \mathbb{K}[s]^{m \times n}$. Then there exist nonsingular $P \in \mathbb{C}^{m \times m}$ and $Q \in \mathbb{C}^{n \times n}$ such that $P(sE - A)Q = \operatorname{diag}(\mathcal{C}_1, \ldots, \mathcal{C}_k)$ where each $\mathcal{C}_j$, $j = 1, \ldots, k$ is of one of the following structures:

Type 2 (infinite eigenvalues):

$$s \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix} - \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} \in \mathbb{C}^{\ell_j \times \ell_j}.$$

Kronecker Canonical Form

Definition: Kronecker canonical form

Let $sE - A \in \mathbb{K}[s]^{m \times n}$. Then there exist nonsingular $P \in \mathbb{C}^{m \times m}$ and $Q \in \mathbb{C}^{n \times n}$ such that $P(sE - A)Q = \operatorname{diag}(\mathcal{C}_1, \ldots, \mathcal{C}_k)$ where each $\mathcal{C}_j$, $j = 1, \ldots, k$ is of one of the following structures:

Type 3 & 4 (singular structure):

$$s \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix} \in \mathbb{C}^{\ell_j \times (\ell_j + 1)},$$

$$s \begin{bmatrix} 0 & & \\ 1 & \ddots & \\ & \ddots & 0 \\ & & 1 \end{bmatrix} - \begin{bmatrix} 1 & & \\ 0 & \ddots & \\ & \ddots & 1 \\ & & 0 \end{bmatrix} \in \mathbb{C}^{(\ell_j + 1) \times \ell_j}.$$

Even Kronecker Canonical Form

Definition: even Kronecker canonical form [Thompson '76]

Let $s\mathcal{E} - \mathcal{A} \in \mathbb{K}[s]^{n \times n}$ be an even pencil. Then there exists a nonsingular $U \in \mathbb{C}^n$ such that $U^*(s\mathcal{E} - \mathcal{A})U = \operatorname{diag}(\mathcal{D}_1, \ldots, \mathcal{D}_k)$ where each $\mathcal{D}_j$, $j = 1, \ldots, k$ is of one of the following structures:

Even Kronecker Canonical Form

Definition: even Kronecker canonical form [Thompson '76]

Let $s\mathcal{E} - \mathcal{A} \in \mathbb{K}[s]^{n \times n}$ be an even pencil. Then there exists a nonsingular $U \in \mathbb{C}^n$ such that $U^*(s\mathcal{E} - \mathcal{A})U = \operatorname{diag}(\mathcal{D}_1, \ldots, \mathcal{D}_k)$ where each $\mathcal{D}_j$, $j = 1, \ldots, k$ is of one of the following structures:

Type 1 (finite non-imaginary eigenvalues):

$$\left[\begin{array}{cccc|cccc} & & & & -s+\mu_j & -1 & & \\ & & & & & \ddots & \ddots & \\ & & & & & & \ddots & -1 \\ & & & & & & & -s+\mu_j \\ \hline s+\bar{\mu}_j & & & & & & & \\ -1 & \ddots & & & & & & \\ & \ddots & \ddots & & & & & \\ & & -1 & s+\bar{\mu}_j & & & & \end{array}\right] \in \mathbb{C}^{2\ell_j \times 2\ell_j},$$

with finite non-imaginary eigenvalues $\mu_j \in \mathbb{C}^+$, $-\bar{\mu}_j \in \mathbb{C}^-$.

