

A Greedy Subspace Method for Computing the \mathcal{H}_∞ -Norm

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Motivation

Consider a function

$$G : \Omega \rightarrow \mathbb{C}^{p \times m}, \quad G(s) := C(s)D(s)^{-1}B(s),$$

where $\Omega \supseteq \mathbb{C}^+ := \{s \in \mathbb{C} : \operatorname{Re}(s) > 0\}$ and

$$\begin{aligned} B : \Omega &\rightarrow \mathbb{C}^{n \times m}, & B(s) &:= f_1(s)B_1 + \cdots + f_{\kappa_B}(s)B_{\kappa_B}, \\ C : \Omega &\rightarrow \mathbb{C}^{p \times n}, & C(s) &:= g_1(s)C_1 + \cdots + g_{\kappa_C}(s)C_{\kappa_C}, \\ D : \Omega &\rightarrow \mathbb{C}^{n \times n}, & D(s) &:= h_1(s)D_1 + \cdots + h_{\kappa_D}(s)D_{\kappa_D}, \end{aligned}$$

and all f_i , g_i , and h_i are meromorphic in Ω .

Typical Examples

Transfer functions of ...

1. differential-algebraic control systems/descriptor systems:

$$G(s) = C(sE - A)^{-1}B,$$

2. higher-order systems:

$$G(s) = C (s^2M + sD + K)^{-1} B,$$

3. delay differential-algebraic equations:

$$G(s) = C (sE - A_0 - e^{-s\tau_1}A_1 - e^{-s\tau_2}A_2)^{-1} B,$$

4. systems with input/output delays:

$$G(s) = C(sE - A)^{-1}Be^{-s\tau},$$

5. spatially discretized Maxwell's equation:

$$G(s) = sB^T (s^2I_n - \sqrt{s}D_2 + D_3)^{-1}B.$$

Define the Hardy space

$$\mathcal{H}_\infty^{p \times m} := \left\{ G : \mathbb{C}^+ \rightarrow \mathbb{C}^{p \times m} \mid G \text{ is analytic and } \sup_{s \in \mathbb{C}^+} \|G(s)\|_2 < \infty \right\},$$

with the \mathcal{H}_∞ -norm

$$\|G\|_{\mathcal{H}_\infty} := \sup_{s \in \mathbb{C}^+} \|G(s)\|_2 = \sup_{\omega \in \mathbb{R}} \sigma_{\max}(G(i\omega)).$$

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Goal of this talk

- computation of the \mathcal{H}_∞ -norm for this general class of functions;
- focus on large-scale systems \implies use ideas from model order reduction.

- 1 Introduction
- 2 Applications
- 3 Algorithm
- 4 Analysis
- 5 Numerical Results
- 6 Conclusions and Outlook

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Application 1: Assessing Robust Stability of Delay DAEs

Consider a delay DAE

$$\frac{d}{dt}Ex(t) = A_0x(t) + A_1x(t - \tau).$$

Introduce a delay DAE with complex perturbations Δ_i , $i = 1, 2, 3$:

$$\frac{d}{dt}(E + B_1\Delta_1C)x(t) = (A_0 + B_2\Delta_2C)x(t) + (A_1 + B_3\Delta_3C)x(t - \tau),$$

Let $\Delta := [\Delta_1^H \quad \Delta_2^H \quad \Delta_3^H]^H$.

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Question:

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Question:

$$r := \inf \{ \|\Delta\|_2 \mid \text{perturbed system is not exponentially stable} \} = ??$$

Answer: Define

$$G(s) := C(sE - A_0 - e^{-s\tau}A_1)^{-1} [-sB_1 \quad B_2 \quad e^{-s\tau}B_3].$$

Under some technical conditions, $r := \|G\|_{\mathcal{H}_\infty}^{-1}$.

