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The Kalman-Yakubovich-Popov Lemma for Differential-Algebraic Equations with Applications

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2 Kalman-Yakubovich-Popov Lemma

3 Characterization via Even Matrix Pencils

4 Generalized Lur'e Equations and Singular Optimal Control

Introduction & Preliminaries

Differential-algebraic equations

Consider the differential-algebraic equation

$$E\dot{x}(t) = Ax(t) + Bu(t), \quad Ex(0) = Ex_0$$

with $E, A \in \mathbb{K}^{n \times n}$, $B \in \mathbb{K}^{n \times m}$, descriptor vector $x(t) \in \mathbb{K}^n$, and input vector $u(t) \in \mathbb{K}^m$.

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Spectral density functions

Define the spectral density function by

$$\Phi(s) = \begin{bmatrix} (-\bar{s}E - A)^{-1}B \\ I_m \end{bmatrix}^* \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{bmatrix} (sE - A)^{-1}B \\ I_m \end{bmatrix},$$

with $Q = Q^* \in \mathbb{K}^{n \times n}$, $S \in \mathbb{K}^{n \times m}$, $R = R^* \in \mathbb{K}^{m \times m}$.

Objective and Applications

Spectral density function

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Question:

$$\Phi(\mathrm{i}\omega) \geq 0 \quad \forall\, \mathrm{i}\omega \notin \Lambda(E, A)?$$

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Goal of this Talk: Present Equivalent Algebraic Conditions via

- linear matrix inequalities,
- algebraic matrix equations (Riccati, Lur'e),
- structured matrices and pencils.

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- linear matrix inequalities,
- algebraic matrix equations (Riccati, Lur'e),
- structured matrices and pencils.

Why is this important?

- Characterization of system properties such as dissipativity,
- structure-preserving model order reduction,
- feasibility and solution of optimal control problems.

The ODE Case ($E = I_n$) with Nonsingular R

Spectral density function

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Kalman-Yakubovich-Popov lemma

Let $[A, B]$ be controllable. Then $\Phi(\mathrm{i}\omega) \geq 0 \; \forall \; \mathrm{i}\omega \notin \Lambda(A)$ if and only if there exists $X = X^* \in \mathbb{K}^{n \times n}$ with

$$\begin{bmatrix} A^*X + XA + Q & XB + S \\ B^*X + S^* & R \end{bmatrix} \geq 0.$$

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The algebraic Riccati equation

Solvability of the LMI is equivalent to solvability of

$$A^*X + XA + Q - (XB + S)R^{-1}(B^*X + S^*) = 0, \quad X = X^*.$$

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The Hamiltonian matrix

Solvability criteria can be given in terms of the spectrum of

$$\begin{bmatrix} A - BR^{-1}S & -BR^{-1}B^* \\ S^*R^{-1}S - Q & -\left(A - BR^{-1}S\right)^* \end{bmatrix}.$$

The ODE Case with Singular R [Reis '11]

The Kalman-Yakubovich-Popov lemma still holds, but neither the algebraic Riccati equation nor the Hamiltonian matrix can be formulated!

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The Lur'e equation

Solvability of the LMI is equivalent to solvability of

$$\begin{aligned} A^*X + XA + Q &= K^*K, \\ XB + S &= K^*L, \quad X = X^* \\ R &= L^*L, \end{aligned}$$

for $(X, K, L) \in \mathbb{K}^{n\times n} \times \mathbb{K}^{p\times n} \times \mathbb{K}^{p\times m}$ and minimal p.