Even Kronecker Canonical Form

Definition: even Kronecker canonical form [THOMPSON '76]

Let $s\mathcal{E} - \mathcal{A} \in \mathbb{K}[s]^{n \times n}$ be an even pencil. Then there exists a nonsingular $U \in \mathbb{C}^n$ such that $U^*(s\mathcal{E} - \mathcal{A})U = \operatorname{diag}(\mathcal{D}_1, \ldots, \mathcal{D}_k)$ where each $\mathcal{D}_j$, $j = 1, \ldots, k$ is of one of the following structures:

Type 2 (finite imaginary eigenvalues):

$$\varepsilon_j \begin{bmatrix} & & & 1 & -s\mathrm{i} - \mu_j \\ & & \iddots & \iddots & \\ & & \iddots & & \\ 1 & \iddots & & & \\ -s\mathrm{i} - \mu_j & & & & \end{bmatrix} \in \mathbb{C}^{\ell_j \times \ell_j},$$

with finite imaginary eigenvalues $-\mathrm{i}\mu_j \in \mathrm{i}\mathbb{R}$ and block signature $\varepsilon_j \in \{-1, 1\}$.

Even Kronecker Canonical Form

Definition: even Kronecker canonical form [Thompson '76]

Let $s\mathcal{E} - \mathcal{A} \in \mathbb{K}[s]^{n \times n}$ be an even pencil. Then there exists a nonsingular $U \in \mathbb{C}^n$ such that $U^*(s\mathcal{E} - \mathcal{A})U = \operatorname{diag}(\mathcal{D}_1, \ldots, \mathcal{D}_k)$ where each $\mathcal{D}_j$, $j = 1, \ldots, k$ is of one of the following structures:

Type 3 (infinite eigenvalues):

$$\varepsilon_j \begin{bmatrix} & & s\mathrm{i} & 1 \\ & \iddots & \iddots & \\ s\mathrm{i} & \iddots & & \\ 1 & & & \end{bmatrix} \in \mathbb{C}^{\ell_j \times \ell_j},$$

with block signature $\varepsilon_j \in \{-1, 1\}$.

Even Kronecker Canonical Form

Definition: even Kronecker canonical form [Thompson '76]

Let $s\mathcal{E} - \mathcal{A} \in \mathbb{K}[s]^{n \times n}$ be an even pencil. Then there exists a nonsingular $U \in \mathbb{C}^n$ such that $U^*(s\mathcal{E} - \mathcal{A})U = \operatorname{diag}(\mathcal{D}_1, \ldots, \mathcal{D}_k)$ where each $\mathcal{D}_j$, $j = 1, \ldots, k$ is of one of the following structures:

Type 4 (singular structure):

$$\left[\begin{array}{cccc|cccc} & & & & 1 & -s & & \\ & & & & & \ddots & \ddots & \\ & & & & & & 1 & -s \\ \hline 1 & & & & & & & \\ s & \ddots & & & & & & \\ & \ddots & 1 & & & & & \\ & & s & & & & & \end{array}\right] \in \mathbb{C}^{(2\ell_j+1)\times(2\ell_j+1)}.$$

Main Result

Theorem

Let the matrices $E, A, Q \in \mathbb{K}^{n\times n}$, $B, S \in \mathbb{K}^{n\times m}$ and $R \in \mathbb{K}^{m\times m}$ be given such that $sE - A \in \mathbb{K}[s]^{n\times n}$ is regular with $\operatorname{rank} E = r$. Furthermore, let $\Phi(s)$ and $s\mathcal{E} - \mathcal{A}$ with $\operatorname{rank}_{\mathbb{K}[s]} \Phi(s) = d$ be defined as before. Then the following statements are equivalent:

(i) For all $\omega \in \mathbb{R}$ with $\det(\mathrm{i}\omega E - A) \neq 0$ holds $\Phi(i\omega) \geq 0$.

(ii) In the EKCF of $s\mathcal{E} - \mathcal{A}$, the blocks have the following structure:

- All blocks of type 2 have even size and negative signature.
- There exist exactly d blocks of type 3 with odd size and positive signature.
- There remain exactly $2(n - r)$ blocks of type 3 which are either of even size; or the number of odd-sized blocks with positive and negative signature is equal.
- There exist exactly $m - d$ blocks of type 4.

