Numerical Methods to Compute the $\mathcal{H}_\infty$-Norm of Large-Scale Descriptor Systems

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Descriptor systems

Consider the descriptor system

$$E\dot{x}(t) = Ax(t) + Bu(t),$$
$$y(t) = Cx(t),$$

with $E, A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, and regular $sE - A$. The associated transfer function is given by

$$G(s) = C(sE - A)^{-1}B.$$

$\mathcal{H}_\infty$-spaces

$\mathcal{RH}_\infty^{p\times m}$:= Banach space of $p \times m$ transfer functions $G(s)$ that are:

a) well-defined, i.e., $sE - A$ is a regular matrix pencil;

b) stable, i.e., all poles of $G(s)$ are in the open left half-plane;

c) proper, i.e., $G(\lambda)$ is bounded for $|\lambda| \to \infty$.

Induced norm:

$$\|G\|_{\mathcal{H}_\infty} := \sup_{\omega\in\mathbb{R}} \sigma_{\max}(G(\mathrm{i}\omega)).$$

Problem Setting

Problem

Computation of the $\mathcal{H}_\infty$-norm for systems with large and sparse E, A, B, C

Applications:

a) error measure for model order reduction;

b) performance measure in robust control.

Method 1: Structured Pseudospectra

Structured stability radius

Consider the perturbed transfer function

$$G_\Delta(s) := C(sE - (A + B\Delta C))^{-1}B$$

with $\Delta \in \mathbb{C}^{m\times p}$. Define

$$r_{\mathbb{C}}^{\mathrm{f}}(E,A,B,C) := \inf\{\|\Delta\|_2 : G_\Delta(s) \text{ is not stable}\},$$
$$r_{\mathbb{C}}^{\infty}(E,A,B,C) := \inf\{\|\Delta\|_2 : G_\Delta(s) \text{ is not well-defined or improper}\}.$$

Then

$$r_{\mathbb{C}}(E,A,B,C) := \min\left\{r_{\mathbb{C}}^{\mathrm{f}}(E,A,B,C), r_{\mathbb{C}}^{\infty}(E,A,B,C)\right\}$$

is called the structured stability radius.

Theorem

$$\|G\|_{\mathcal{H}_\infty} = \begin{cases} 1/r_{\mathbb{C}}(E,A,B,C) & \text{if } G \not\equiv 0, \\ 0 & \text{if } G \equiv 0. \end{cases}$$

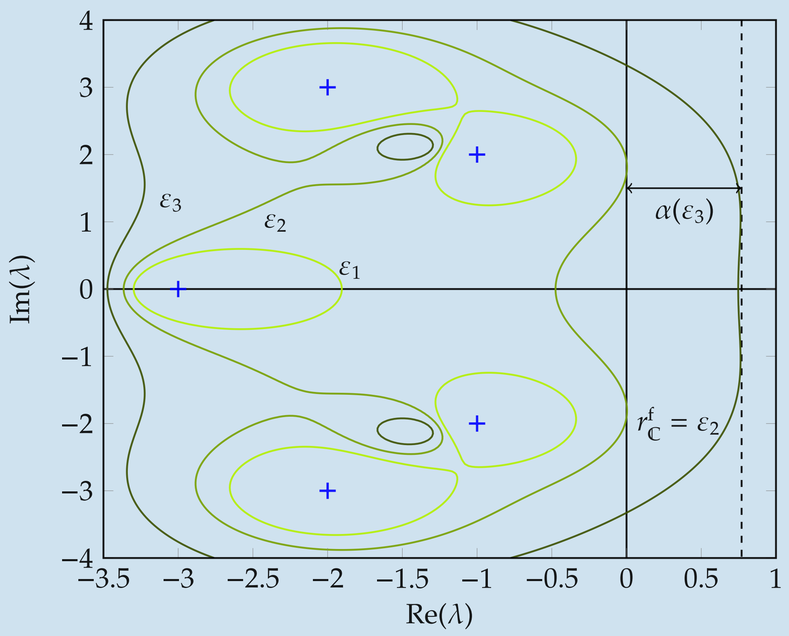
Structured pseudospectra

- set of (finite) poles of $G(s)$ denoted by $\Pi_{\mathrm{f}}(E,A,B,C)$;
- structured pseudospectrum:

$$\Pi_\varepsilon(E,A,B,C) := \Big\{\lambda \in \mathbb{C} : \lambda \in \Pi_{\mathrm{f}}(E, A + B\Delta C, B, C) \text{ for } \Delta \in \mathbb{C}^{m\times p} \text{ with } \|\Delta\|_2 < \varepsilon\Big\};$$

- structured pseudospectral abscissa:

$$\alpha(\varepsilon) := \max\{\mathrm{Re}(\lambda) : \lambda \in \Pi_\varepsilon(E,A,B,C)\}.$$



Structured pseudospectra

Algorithm outline

To compute the $\mathcal{H}_\infty$-norm we need to find the zero of $\alpha(\cdot)$.