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The even matrix pencil

Solvability criteria can be given in terms of the spectrum of

$$\begin{bmatrix} 0 & -sI_n + A & B \\ sI_n + A^* & Q & S \\ B^* & S^* & R \end{bmatrix}.$$

Controllability Concepts for DAEs

Controllability of DAEs [Rosenbrock '74]

Further, let $\operatorname{rank} E = r$ and $S_\infty \in \mathbb{K}^{n,n-r}$ be a matrix with $\operatorname{im} S_\infty = \ker E$. Then, $[E, A, B]$ is called

(a) R-controllable, if $\operatorname{rank} \begin{bmatrix} sE - A & B \end{bmatrix} = n$ for all $s \in \mathbb{C}$;

(b) impulse controllable, if $\operatorname{rank} \begin{bmatrix} E & AS_\infty & B \end{bmatrix} = n$;

(c) strongly controllable, if it is R-controllable and impulse controllable.

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Why impulse controllability? [Bunse-Gerstner, Mehrmann, Nichols '94]

Regularization under feedback transformations:
Impulse controllability $\Leftrightarrow$ It exists $F \in \mathbb{K}^{m \times n}$ such that $sE - (A + BF)$ is regular and of index at most 1.

Kalman-Yakubovich-Popov Lemma

Kalman-Yakubovich-Popov lemma

Assume that $[E, A, B]$ is strongly controllable. Furthermore, let

$$\mathcal{V} = \left\{ \begin{pmatrix} x \\ u \end{pmatrix} \in \mathbb{K}^{n+m} \;:\; Ax + Bu \in \operatorname{im} E \right\}.$$

Then the following two statements are equivalent:

(i) $\Phi(i\omega) \geq 0$ for all $\mathrm{i}\omega \notin \Lambda(E, A)$.

(ii) There exists some $X \in \mathbb{K}^{n \times n}$ such that

$$\begin{pmatrix} x \\ u \end{pmatrix}^* \begin{bmatrix} A^*X + X^*A + Q & X^*B + S \\ B^*X + S^* & R \end{bmatrix} \begin{pmatrix} x \\ u \end{pmatrix} \geq 0, \quad E^*X = X^*E.$$

for all $\begin{pmatrix} x \\ u \end{pmatrix} \in \mathcal{V}$.

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Characterization via Even Matrix Pencils

Even matrix pencils

A matrix pencil $s\mathcal{E} - \mathcal{A}$ is called even, if $\mathcal{E}^* = -\mathcal{E}$ and $\mathcal{A}^* = \mathcal{A}$.
Spectral Property: Hamiltonian eigensymmetry, i.e.,

$$\lambda \in \Lambda(\mathcal{E}, \mathcal{A}) \Longrightarrow -\overline{\lambda} \in \Lambda(\mathcal{E}, \mathcal{A}).$$

Relation to even matrix pencils

We consider the matrix pencil

$$s\mathcal{E} - \mathcal{A} = \begin{bmatrix} 0 & -sE + A & B \\ sE^* + A^* & Q & S \\ B^* & S^* & R \end{bmatrix}.$$

Even Kronecker Canonical Form

Definition: even Kronecker canonical form [THOMPSON '76]

Let $s\mathcal{E} - \mathcal{A}$ be an even pencil. Then there exists a nonsingular $U \in \mathbb{C}^n$ such that $U^*(s\mathcal{E} - \mathcal{A})U = \operatorname{diag}(\mathcal{D}_1, \ldots, \mathcal{D}_k)$ where each $\mathcal{D}_j$, $j = 1, \ldots, k$ is of one of the following structures:

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Type 1 (finite non-imaginary eigenvalues):

$$\left[\begin{array}{cccc|cccc} & & & & -s+\mu_j & -1 & & \\ & & & & & \ddots & \ddots & \\ & & & & & & \ddots & -1 \\ & & & & & & & -s+\mu_j \\ \hline s+\bar{\mu}_j & & & & & & & \\ -1 & \ddots & & & & & & \\ & \ddots & \ddots & & & & & \\ & & -1 & s+\bar{\mu}_j & & & & \end{array}\right] \in \mathbb{C}[s]^{2\ell_j \times 2\ell_j},$$

with finite non-imaginary eigenvalues $\mu_j \in \mathbb{C}^+$, $-\bar{\mu}_j \in \mathbb{C}^-$.