Application 2: Robust Control

Consider a DAE (a plant P) with inputs and outputs

$$\begin{aligned}\frac{d}{dt}Ex(t) &= Ax(t) + B_1u(t) + B_2w(t), \\ y(t) &= C_1x(t), \\ z(t) &= C_2x(t),\end{aligned}$$

with $B_i \in \mathbb{R}^{n \times m_i}$, $C_i \in \mathbb{R}^{p_i \times n}$.

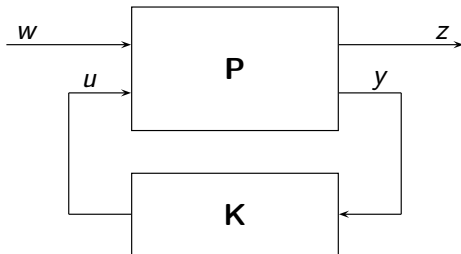
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with $B_i \in \mathbb{R}^{n \times m_i}$, $C_i \in \mathbb{R}^{p_i \times n}$.

Task: design a controller $K \in \mathbb{R}^{m \times n}$ such that $u(t) = Ky(t)$.



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Closed-loop system:

$$\begin{aligned} \frac{d}{dt}Ex(t) &= (A + B_1KC_1)x(t) + B_2w(t), \\ z(t) &= C_2x(t). \end{aligned}$$

Want: controller K such that the influence on stability of the closed loop system by disturbances of the form $w(t) = \Delta z(t)$ is as small as possible \implies find K that minimizes the \mathcal{H}_∞ -norm of the closed loop transfer function

$$G_K(s) := C_2(sE - (A + B_1KC_1))^{-1}B_2.$$

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Algorithm

We have

$$G(s) = C(s)D(s)^{-1}B(s). \quad ((p \times n) \cdot (n \times n) \cdot (n \times m))$$

Often, $n \gg p, m$, i. e., the large-scale part of the problem is the **middle factor**.
Do model reduction to make the middle factor small, i. e., determine $V, W \in \mathbb{R}^{n \times k}$ such that

$$\tilde{G}(s) = \tilde{C}(s)\tilde{D}(s)^{-1}\tilde{B}(s),$$

with

$$\tilde{C}(s) = C(s)V, \quad \tilde{D}(s) = W^H D(s)V, \quad \tilde{B}(s) = W^H B(s).$$

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Question: How to choose the right subspaces V and W ?

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Question: How to choose the right subspaces V and W ?

Answer: Do interpolation of the maximum singular values and update interpolation points using information of the current reduced order model.

Interpolatory Model Reduction

Theorem: Tangential Interpolation

[BEATTIE, GUGERCIN '08]

Let $\mu \in \mathbb{C}$ be such that $C(\mu)$, $D(\mu)$, and $B(\mu)$ are analytic and both $D(\mu)$ and $\tilde{D}(\mu)$ are invertible. Suppose also that $b \in \mathbb{C}^m \setminus \{0\}$ and $c \in \mathbb{C}^p \setminus \{0\}$ are given. Let $\mathcal{V} = \text{Col}(V)$ and $\mathcal{W} = \text{Col}(W)$. Then the following statements hold:

- a) $D(\mu)^{-1}B(\mu)b \in \mathcal{V} \implies G(\mu)b = \tilde{G}(\mu)b,$
- b) $(c^H C(\mu)D(\mu)^{-1})^H \in \mathcal{W} \implies c^H G(\mu) = c^H \tilde{G}(\mu),$
- c) $D(\mu)^{-1}B(\mu)b \in \mathcal{V}, (c^H C(\mu)D(\mu)^{-1})^H \in \mathcal{W} \implies c^H G'(\mu)b = c^H \tilde{G}'(\mu)b.$

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Let $\sigma := \sigma_{\max}(G(\mu))$ be the largest singular value of $G(\mu)$ with right and left singular vectors $v \in \mathbb{C}^m$ and $w \in \mathbb{C}^p$. Then we have

$$\begin{aligned} G(\mu)v &= \sigma w \\ w^H G(\mu) &= v^H \sigma \end{aligned}$$