Main Steps:

1. Choose initial ε.
2. Compute rightmost pseudopole to get $\alpha(\varepsilon)$.
3. Update ε via Newton's method and repeat Steps 2 and 3.

Software

MATLAB software is available under

`http://www2.mpi-magdeburg.mpg.de/mpcsc/voigtm/software/infnorm.php?lang=en`.

Method 2: Even Matrix Pencils

Theorem

Consider the even matrix pencil

$$\mathcal{H}_\gamma(s) := \left[\begin{array}{cc|cc} 0 & -sE^T - A^T & -C^T & 0 \\ sE - A & 0 & 0 & -B \\ \hline -C & 0 & \gamma I_p & 0 \\ 0 & -B^T & 0 & \gamma I_m \end{array}\right].$$

Assume that $sE - A$ has no purely imaginary eigenvalues and let $\gamma > \inf_{\omega\in\mathbb{R}} \sigma_{\max}(G(\mathrm{i}\omega))$. Then

a) γ is a singular value of $G(\mathrm{i}\omega_0) \iff \mathcal{H}_\gamma(s)$ has the eigenvalue $\mathrm{i}\omega_0$;

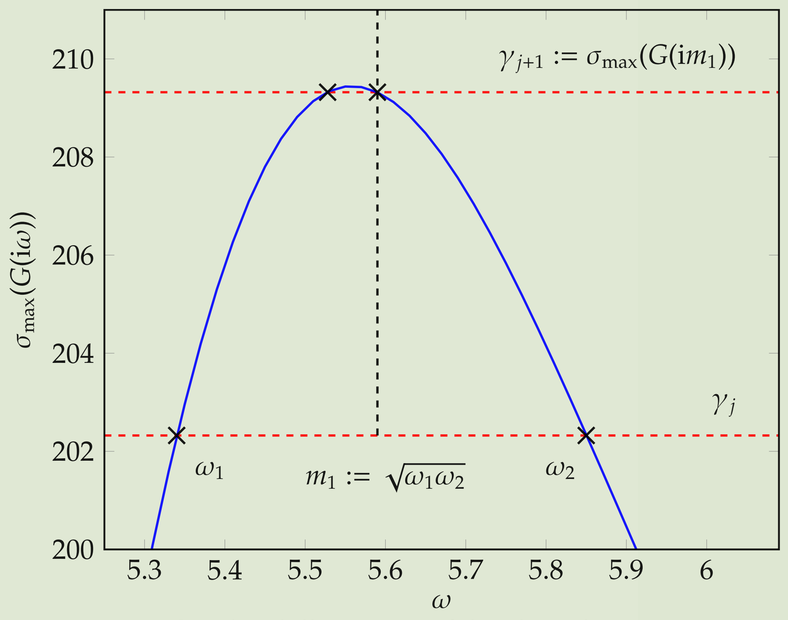
b) $\|G\|_{\mathcal{H}_\infty} \geq \gamma \iff \mathcal{H}_\gamma(s)$ has purely imaginary eigenvalues.

Eigenvalue computation

- Eigenvalues computed using the structure-preserving even IRA method;
- even IRA computes some eigenvalues close to a prespecified shift;
- heuristically, the $\mathcal{H}_\infty$-norm is attained close to a dominant pole! $\implies$ use the imaginary parts of the dominant poles as initial shifts!

Algorithm outline

1. Choose an initial value for γ.
2. Determine purely imaginary eigenvalues of $\mathcal{H}_\gamma(s)$.
3. If imaginary eigenvalues exist, increase γ and repeat Step 2.
4. If no imaginary eigenvalues exist, γ is an upper bound for $\|G\|_{\mathcal{H}_\infty}$.