Main Result

Theorem

Let the matrices $E, A, Q \in \mathbb{K}^{n\times n}$, $B, S \in \mathbb{K}^{n\times m}$ and $R \in \mathbb{K}^{m\times m}$ be given such that $sE - A \in \mathbb{K}[s]^{n\times n}$ is regular with $\operatorname{rank} E = r$. Furthermore, let $\Phi(s)$ and $s\mathcal{E} - \mathcal{A}$ with $\operatorname{rank}_{\mathbb{K}[s]} \Phi(s) = d$ be defined as before. Then the following statements are equivalent:

(i) For all $\omega \in \mathbb{R}$ with $\det(\mathrm{i}\omega E - A) \neq 0$ holds $\Phi(i\omega) \geq 0$.

(ii) In the EKCF of $s\mathcal{E} - \mathcal{A}$, the blocks have the following structure:

- All blocks of type 2 have even size and negative signature.
- There exist exactly d blocks of type 3 with odd size and positive signature.
- There remain exactly $2(n - r)$ blocks of type 3 which are either of even size; or the number of odd-sized blocks with positive and negative signature is equal.
- There exist exactly $m - d$ blocks of type 4.

Main Result

Theorem

Let the matrices $E, A, Q \in \mathbb{K}^{n\times n}$, $B, S \in \mathbb{K}^{n\times m}$ and $R \in \mathbb{K}^{m\times m}$ be given such that $sE - A \in \mathbb{K}[s]^{n\times n}$ is regular with $\operatorname{rank} E = r$. Furthermore, let $\Phi(s)$ and $s\mathcal{E} - \mathcal{A}$ with $\operatorname{rank}_{\mathbb{K}[s]} \Phi(s) = d$ be defined as before. Then the following statements are equivalent:

(i) For all $\omega \in \mathbb{R}$ with $\det(\mathrm{i}\omega E - A) \neq 0$ holds $\Phi(i\omega) \geq 0$.

(ii) In the EKCF of $s\mathcal{E} - \mathcal{A}$, the blocks have the following structure:

- All blocks of type 2 have even size.
- There exist exactly d blocks of type 3 with odd size and positive signature.
- There remain exactly $2(n - r)$ blocks of type 3 which are either of even size; or the number of odd-sized blocks with positive and negative signature is equal.
- There exist exactly $m - d$ blocks of type 4.

Behavior and Cost Functionals

Behavior of the system

$$\mathfrak{B}_{[E,A,B]} := \Big\{ (x(\cdot), u(\cdot)) \in \mathcal{L}^2_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^n) \times \mathcal{L}^2_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^m) \;\Big|\; Ex(\cdot) \in \mathcal{H}^1_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^n) \text{ and } (x,u) \text{ fulfills } E\dot{x}(t) = Ax(t) + Bu(t)\ \forall\, t \in \mathbb{R} \Big\}.$$

Cost functionals

$$\mathcal{J}(x,u) = \int_0^\infty \begin{pmatrix} x(t) \\ u(t) \end{pmatrix}^* \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{pmatrix} x(t) \\ u(t) \end{pmatrix} \mathrm{d}t.$$

Optimization problems

(a) $V_f^+(Ex_0) = \inf \left\{ \mathcal{J}(x,u) \,:\, (x,u) \in \mathfrak{B}_{[E,A,B]} \text{ with } Ex(0) = Ex_0 \right\},$

(b) $V^+(Ex_0) = \inf \left\{ \mathcal{J}(x,u) \,:\, (x,u) \in \mathfrak{B}_{[E,A,B]} \text{ with } Ex(0) = Ex_0 \text{ and } \lim\limits_{t\to\infty} Ex(t) = 0 \right\}.$

Dissipation Inequalities

Goals:

- Conditions for finiteness of $V_f^+(Ex_0)$, $V^+(Ex_0)$,
- values of $V_f^+(Ex_0)$, $V^+(Ex_0)$.

Dissipation Inequalities

Goals:

- Conditions for finiteness of $V_f^+(Ex_0)$, $V^+(Ex_0)$,
- values of $V_f^+(Ex_0)$, $V^+(Ex_0)$.