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Let $\sigma := \sigma_{\max}(G(\mu))$ be the largest singular value of $G(\mu)$ with right and left singular vectors $v \in \mathbb{C}^m$ and $w \in \mathbb{C}^p$. Then we have

$$\begin{aligned} G(\mu)v &= \sigma w = \tilde{G}(\mu)v, \\ w^H G(\mu) &= v^H \sigma = w^H \tilde{G}(\mu), \end{aligned}$$

but we do **not** know, whether $\sigma = \sigma_{\max}(\tilde{G}(\mu))$.

Interpolatory Model Reduction

Solution: Do full interpolation, i. e., instead of one pair singular vectors, put all of them into the projection spaces:

$$\text{im} (D(\mu)^{-1}B(\mu)) \subseteq \mathcal{V} \implies G(\mu) = \tilde{G}(\mu),$$

$$\text{im} (C(\mu)D(\mu)^{-1})^H \subseteq \mathcal{W} \implies G(\mu) = \tilde{G}(\mu),$$

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What Next? Use a greedy method to collect more interpolation data!

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What Next? Use a greedy method to collect more interpolation data!

- Compute the maximum singular value of the “low-dimensional” transfer function \tilde{G} on the imaginary axis and the optimal frequency ω_* using established methods for the small-scale case:

- Boyd-Balakrishnan algorithm for linear systems: based on eigenvalue computations for Hamiltonian matrices/even matrix pencils

[BOYD, BALAKRISHNAN '90], [BENNER, SIMA, V. '12]

- eigenvalue optimization methods for the general problem

[MENGI, YILDIRIM, KILIÇ '14]

- add subspaces from the current optimizer ω_* to get better local information.

Algorithm

Main steps of the algorithm:

- a) create an initial reduced-order model with transfer function \tilde{G} and projection matrices V, W ,
- b) compute $\sup_{\omega \in \mathbb{R}} \|\tilde{G}(i\omega)\|_2$ with global optimizer ω_* ,
- c) update V and W with subspaces at $i\omega_*$:
 - if $m = p$: set $\tilde{V} := D(i\omega_*)^{-1}B(i\omega_*)$, $\tilde{W} := (C(i\omega_*)D(i\omega_*)^{-1})^H$,
 - if $m < p$: set $\tilde{V} := D(i\omega_*)^{-1}B(i\omega_*)$, $\tilde{W} := (C(i\omega_*)D(i\omega_*)^{-1})^H G(i\omega_*)$,
 - if $m > p$: set $\tilde{V} := D(i\omega_*)^{-1}B(i\omega_*)G(i\omega_*)^H$, $\tilde{W} := (C(i\omega_*)D(i\omega_*)^{-1})^H$,
 - set $V := \begin{bmatrix} V & \tilde{V} \end{bmatrix}$ and $W := \begin{bmatrix} W & \tilde{W} \end{bmatrix}$ and update \tilde{G} ,
- d) if not converged, repeat b), c), and d).

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Analysis

Lemma

Let V, W be constructed as above at interpolation points $i\omega_1, \dots, i\omega_k$. Then the following statements hold:

- a) $\sigma_{\max}(G(i\omega_j)) = \sigma_{\max}(\tilde{G}(i\omega_j)), \quad j = 1, \dots, k.$
- b) If $\sigma_{\max}(G(i\cdot))$ is simple at $\omega_1, \dots, \omega_k$, then $\sigma_{\max}(G(i\cdot)), \sigma_{\max}(\tilde{G}(i\cdot))$ are differentiable at $\omega_1, \dots, \omega_k$ and

$$\sigma'_{\max}(G(i\omega_j)) = \sigma'_{\max}(\tilde{G}(i\omega_j)), \quad j = 1, \dots, k.$$

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Superlinear Convergence

[ALIYEV, BENNER, MENGI, V. '16]