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Type 2 (finite imaginary eigenvalues):

$$\varepsilon_j \begin{bmatrix} & & 1 & -s\mathrm{i} - \mu_j \\ & \iddots & \iddots & \\ 1 & \iddots & & \\ -s\mathrm{i} - \mu_j & & & \end{bmatrix} \in \mathbb{C}[s]^{\ell_j \times \ell_j},$$

with finite imaginary eigenvalues $\mathrm{i}\mu_j \in \mathrm{i}\mathbb{R}$ and block signature $\varepsilon_j \in \{-1, 1\}$.

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Type 3 (infinite eigenvalues):

$$\varepsilon_j \begin{bmatrix} & & s\mathrm{i} & 1 \\ & \iddots & \iddots & \\ s\mathrm{i} & \iddots & & \\ 1 & & & \end{bmatrix} \in \mathbb{C}[s]^{\ell_j \times \ell_j},$$

with block signature $\varepsilon_j \in \{-1, 1\}$.

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Type 4 (singular structure):

$$\left[\begin{array}{cccc|cccc} & & & & 1 & -s & & \\ & & & & & \ddots & \ddots & \\ & & & & & & 1 & -s \\ \hline 1 & & & & & & & \\ s & \ddots & & & & & & \\ & \ddots & 1 & & & & & \\ & & s & & & & & \end{array}\right] \in \mathbb{C}[s]^{(2\ell_j+1)\times(2\ell_j+1)}.$$

Spectral conditions

Theorem

Let $\operatorname{rank} E = r$ and $\operatorname{normalrank} \Phi = p$. Then the following statements are equivalent:

(i) $\Phi(i\omega) \geq 0$ for all $\mathrm{i}\omega \notin \Lambda(E, A)$.

(ii) In the EKCF of $s\mathcal{E} - \mathcal{A}$, the blocks have the following structure:

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- All blocks of Type 2 (imaginary evs) have even size and negative signature.

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- There exist exactly $2(n-r)+p$ blocks of Type 3 (infinite evs).

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- There exist exactly $2(n - r) + p$ blocks of Type 3 (infinite evs).
- There exist p blocks of Type 3 with positive signature.

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- There exist p blocks of Type 3 with positive signature.
- The remaing $2(n - r)$ blocks of Type 3 are either of even size; or the number of odd-sized blocks with positive and negative signature is equal.

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- There exist p blocks of Type 3 with positive signature.
- The remaing $2(n-r)$ blocks of Type 3 are either of even size; or the number of odd-sized blocks with positive and negative signature is equal.
- There exist exactly $m-p$ blocks of Type 4 (singular structure).

The Optimal Control Problem (Regular ODE Case)

Cost functionals

$$\mathcal{J}(x,u) = \int_0^\infty \begin{pmatrix} x(t) \\ u(t) \end{pmatrix}^* \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{pmatrix} x(t) \\ u(t) \end{pmatrix} \mathrm{d}t.$$

Optimization problem

$$V^+(x_0) = \inf \Big\{ \mathcal{J}(x,u) \ : \ \dot{x}(t) = Ax(t) + Bu(t) \\ \text{with } x(0) = x_0 \text{ and } \lim_{t\to\infty} x(t) = 0 \Big\}.$$

Questions

Under which conditions is the optimization problem feasible, i.e., $V^+(x_0) > -\infty$ for all $x_0 \in \mathbb{K}^n$? If yes, how to compute the optimal solution?

Boundedness

Theorem [Willems '71]

Let $[A, B]$ be controllable. Then the following statements are equivalent:

1. The optimization problem is feasible.
2. It exists a solution $X = X^* \in \mathbb{K}^{n \times n}$ that solves the LMI

$$\begin{bmatrix} A^*X + XA + Q & XB + S \\ B^*X + S^* & R \end{bmatrix} \geq 0.$$

Optimal Solution

Theorem [Lancaster, Rodman '95]

The optimal costs are given by

$$V(x_0) = x_0^* X^+ x_0,$$

where X^+ is the unique maximal and semi-stabilizing solution of the algebraic Riccati equation

$$A^*X + XA + Q - (XB + S)R^{-1}(B^*X + S^*) = 0, \quad X = X^*.$$

Maximalility: $X \leq X^+$ for all other solutions X of the ARE;
Semi-Stabilization: Closed-loop matrix $A - BR^{-1}(B^*X^+ + S^*)$ has only eigenvalues in the closed left half-plane.