Dissipation inequalities

We consider functions $V : \operatorname{im} E \longrightarrow \mathbb{R}$ that fulfill

$$\int_{t_0}^{t_1} \begin{pmatrix} x(t) \\ u(t) \end{pmatrix}^* \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{pmatrix} x(t) \\ u(t) \end{pmatrix} \mathrm{d}t + V(Ex(t_1)) \geq V(Ex(t_0))$$

for all $(x, u) \in \mathfrak{B}_{[E,A,B]}$, $t_0, t_1 \in \mathbb{R}$ with $t_0 \leq t_1$.

Conditions for Finiteness [Willems '71]

Let $sE - A$ be regular and $[E, A, B]$ R-controllable & impulse controllable.

Equivalent conditions for finiteness of $V_f^+(Ex_0)$

(a) $V_f^+(Ex_0) > -\infty$ for all $x_0 \in \mathbb{R}^n$;

(b) $\int_0^T \begin{pmatrix} x(t) \\ u(t) \end{pmatrix}^* \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{pmatrix} x(t) \\ u(t) \end{pmatrix} \mathrm{d}t \geq 0$ for all $T \geq 0$, $(x, u) \in \mathfrak{B}_{[E,A,B]}$ with $Ex(0) = 0$;

(c) There exists $V : \mathrm{im}(E) \to \mathbb{R}_-$ that satisfies the dissipation inequality.

Equivalent conditions for finiteness of $V^+(Ex_0)$

(a) $V^+(Ex_0) > -\infty$ for all $x_0 \in \mathbb{R}^n$;

(b) $\int_0^T \begin{pmatrix} x(t) \\ u(t) \end{pmatrix}^* \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{pmatrix} x(t) \\ u(t) \end{pmatrix} \mathrm{d}t \geq 0$ for all $T \geq 0$, $(x, u) \in \mathfrak{B}_{[E,A,B]}$ with $Ex(0) = 0$ and $\lim_{t\to\infty} Ex(t) = 0$;

(c) There exists $V : \mathrm{im}(E) \to \mathbb{R}$ that satisfies the dissipation inequality.

Equivalence of Dissipation Inequality and LMI

- $\int_{t_0}^{t_1} \begin{pmatrix} x(t) \\ u(t) \end{pmatrix}^* \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{pmatrix} x(t) \\ u(t) \end{pmatrix} \mathrm{d}t + V(Ex(t_1)) \geq V(Ex(t_0))$
  for all $(x, u) \in \mathfrak{B}_{[E,A,B]},\ t_0, t_1 \in \mathbb{R}$ with $t_0 \leq t_1$.

Equivalence of Dissipation Inequality and LMI

- $\int_{t_0}^{t_1} \begin{pmatrix} x(t) \\ u(t) \end{pmatrix}^* \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{pmatrix} x(t) \\ u(t) \end{pmatrix} \mathrm{d}t + V(Ex(t_1)) \geq V(Ex(t_0))$
  for all $(x, u) \in \mathfrak{B}_{[E,A,B]}$, $t_0, t_1 \in \mathbb{R}$ with $t_0 \leq t_1$.
- $\begin{pmatrix} x(t) \\ u(t) \end{pmatrix}^* \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{pmatrix} x(t) \\ u(t) \end{pmatrix} \geq -V'(Ex(t))E\dot{x}(t)$
  for all $(x, u) \in \mathfrak{B}_{[E,A,B]}$, $t \in \mathbb{R}$.