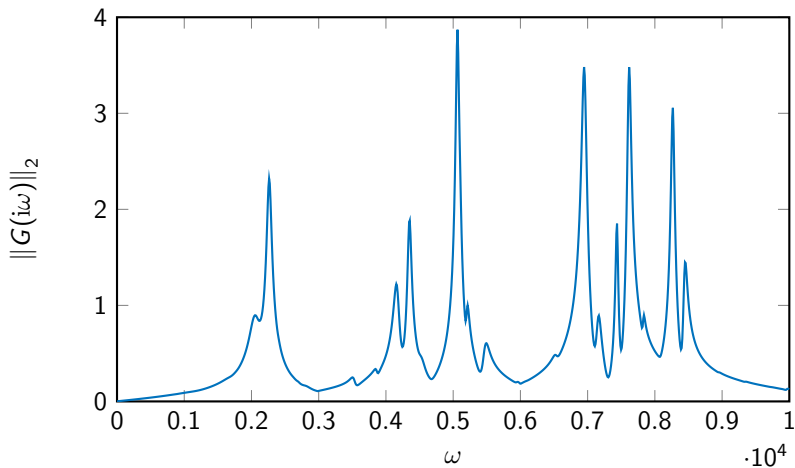
Let ω_* be a local maximizer of $\sigma_{\max}(G(i\cdot))$ such that $\sigma_{\max}(G(i\omega_*))$ is simple. If for some $k \geq 2$, ω_{k-1} , ω_k , and ω_{k+1} are sufficiently close to ω_* , then it holds

$$|\omega_{k+1} - \omega_*| \leq \nu \cdot \max\{|\omega_{k-1} - \omega_*|, |\omega_k - \omega_*|\} \cdot |\omega_k - \omega_*|.$$

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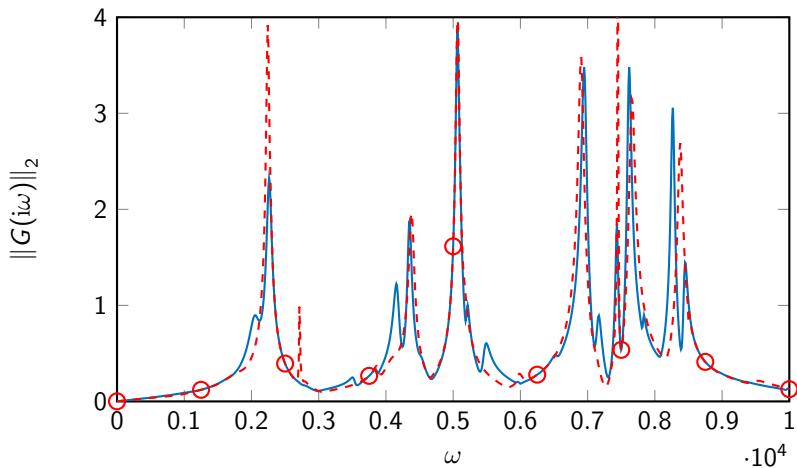
Example 1: M20PI_n

Model with $n = 1182$, $m = p = 3$.