Optimal control signal

$$\hat{u}(t) = -R^{-1}(B^*X^+ + S^*)x(t).$$

Construction of X^+

Construction of X^+ [Lancaster, Rodman '95]

Let $[A, B]$ be controllable and

$$\begin{bmatrix} A - BR^{-1}S & -BR^{-1}B^* \\ S^*R^{-1}S - Q & -\left(A - BR^{-1}S\right)^* \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} T,$$

with $Y := \begin{bmatrix} Y_1^* & Y_2^* \end{bmatrix}^* \in \mathbb{K}^{n+m,n}$ and $T \in \mathbb{K}^{n \times n}$. If Y spans a semi-stable Lagrangian invariant subspace, then

1. $X^+ = Y_2 Y_1^{-1}$;
2. $T = A - BR^{-1}(B^*X^+ + S^*)$ with $\Lambda(T) \subset \mathbb{C}^- \cup \mathrm{i}\mathbb{R}$.

The Singular Optimal Control Problem for DAEs

Cost functionals

$$\mathcal{J}(x,u) = \int_0^\infty \begin{pmatrix} x(t) \\ u(t) \end{pmatrix}^* \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{pmatrix} x(t) \\ u(t) \end{pmatrix} \mathrm{d}t.$$

Optimization problem

$$V^+(Ex_0) = \inf \Big\{ \mathcal{J}(x,u) \,:\, E\dot{x}(t) = Ax(t) + Bu(t) \text{ with } Ex(0) = Ex_0 \text{ and } \lim_{t\to\infty} Ex(t) = 0 \Big\}.$$

Questions

Under which conditions is the optimization problem feasible, i.e., $V^+(Ex_0) > -\infty$ for all $Ex_0 \in \mathbb{K}^n$? If yes, how to compute the optimal solution?

Boundedness

Theorem [Reis, V. '13]

Let $[E, A, B]$ be strongly controllable. Then the following statements are equivalent:

1. $V^+(Ex_0) > -\infty$ for all $Ex_0 \in \mathbb{K}^n$.
2. It exists a solution $X \in \mathbb{K}^{n \times n}$ that solves the LMI

$$\begin{bmatrix} A^*X + X^*A + Q & X^*B + S \\ B^*X + S^* & R \end{bmatrix} \geq_{\mathcal{V}} 0, \quad E^*X = X^*E$$

with

$$\mathcal{V} = \left\{ \begin{pmatrix} x \\ u \end{pmatrix} \in \mathbb{K}^{n+m} \ : \ Ax + Bu \in \operatorname{im} E \right\}.$$

Optimal Solution

Theorem [Reis, V. '13]

The optimal costs are given by

$$V(Ex_0) = x_0^* E^* X^+ x_0,$$

where $(X^+, K^+, L^+, V_\infty^+, W_\infty^+, \Sigma^+) \in \mathbb{K}^{n\times n} \times \mathbb{K}^{p\times n} \times \mathbb{K}^{p\times m} \times \mathbb{K}^{n-r\times n} \times \mathbb{K}^{n-r\times m} \times \mathbb{K}^{n-r\times n-r}$ with minimal p is a maximal and semi-stabilizing solution of the generalized Lur'e equation

$$\begin{aligned} A^* X + X^* A + Q &= K^* K + V_\infty^* \Sigma V_\infty, \\ X^* B + S &= K^* L + V_\infty^* \Sigma W_\infty, \\ R &= L^* L + W_\infty^* \Sigma W_\infty, \\ E^* X &= X^* E \end{aligned}$$

with $\ker \begin{bmatrix} V_\infty & W_\infty \end{bmatrix} = \mathcal{V}$ and a signature matrix Σ.