Equivalence of Dissipation Inequality and LMI

- $\displaystyle\int_{t_0}^{t_1} \begin{pmatrix} x(t) \\ u(t) \end{pmatrix}^* \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{pmatrix} x(t) \\ u(t) \end{pmatrix} \mathrm{d}t + V(Ex(t_1)) \geq V(Ex(t_0))$
  for all $(x, u) \in \mathfrak{B}_{[E,A,B]},\ t_0, t_1 \in \mathbb{R}$ with $t_0 \leq t_1$.
- $\begin{pmatrix} x(t) \\ u(t) \end{pmatrix}^* \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{pmatrix} x(t) \\ u(t) \end{pmatrix} \geq -V'(Ex(t))E\dot{x}(t)$
  for all $(x, u) \in \mathfrak{B}_{[E,A,B]},\ t \in \mathbb{R}$.
- Choose quadratic V, i.e., $V(Ex) = x^*E^*Xx$ with $E^*X = X^*E$:
  $\begin{pmatrix} x(t) \\ u(t) \end{pmatrix}^* \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{pmatrix} x(t) \\ u(t) \end{pmatrix} \geq -2x(t)^*X^*(Ax(t) + Bu(t))$
  for all $(x, u) \in \mathfrak{B}_{[E,A,B]},\ t \in \mathbb{R}$.

Equivalence of Dissipation Inequality and LMI

- $\displaystyle\int_{t_0}^{t_1} \begin{pmatrix} x(t) \\ u(t) \end{pmatrix}^* \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{pmatrix} x(t) \\ u(t) \end{pmatrix} \mathrm{d}t + V(Ex(t_1)) \geq V(Ex(t_0))$
  for all $(x,u) \in \mathfrak{B}_{[E,A,B]},\ t_0, t_1 \in \mathbb{R}$ with $t_0 \leq t_1$.
- $\begin{pmatrix} x(t) \\ u(t) \end{pmatrix}^* \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{pmatrix} x(t) \\ u(t) \end{pmatrix} \geq -V'(Ex(t))E\dot{x}(t)$
  for all $(x,u) \in \mathfrak{B}_{[E,A,B]},\ t \in \mathbb{R}$.
- Choose quadratic V, i.e., $V(Ex) = x^*E^*Xx$ with $E^*X = X^*E$:
  $\begin{pmatrix} x(t) \\ u(t) \end{pmatrix}^* \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{pmatrix} x(t) \\ u(t) \end{pmatrix} \geq -2x(t)^*X^*(Ax(t)+Bu(t))$
  for all $(x,u) \in \mathfrak{B}_{[E,A,B]},\ t \in \mathbb{R}$.
- $\begin{pmatrix} x(t) \\ u(t) \end{pmatrix}^* \begin{bmatrix} A^*X + X^*A + Q & X^*B + S \\ B^*X + S^* & R \end{bmatrix} \begin{pmatrix} x(t) \\ u(t) \end{pmatrix} \geq 0,\ E^*X = X^*E$
  for all $(x,u) \in \mathfrak{B}_{[E,A,B]},\ t \in \mathbb{R}$.

Equivalence of Dissipation Inequality and LMI

- $\int_{t_0}^{t_1} \begin{pmatrix} x(t) \\ u(t) \end{pmatrix}^* \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{pmatrix} x(t) \\ u(t) \end{pmatrix} \mathrm{d}t + V(Ex(t_1)) \geq V(Ex(t_0))$
  for all $(x,u) \in \mathfrak{B}_{[E,A,B]},\ t_0, t_1 \in \mathbb{R}$ with $t_0 \leq t_1$.
- $\begin{pmatrix} x(t) \\ u(t) \end{pmatrix}^* \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{pmatrix} x(t) \\ u(t) \end{pmatrix} \geq -V'(Ex(t))E\dot{x}(t)$
  for all $(x,u) \in \mathfrak{B}_{[E,A,B]},\ t \in \mathbb{R}$.
- Choose quadratic V, i.e., $V(Ex) = x^*E^*Xx$ with $E^*X = X^*E$:
  $\begin{pmatrix} x(t) \\ u(t) \end{pmatrix}^* \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{pmatrix} x(t) \\ u(t) \end{pmatrix} \geq -2x(t)^*X^*(Ax(t) + Bu(t))$
  for all $(x,u) \in \mathfrak{B}_{[E,A,B]},\ t \in \mathbb{R}$.
- $\begin{pmatrix} x(t) \\ u(t) \end{pmatrix}^* \begin{bmatrix} A^*X + X^*A + Q & X^*B + S \\ B^*X + S^* & R \end{bmatrix} \begin{pmatrix} x(t) \\ u(t) \end{pmatrix} \geq 0,\ E^*X = X^*E$
  for all $(x,u) \in \mathfrak{B}_{[E,A,B]},\ t \in \mathbb{R}$.
- $\begin{bmatrix} A^*X + X^*A + Q & X^*B + S \\ B^*X + S^* & R \end{bmatrix} \geq_{\mathcal{V}} 0,\ E^*X = X^*E.$

LMI Conditions for Finiteness

Let $sE - A$ be regular and $[E, A, B]$ R-controllable & impulse controllable.