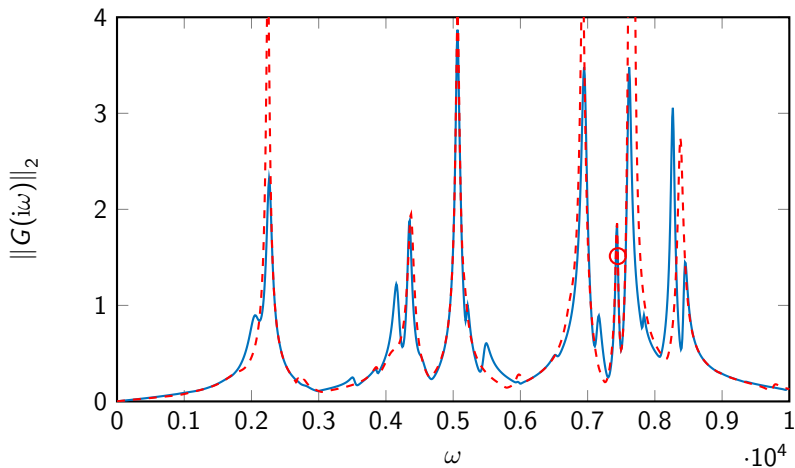
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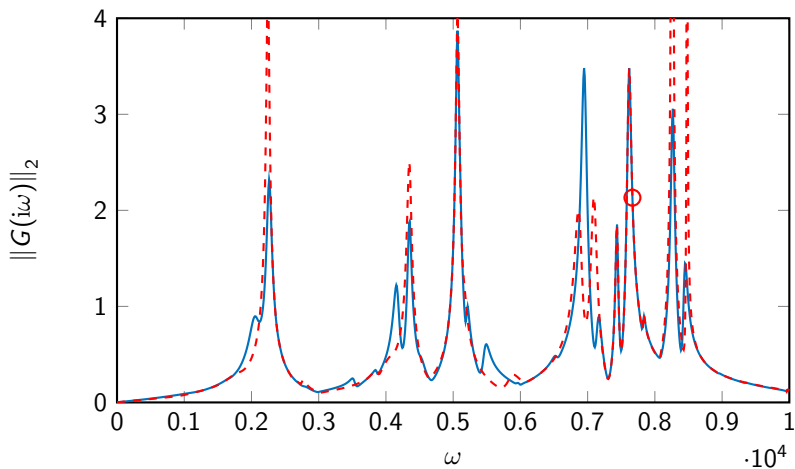
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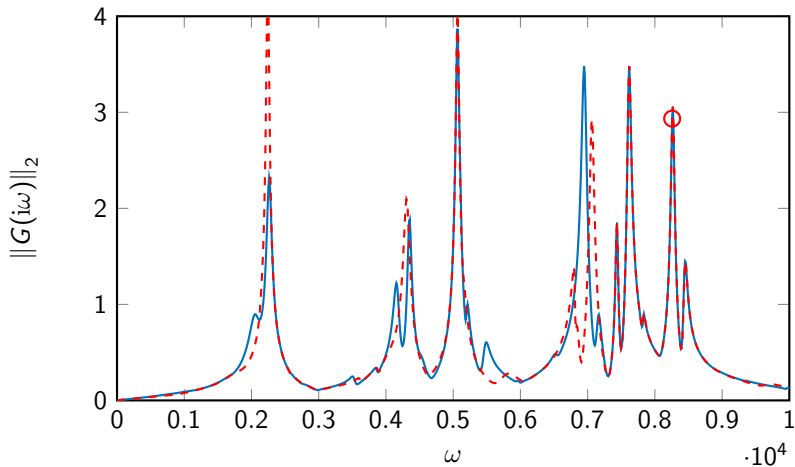
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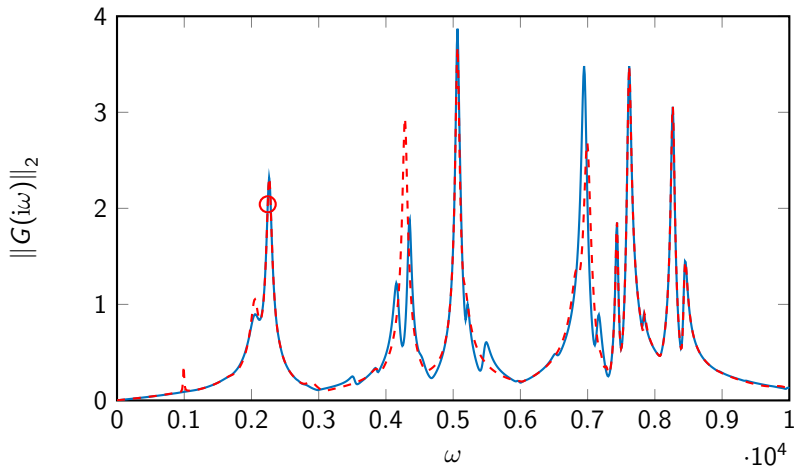
