

# Inner-Outer Factorization for Differential-Algebraic Equations

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# Differential-Algebraic Systems

Linear time-invariant differential-algebraic systems

$$\begin{aligned}\frac{d}{dt}Ex(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t),\end{aligned}$$

where  $sE - A \in \mathbb{R}[s]^{n \times n}$  is regular,  $B \in \mathbb{R}^{n \times m}$ ,  $C \in \mathbb{R}^{p \times n}$ ,  $D \in \mathbb{R}^{p \times m}$  and with the transfer function

$$G(s) = C(sE - A)^{-1}B + D.$$

# Inner-Outer Factorizations

**Definition:** A rational function  $G(s) \in \mathbb{R}(s)^{p \times m}$  is called

- a) **outer** if  $p = \text{rank}_{\mathbb{R}(s)} G(s)$  and  $G(s)$  has no zeros in  $\mathbb{C}^+$ ;
- b) **inner** if  $G(s)$  has no poles in  $\mathbb{C}^+$  and  $G^H(-\bar{s})G(s) = I_m$ .

## Goal of this talk

Determination of a factorization of the form

$$G(s) = G_i(s)G_o(s),$$

where  $G_i(s)$  is **inner** and  $G_o(s)$  is **outer**.

# Terminology

a) behavior:

$$\mathfrak{B}_{[E,A,B]} := \{(x, u) \in \mathcal{L}_{\text{loc}}^2(\mathbb{R}, \mathbb{R}^n) \times \mathcal{L}_{\text{loc}}^2(\mathbb{R}, \mathbb{R}^m) : \frac{d}{dt}Ex = Ax + Bu\}.$$

b) system space: smallest subspace in  $\mathbb{R}^{n+m}$  such that

$$\begin{pmatrix} x(t) \\ u(t) \end{pmatrix} \in \mathcal{V}_{\text{sys}} \quad \forall (x, u) \in \mathfrak{B}_{[E,A,B]} \text{ and almost all } t \in \mathbb{R}.$$

c) space of consistent initial differential variables:

$$\mathcal{V}_{\text{diff}} := \{x_0 \in \mathbb{K}^n : \exists (x, u) \in \mathfrak{B}_{[E,A,B]} \text{ with } Ex(0) = Ex_0\}.$$

# Lur'e Equations

For  $Q = Q^T \in \mathbb{R}^{n \times n}$ ,  $S \in \mathbb{R}^{n \times m}$ ,  $R = R^T \in \mathbb{R}^{m \times m}$ , the **Lur'e equation** is given by

$$\begin{bmatrix} A^T X E + E^T X A + Q & E^T X B + S \\ B^T X E + S^T & R \end{bmatrix} = \nu_{\text{sys}} \begin{bmatrix} K^T \\ L^T \end{bmatrix} \begin{bmatrix} K & L \end{bmatrix}, \quad X = X^T.$$

A triple  $(X, K, L) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{q \times n} \times \mathbb{R}^{q \times m}$  for some  $q \in \mathbb{N}_0$  is called **solution** if

$$\text{rank}_{\mathbb{R}(s)} \begin{bmatrix} -sE + A & B \\ K & L \end{bmatrix} = n + q.$$

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**Remark:** For  $E = I_n$  and invertible  $R$ , this is equivalent to the algebraic Riccati equation

$$A^T X + X A + Q - (X B + S) R^{-1} (X B + S)^T = 0, \quad X = X^T.$$

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A solution is called

a) **stabilizing**, if

$$\text{rank} \begin{bmatrix} -\lambda E + A & B \\ K & L \end{bmatrix} = n + q \quad \forall \lambda \in \mathbb{C}^+;$$

b) **nonnegative**, if

$$E^T X E \geq \nu_{\text{diff}} 0.$$

## Theorem

Let  $[E, A, B]$  be behaviorally stabilizable. If

$$\begin{bmatrix} A^T P E + E^T P A + Q & E^T P B + S \\ B^T P E + S^T & R \end{bmatrix} \geq_{\mathcal{V}_{\text{sys}}} 0,$$

is solvable, then the Lur'e equation has a stabilizing solution  $(X, K, L) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{q \times n} \times \mathbb{R}^{q \times m}$  with the properties:

- a) **Maximality:**  $E^T X E \geq_{\mathcal{V}_{\text{diff}}} E^T P E$  for all  $P \in \mathbb{R}^{n \times n}$  fulfilling the LMI.  
 b) **Spectral factorization:** Let

$$\Phi(s) := \begin{bmatrix} (-\bar{s}E - A)^{-1}B \\ I_m \end{bmatrix}^H \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} (sE - A)^{-1}B \\ I_m \end{bmatrix} \in \mathbb{R}(s)^{m \times m}.$$

Then  $\Phi(s) = W^H(-\bar{s})W(s)$  for **outer**

$$W(s) = K(sE - A)^{-1}B + L \in \mathbb{R}(s)^{q \times m}.$$

- c) **Rank:**  $q = \text{rank}_{\mathbb{R}(s)} \Phi(s)$ .



# Construction of the Factors

## Theorem

Let  $[E, A, B, C, D]$  be behaviorally stabilizable with transfer function  $G(s) \in \mathbb{R}(s)^{p \times m}$ . Let  $q = \text{rank}_{\mathbb{R}(s)} G(s)$  and  $Z \in \mathbb{R}^{m \times q}$  be a matrix with  $\text{rank}_{\mathbb{R}(s)} G(s)Z = q$ . Let  $(X, K, L)$  be a stabilizing solution of the Lur'e equation

$$\begin{bmatrix} A^T X E + E^T X A + C^T C & E^T X B + C^T D \\ B^T X E + D^T C & D^T D \end{bmatrix} = \nu_{\text{sys}} \begin{bmatrix} K^T \\ L^T \end{bmatrix} \begin{bmatrix} K & L \end{bmatrix}, \quad X = X^T.$$

Then  $G_i(s) \in \mathbb{R}(s)^{p \times q}$  is the transfer function of

$$[E_i, A_i, B_i, C_i, D_i] := \left[ \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A & BZ \\ K & LZ \end{bmatrix}, \begin{bmatrix} 0 \\ -I_q \end{bmatrix}, [C \quad DZ], 0_{p \times q} \right]$$

and  $G_o(s) \in \mathbb{R}(s)^{q \times m}$  is the transfer function of

$$[E_o, A_o, B_o, C_o, D_o] := [E, A, B, K, L].$$

# Proof Idea

**Step 1:**  $G_o(s) = K(sE - A)^{-1}B + L$  is outer:

Follows from the fact, that  $(X, K, L)$  is a stabilizing solution.

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Follows from

$$\Phi(i\omega) = G(i\omega)^H G(i\omega) = G_o(i\omega)^H G_o(i\omega) \quad \forall i\omega \notin \Lambda(E, A)$$

and

$$\|G(i\omega)v(i\omega)\|_2 = 0 \quad \Leftrightarrow \quad \|G_o(i\omega)v(i\omega)\|_2 = 0 \quad \forall i\omega \notin \Lambda(E, A) \cup \mathfrak{P}(v).$$

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**Step 3:**  $G_o(s)Z$  is invertible:

- Step 2  $\Rightarrow$   $\text{rank}_{\mathbb{R}(s)} G_o(s) = \text{rank}_{\mathbb{R}(s)} G(s) = q$ ,
- $G_o(s)$  outer  $\Rightarrow$   $G_o(s) \in \mathbb{R}(s)^{q \times m}$ ,
- $(G(i\omega)Z)^H (G(i\omega)Z) = (G_o(i\omega)Z)^H (G_o(i\omega)Z) \quad \forall i\omega \notin \Lambda(E, A)$ .

Then the statement follows from  $\text{rank}_{\mathbb{R}(s)} G(s)Z = q$ .