Equivalent conditions for finiteness of $V_f^+(Ex_0)$

(a) $V_f^+(Ex_0) > -\infty$ for all $x_0 \in \mathbb{R}^n$;

(b) $\int_0^T \begin{pmatrix} x(t) \\ u(t) \end{pmatrix}^* \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{pmatrix} x(t) \\ u(t) \end{pmatrix} \mathrm{d}t \geq 0$ for all $T \geq 0$, $(x, u) \in \mathfrak{B}_{[E,A,B]}$ with $Ex(0) = 0$;

(c) There exists an X with $E^*X = X^*E \leq 0$ that satisfies the LMI.

Equivalent conditions for finiteness of $V^+(Ex_0)$

(a) $V^+(Ex_0) > -\infty$ for all $x_0 \in \mathbb{R}^n$;

(b) $\int_0^T \begin{pmatrix} x(t) \\ u(t) \end{pmatrix}^* \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{pmatrix} x(t) \\ u(t) \end{pmatrix} \mathrm{d}t \geq 0$ for all $T \geq 0$, $(x, u) \in \mathfrak{B}_{[E,A,B]}$ with $Ex(0) = 0$ and $\lim_{t\to\infty} Ex(t) = 0$;

(c) There exists an X that satisfies the LMI.

Values of $V_f^+(Ex_0)$ and $V^+(Ex_0)$

Extremal solutions of the LMI

The set of solution of the LMI contains extremal elements X^- and X^+ which satisfy

$$x_0^* E^* X^- x_0 \leq x_0^* E^* X x_0 \leq x_0^* E^* X^+ x_0 \quad \forall x_0 \in \mathbb{K}^n.$$

Values of $V_f^+(Ex_0)$ and $V^+(Ex_0)$

- $V^+(Ex_0) = x_0^* E^* X^+ x_0$,
- no easy expression for $V_f^+(Ex_0)$.

Descriptor Systems and Transfer Functions

Descriptor systems

We now consider descriptor systems

$$\Sigma : \begin{cases} E\dot{x}(t) = Ax(t) + Bu(t), \\ \quad y(t) = Cx(t) + Du(t). \end{cases}$$

Transfer functions

The transfer function is given by

$$G(s) = C(sE - A)^{-1}B + D.$$

Passivity and Positive Realness

Passivity

A descriptor system Σ is called passive if $p = m$ and for all trajectories $(x(\cdot), u(\cdot), y(\cdot))$ of Σ with $Ex(0) = 0$ and all $T \geq 0$, there holds

$$\int_0^T \mathrm{Re}\,(u^*(t)y(t))\,\mathrm{d}t \geq 0.$$

Positive realness

A rational function $G(s) \in \mathbb{K}(s)^{p \times m}$ is called positive real if $p = m$,

- $G(s)$ is analytic in $\mathbb{C}^+$,
- $G(\bar{s}) = \overline{G(s)}\ \forall\, s \in \mathbb{C}$, and
- $G(s) + G^*(s) \geq 0$ for all $s \in \mathbb{C}^+$.

Equivalence

The system Σ is passive if and only if $G(s)$ is positive real.