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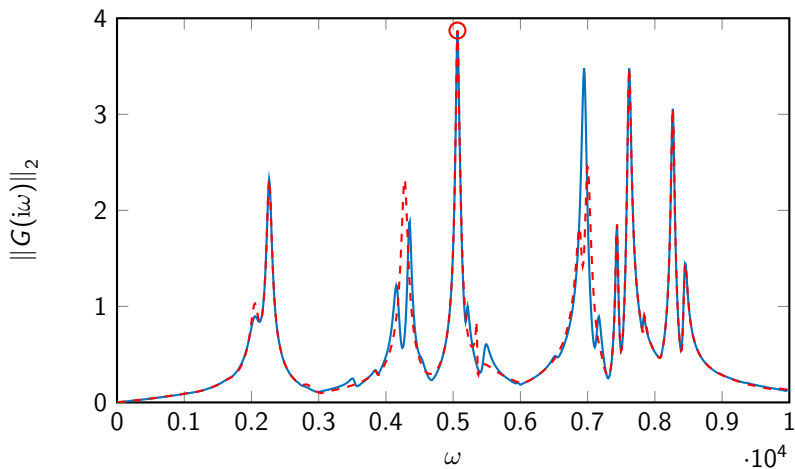
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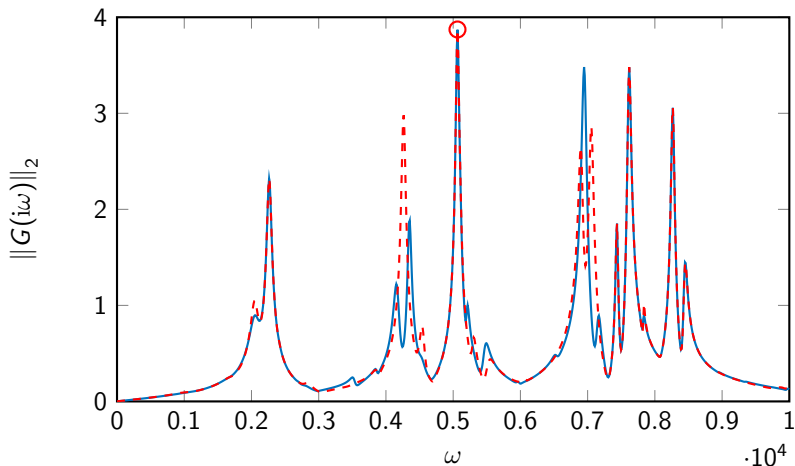
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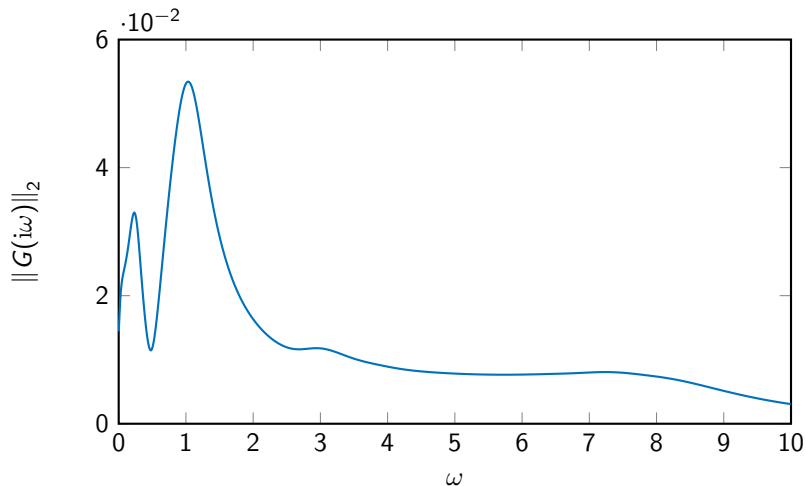
Model with $n = 1182$, $m = p = 3$.