Optimal Solution

Theorem [Reis, V. '13]

$$
\begin{aligned}
A^*X + X^*A + Q &= K^*K + V_\infty^* \Sigma V_\infty, \\
X^*B + S &= K^*L + V_\infty^* \Sigma W_\infty, \\
R &= L^*L + W_\infty^* \Sigma W_\infty, \\
E^*X &= X^*E
\end{aligned}
$$

Maximality: $E^*X \leq E^*X^+$ for all other solutions $(X, K, L, V_\infty, W_\infty, \Sigma)$ of the generalized Lur'e equation;
Uniqueness: When X_1^+ and X_2^+ are parts of two maximal solutions then $E^*X_1^+ = E^*X_2^+$.
Semi-Stabilization: Closed-loop matrix pencil

$$
\begin{bmatrix} -sE + A & B \\ K^+ & L^+ \end{bmatrix}
$$

has only eigenvalues in the closed left half-plane.

Optimal Control Signal [Reis, V. '13]

Optimal control signal

The optimal control signal $\hat{u}(\cdot)$ fulfills the DAE

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + B\hat{u}(t) + \delta_0 E x_0, \\ 0 &= K^+ x(t) + L^+ \hat{u}(t) \end{aligned}$$

(in the distributional sense).

Construction of X^+ [Reis, V. '13]

Let $[E, A, B]$ be strongly controllable and

$$\begin{bmatrix} 0 & -sE+A & B \\ sE^*+A^* & Q & S \\ B^* & S^* & R \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix} = \begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \end{bmatrix} \left(s\tilde{\mathcal{E}} - \tilde{\mathcal{A}}\right)$$

with $Y := \begin{bmatrix} Y_1^* & Y_2^* & Y_3^* \end{bmatrix}^* \in \mathbb{K}^{2n+m \times n+m}$ and $Z := \begin{bmatrix} Z_1^* & Z_2^* & Z_3^* \end{bmatrix}^* \in \mathbb{K}^{2n+m \times n+p}$ and $s\tilde{\mathcal{E}} - \tilde{\mathcal{A}} \in \mathbb{K}[s]^{n+p \times n+m}$. If Y spans a semi-stable $\mathcal{E}$-neutral right deflating subspace of dimension $n+m$, then

1. $X^+ = Y_2 Y_1^-$ with an arbitrary right inverse Y_1^- of Y_1;
2. $s\tilde{\mathcal{E}} - \tilde{\mathcal{A}} = \begin{bmatrix} -sE+A & B \\ K^+ & L^+ \end{bmatrix}$ with $\Lambda\left(\tilde{\mathcal{E}}, \tilde{\mathcal{A}}\right) \subset \mathbb{C}^- \cup \mathrm{i}\mathbb{R} \cup \{\infty\}$.

Conclusions

What have we shown in this talk

- Kalman-Yakubovich-Popov lemma for DAEs;
- relation to even matrix pencils;
- linear-quadratic optimal control using generalized Lur'e equations.

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- Kalman-Yakubovich-Popov lemma for DAEs;
- relation to even matrix pencils;
- linear-quadratic optimal control using generalized Lur'e equations.

What have we not shown in this talk

- Factorization of spectral density functions (spectral factorization);
- application to dissipative systems.

Conclusions

What have we shown in this talk

- Kalman-Yakubovich-Popov lemma for DAEs;
- relation to even matrix pencils;
- linear-quadratic optimal control using generalized Lur'e equations.

What have we not shown in this talk

- Factorization of spectral density functions (spectral factorization);
- application to dissipative systems.

Open problems

- How to solve generalized Lur'e equations;
- generalization to $\mathcal{J}$-spectral factorization.

Thanks for your Attention!

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