Contractivity and Bounded Realness

Contractivity

A descriptor system Σ is called contractive if for all trajectories $(x(\cdot), u(\cdot), y(\cdot))$ of Σ with $Ex(0) = 0$ and all $T \geq 0$, there holds

$$\int_0^T (u^*(t)u(t) - y^*(t)y(t))\,\mathrm{d}t \geq 0.$$

Bounded realness

A rational function $G(s) \in \mathbb{K}(s)^{p \times m}$ is called bounded real if

- $G(s)$ is analytic in $\mathbb{C}^+$,
- $G(\overline{s}) = \overline{G(s)}\ \forall\ s \in \mathbb{C}$, and
- $I_m - G^*(s)G(s) \geq 0$ for all $s \in \mathbb{C}^+$.

Equivalence

The system Σ is contractive if and only if $G(s)$ is bounded real.

New Positive Real Lemma

Positive real lemma for descriptor systems

Let $E, A \in \mathbb{K}^{n\times n}$, B, $C^* \in \mathbb{K}^{n\times m}$ and $D \in \mathbb{K}^{m\times m}$ be given such that $sE - A$ is regular. Furthermore assume that $[E, A, B]$ is R-controllable and impulse controllable. Then $G(s)$ is positive real if and only if for

$$\mathcal{V} = \left\{ \begin{pmatrix} x \\ u \end{pmatrix} \in \mathbb{K}^{n+m} \ : \ Ax + Bu \in \operatorname{im} E \right\}$$

there exists some $X \in \mathbb{K}^{n\times n}$ such that

$$\begin{bmatrix} A^*X + X^*A & X^*B - C^* \\ B^*X - C & -D - D^* \end{bmatrix} \leq_{\mathcal{V}} 0,$$
$$E^*X = X^*E \geq 0.$$

New Bounded Real Lemma

Bounded real lemma for descriptor systems

Let $E, A \in \mathbb{K}^{n \times n}$, $B \in \mathbb{K}^{n \times m}$, $C \in \mathbb{K}^{p \times n}$ and $D \in \mathbb{K}^{m \times m}$ be given such that $sE - A$ is regular. Furthermore assume that $[E, A, B]$ is R-controllable and impulse controllable. Then $G(s)$ is bounded real if and only if for

$$\mathcal{V} = \left\{ \begin{pmatrix} x \\ u \end{pmatrix} \in \mathbb{K}^{n+m} \; : \; Ax + Bu \in \operatorname{im} E \right\}$$

there exists some $X \in \mathbb{K}^{n \times n}$ such that

$$\begin{bmatrix} A^*X + X^*A + C^*C & X^*B + C^*D \\ B^*X + D^*C & D^*D - I_m \end{bmatrix} \leq_{\mathcal{V}} 0,$$

$$E^*X = X^*E \geq 0.$$

Conclusions and Outlook

Conclusions

- New algebraic criteria for semidefiniteness of a spectral density function on the imaginary axis in terms of LMIs and the eigenstructure of even pencils,
- application to infinite time horizon optimal control,
- new characterization of passivity and contractivity of descriptor systems.

Outlook

Develop algorithms to solve generalized Lur'e equations

$$\begin{bmatrix} A^*X + X^*A + Q & X^*B + S \\ B^*X + S^* & R \end{bmatrix} = \begin{bmatrix} K^* \\ L^* \end{bmatrix} \begin{bmatrix} K & L \end{bmatrix} + M_{\mathcal{V}^\perp},$$
$$E^*X = X^*E,$$

with im $M_{\mathcal{V}^\perp} \subset \mathcal{V}^\perp$; characterization via deflating subspaces of $s\mathcal{E} - \mathcal{A}$.

Thanks for your Attention!

References

- REIS '11: *Lur'e equations and even matrix pencils*, Lin. Alg. Appl., 434(1), Jan. 2011, pp. 152–173.
- ROSENBROCK '74: *Structural properties of linear dynamical systems*, Int. J. Control, 20, 1974, pp. 191–202.
- THOMPSON '76: *The characteristic polynomial of a principal subpencil of a Hermitian matrix pencil*, Lin. Alg. Appl., 14(2), 1976, pp. 135–177.
- WILLEMS '71: *Least squares stationary optimal control and the algebraic Riccati equation*, IEEE Trans. Automat. Control, AC-16(6), Dec. 1971, pp. 621–634.