Timings: 0.25s (pseudopole method: 4.17s, even pencil method: 1.74s)

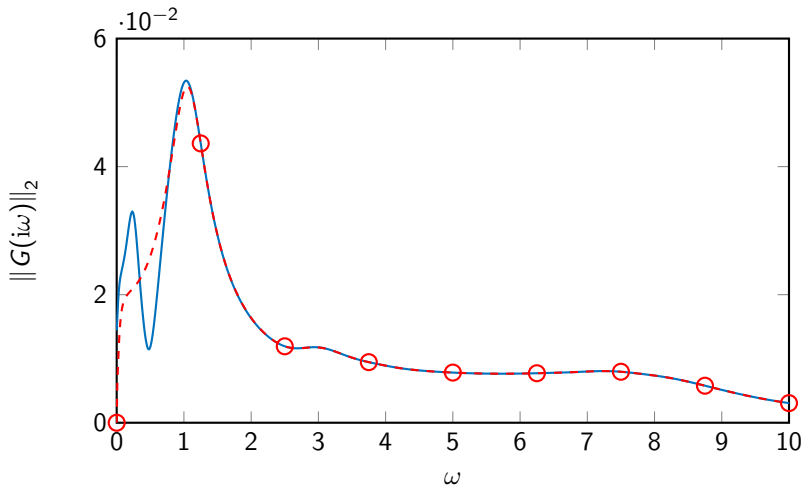
Example 2: mimo8x8_system

Model with $n = 13309$, $m = p = 8$.