Graphical illustration of the algorithm

Dominant poles

Assume that $\lambda E - A$ has only simple eigenvalues λ_k with left and right eigenvectors y_k and x_k such that $y_k^H E x_k = 1$. If $G(s)$ is proper then

$$G(s) = \sum_{k=1}^{n} \frac{R_k}{s - \lambda_k} + R_\infty$$

with residues

$$R_k = Cx_k y_k^H B, \quad R_\infty = \lim_{\omega\to\infty} G(\mathrm{i}\omega).$$

Observation: If λ_j is close to the imaginary axis and $\|R_j\|_2$ is large, then

$$G(\mathrm{i}\omega) \approx \frac{R_j}{-\mathrm{Re}(\lambda_j)} + \sum_{\substack{k=1 \\ k\neq j}}^{n} \frac{R_k}{\mathrm{i}\omega - \lambda_k} + R_\infty$$

for $\omega \approx \mathrm{Im}(\lambda_j)$ and therefore $\|G(\mathrm{i}\omega)\|_2$ is large, too.

Dominant poles:

- = the λ_j with largest $\|R_j\|_2 / \mathrm{Re}(\lambda_j)$;
- computation by the dominant pole algorithm.

Numerical Results and Comparison

Comparison of runtimes

example	n	m	p	time in s Method 1	time in s Method 2
`build`	48	1	1	1.54	0.54
`beam`	348	1	1	52.22	38.15
`M80PI_n`	4182	3	3	10.41	4.51
`bips98_1142`	9735	4	4	69.65	16.37
`xingo_afonso_itaipu`	13250	1	1	39.24	16.80
`mimo8x8_system`	13309	8	8	78.47	23.25
`mimo28x28_system`	13251	28	28	85.36	35.45
`mimo46x46_system`	13250	46	46	115.91	49.13

Summary of Method 1:

+ smaller number of user-defined parameters, easier to use;
– slower than Method 2;
– perturbed pencils are complex ⇝ complex arithmetics;
– no guaranteed global optimization.

Summary of Method 2:

+ faster than Method 1;
+ works under less restrictive conditions than Method 1;
– larger number of user-defined parameters;
– no guaranteed global optimization.

Computation of the rightmost pseudopole

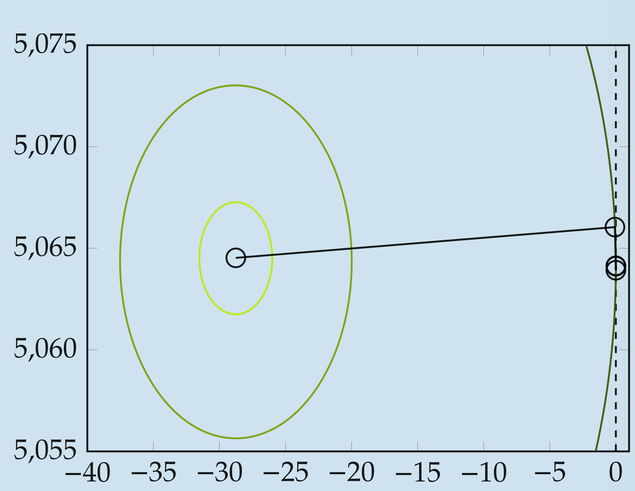
- **Useful fact:** The whole structured pseudospectrum can be obtained by using only rank-1 perturbations, i.e., $\Delta = \varepsilon uv^H$ with vectors u, v with $\|u\|_2 = \|v\|_2 = 1$.
- **Strategy:** Compute a sequence of suitable structured rank-1 perturbed transfer functions

$$G_{\varepsilon uv^H}(s) := C(sE - (A + \varepsilon Buv^H C))^{-1}B$$

such that one of the perturbed poles converges to the rightmost pseudopole of $G(s)$.

- **Optimal perturbation:** Let λ be a simple eigenvalue of $sE - A$ with right/left eigenvectors x, y. Then maximal local growth of the real part of λ for

$$u = B^T y / \|B^T y\|_2, \quad v = Cx / \|Cx\|_2.$$



Graphical illustration of an example run

References

1. P. Benner and M. Voigt: *A structured pseudospectral method for $\mathcal{H}_\infty$-norm computation of large-scale descriptor systems*, Math. Control Signals Systems, 26(1):303–338, 2014.
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3. J. Rommes and N. Martins: *Efficient computation of transfer function dominant poles using subspace acceleration*, IEEE Trans. Power Syst., 21(3):1218–1226, 2006.

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