# Proof Idea

**Step 4:**  $G_i(s)G_o(s) = G(s)$ :

$$G_i(s) = [C \quad DZ] \begin{bmatrix} sE - A & -BZ \\ -K & -LZ \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ -I_q \end{bmatrix} = G(s)Z(G_o(s)Z)^{-1}.$$

Further,  $Z(G_o(s)Z)^{-1}G_o(s)$  is projector along  $\ker_{\mathbb{R}(s)} G(s)$  and thus

$$\begin{aligned} G_i(s) \cdot G_o(s) &= G(s)Z(G_o(s)Z)^{-1} \cdot G_o(s) \\ &= G(s) - \underbrace{G(s)(I_m - Z(G_o(s)Z)^{-1}G_o(s))}_{=0} = G(s). \end{aligned}$$

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**Step 5:**  $G_i(s)$  is inner:

Show that

$$\begin{bmatrix} A_i^T P_i E_i + E_i^T P_i A_i + C_i^T C_i & E_i^T P_i B_i \\ B_i^T P_i E_i & -I_q \end{bmatrix} =_{\nu_{\text{sys},i}} 0, \quad E_i^T P_i E_i \geq_{\nu_{\text{diff},i}} 0,$$

simple calculation, leads to the Lur'e equation.

# Example

Let the behaviorally stabilizable system

$$sE - A = \begin{bmatrix} -1 & s \\ 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} -1 & -1 \\ -1 & 0 \end{bmatrix}, \quad D = 0_{2 \times 1}$$

be given. The transfer function is  $G(s) = \begin{bmatrix} s + 1 \\ s \end{bmatrix} \in \mathbb{R}(s)^{2 \times 1}$ .

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A stabilizing solution of the Lur'e equation

$$\begin{bmatrix} A^T X E + E^T X A + C^T C & E^T X B + C^T D \\ B^T X E + D^T C & D^T D \end{bmatrix} = \nu_{\text{sys}} \begin{bmatrix} K^T \\ L^T \end{bmatrix} \begin{bmatrix} K & L \end{bmatrix}, \quad X = X^T$$

is

$$(X, K, L) = \left( \begin{bmatrix} \sqrt{2}-1 & 0 \\ 0 & 0 \end{bmatrix}, [-\sqrt{2} \quad -1], 0 \right).$$



# Example

Then with  $Z = 1$

$$G_o(s) = [-\sqrt{2} \quad -1] \begin{bmatrix} -1 & s \\ 0 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \sqrt{2}s + 1 \in \mathbb{R}(s),$$

$$G_i(s) = \begin{bmatrix} -1 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} -1 & s & 0 \\ 0 & -1 & -1 \\ \sqrt{2} & 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{s+1}{1+\sqrt{2}s} \\ \frac{s}{1+\sqrt{2}s} \end{bmatrix} \in \mathbb{R}(s)^{2 \times 1}.$$

We have  $G(s) = G_i(s)G_o(s) = \begin{bmatrix} s+1 \\ s \end{bmatrix} \in \mathbb{R}(s)^{2 \times 1}$ . Further,

- $G_o(s)$  is outer, since it has full row rank and the only zero  $\lambda_0 = -\frac{1}{\sqrt{2}} \notin \mathbb{C}^+$ .
- $G_i(s)$  is inner, since the only pole  $\lambda_0 = -\frac{1}{\sqrt{2}} \notin \mathbb{C}^+$  and

$$G_i^H(-\bar{s})G_i(s) = \begin{bmatrix} \frac{-s+1}{1-\sqrt{2}s} & \frac{-s}{1-\sqrt{2}s} \end{bmatrix} \begin{bmatrix} \frac{s+1}{1+\sqrt{2}s} \\ \frac{s}{1+\sqrt{2}s} \end{bmatrix} = 1.$$

# Conclusions

Have shown the construction of inner-outer factorization for arbitrary transfer functions given by behaviorally stabilizable differential-algebraic equations. The construction based on simple formulas using Lur'e equations.

## Final remarks:

- the formulas do not need previous successive reductions of  $[E, A, B, C, D]$  such as in [OARĀ, VARGA '00],
- no restrictive assumptions such as properness, stability, or right invertibility [YEH, WEI '90], [XIN, MITA '98],
- algorithms in [VARGA '98], [OARĀ, VARGA '00].

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Thanks for the Attention!

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