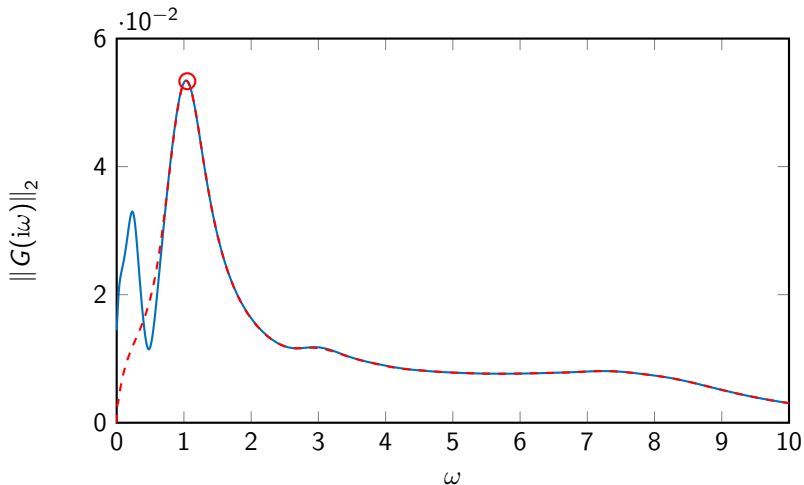
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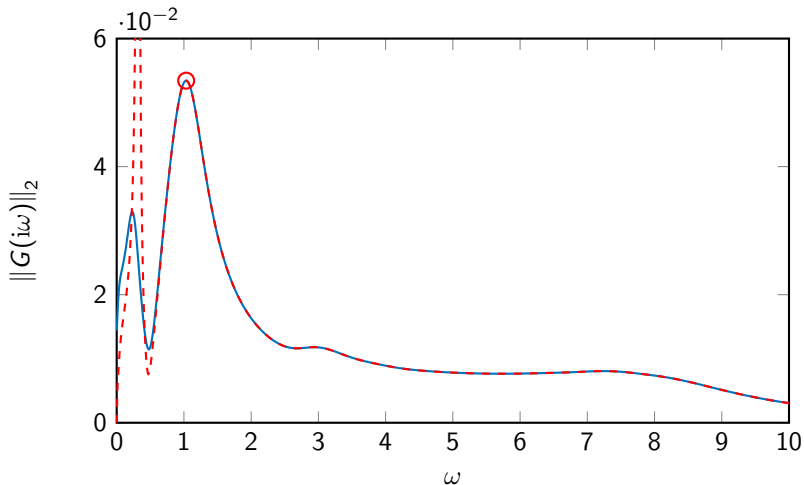
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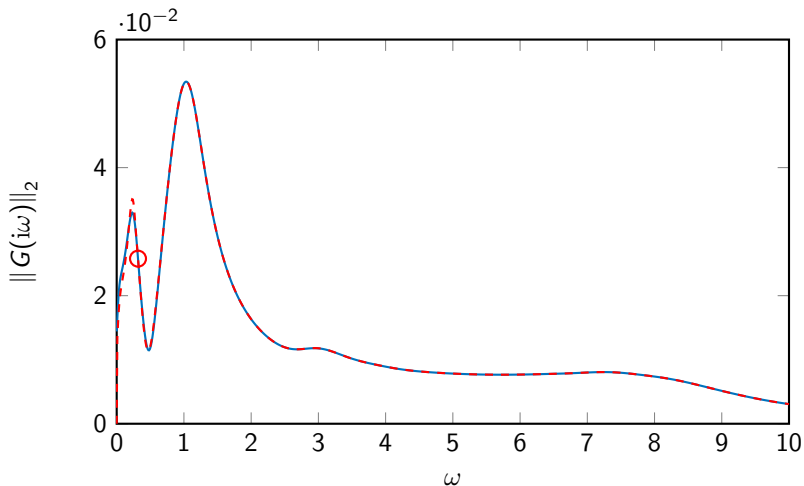
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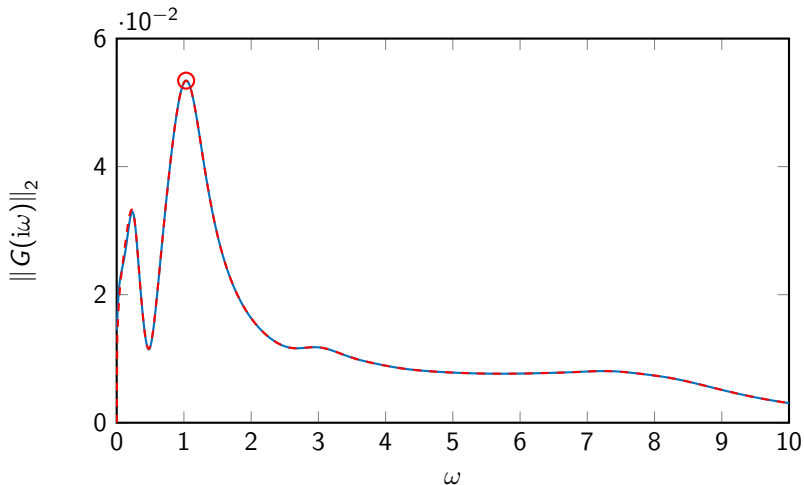
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Timings: 2.09s (pseudopole method: 27.00s, even pencil method: 8.46s)

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Conclusions and Outlook

Conclusions:

- subspace framework for computing the \mathcal{H}_∞ -norm for a general class of functions

$$G(s) = C(s)D(s)^{-1}B(s),$$

- local superlinear convergence.

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Conclusions:

- subspace framework for computing the \mathcal{H}_∞ -norm for a general class of functions

$$G(s) = C(s)D(s)^{-1}B(s),$$

- local superlinear convergence.

Outlook:

- improvement of the implementation (better choice of the initial model, better updates of the subspaces, ...)
- apply similar techniques in the context of \mathcal{H}_∞ -optimization.