51th IEEE Conference on Decision and Control
Section "Stability of Linear Systems"
Wailea, Hawaii, USA
December 10-13, 2012

Numerical Computation of Structured Complex Stability Radii of Large-Scale Matrices and Pencils

Peter Benner **Matthias Voigt**

Computational Methods in Systems and Control Theory
Max Planck Institute for Dynamics of Complex Technical Systems
Magdeburg, Germany

MAX PLANCK INSTITUTE
FOR DYNAMICS OF COMPLEX
TECHNICAL SYSTEMS
MAGDEBURG

Continuous-Time Linear Systems

Given: Continuous-time LTI system

$$\Sigma : \quad \begin{cases} E\dot{x}(t) = Ax(t) + Bu(t), \\ \; y(t) = Cx(t) \end{cases}$$

- E, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, $m, p \ll n$,
- descriptor vector $x(t) \in \mathbb{R}^n$, input vector $u(t) \in \mathbb{R}^m$, output vector $y(t) \in \mathbb{R}^p$,
- E is possibly singular, but $\lambda E - A$ is regular, i.e. $\det(\lambda E - A) \neq 0$.
- Main assumption: all matrices are large and sparse.

Problem of this Talk

Imagine you have designed an output feedback controller K of your system, i.e. $u(t) = Ky(t) = KCx(t)$.

Problem of this Talk

Imagine you have designed an output feedback controller K of your system, i.e. $u(t) = Ky(t) = KCx(t)$.

Closed-loop system

$$E\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t)$$
$$\rightsquigarrow E\dot{x}(t) = (A + BKC)\,x(t) \quad \text{(asymptotically stable)}$$

Problem of this Talk

Question

- **Given:** Asymptotically stable (closed-loop) system $(\lambda E - A, B, C)$, i.e. all *finite* eigenvalues of $\lambda E - A$ in open left half-plane.
- Consider perturbations in the controller, i.e. $K \rightsquigarrow K + \Delta$.
- Corresponding perturbation in the pencil:

$$\lambda E - A \rightsquigarrow \lambda E - (A + B\Delta C)$$

- **Question:** How large is the smallest complex perturbation Δ that destabilizes the system? ($\Rightarrow$ structured complex stability radius $r_{\mathbb{C}}$)

Literature Review

Problem solved for small & dense problems by using that under some conditions

$$r_{\mathbb{C}} > \gamma \Leftrightarrow \lambda \mathcal{S} - \mathcal{H}_\gamma := \lambda \begin{bmatrix} E & 0 \\ 0 & E^T \end{bmatrix} - \begin{bmatrix} A & \gamma B B^T \\ -\gamma C^T C & -A^T \end{bmatrix}$$

has no purely imaginary eigenvalues.

Literature Review

Problem solved for small & dense problems by using that under some conditions

$$r_{\mathbb{C}} > \gamma \Leftrightarrow \lambda \mathcal{S} - \mathcal{H}_\gamma := \lambda \begin{bmatrix} E & 0 \\ 0 & E^T \end{bmatrix} - \begin{bmatrix} A & \gamma B B^T \\ -\gamma C^T C & -A^T \end{bmatrix}$$

has no purely imaginary eigenvalues.

- Byers '88: First work: bisection algorithm for unstructured matrix case,
- Boyd, Balakrishnan, Kabamba '89: structured case,
- Bruinsma, Steinbuch '90: quadratically convergent algorithm,
- Benner, Sima, V. '12: descriptor systems, structure-preserving eigensolver.

Literature Review

Problem solved for small & dense problems by using that under some conditions

$$r_{\mathbb{C}} > \gamma \Leftrightarrow \lambda \mathcal{S} - \mathcal{H}_\gamma := \lambda \begin{bmatrix} E & 0 \\ 0 & E^T \end{bmatrix} - \begin{bmatrix} A & \gamma B B^T \\ -\gamma C^T C & -A^T \end{bmatrix}$$

has no purely imaginary eigenvalues.

- Byers '88: First work: bisection algorithm for unstructured matrix case,
- Boyd, Balakrishnan, Kabamba '89: structured case,
- Bruinsma, Steinbuch '90: quadratically convergent algorithm,
- Benner, Sima, V. '12: descriptor systems, structure-preserving eigensolver.

Drawback of these methods

- Need to compute all purely imaginary eigenvalues of a structured matrix/pencil,
- unsolved problem for large-scale systems,
- need new method that can exploit sparsity structure.

The Way Out: Structured Pseudospectra

Definitions

- structured complex stability radius $r_{\mathbb{C}}$:

$$r_{\mathbb{C}} = \inf \left\{ \|\Delta\|_2 : \Lambda_{\mathrm{f}}(E, A + B\Delta C) \cap \mathrm{i}\mathbb{R} \neq \emptyset \right\},$$

- structured pseudospectrum Λ_ε:

$$\Lambda_\varepsilon = \left\{ s \in \mathbb{C} : s \in \Lambda_{\mathrm{f}}(E, A + B\Delta C) \text{ for } \Delta \in \mathbb{C}^{m \times p} \text{ with } \|\Delta\|_2 < \varepsilon \right\},$$

- structured pseudospectral abscissa $\alpha(\varepsilon)$:

$$\alpha(\varepsilon) := \max \left\{ \operatorname{Re} s : s \in \Lambda_\varepsilon \right\}.$$

The Way Out: Structured Pseudospectra

Definitions

- structured complex stability radius $r_{\mathbb{C}}$:

$$r_{\mathbb{C}} = \inf \left\{ \|\Delta\|_2 : \Lambda_f(E, A + B\Delta C) \cap i\mathbb{R} \neq \emptyset \right\},$$

- structured pseudospectrum Λ_ε:

$$\Lambda_\varepsilon = \left\{ s \in \mathbb{C} : s \in \Lambda_f(E, A + B\Delta C) \text{ for } \Delta \in \mathbb{C}^{m \times p} \text{ with } \|\Delta\|_2 < \varepsilon \right\},$$

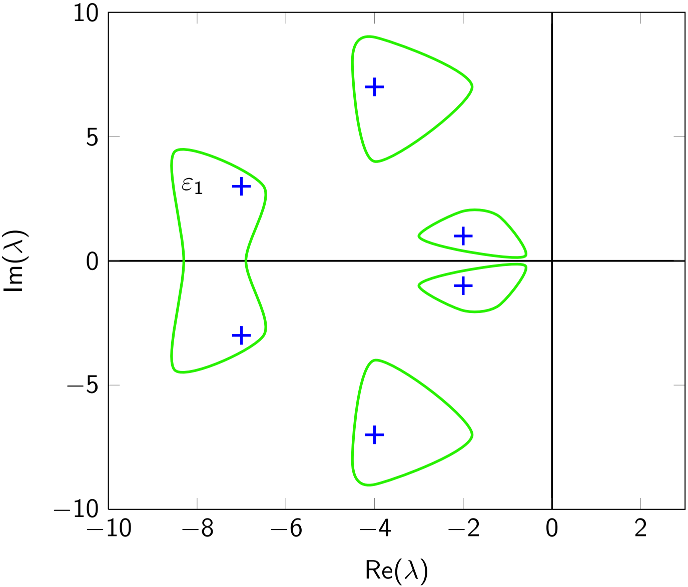
- structured pseudospectral abscissa $\alpha(\varepsilon)$:

$$\alpha(\varepsilon) := \max \left\{ \operatorname{Re} s : s \in \Lambda_\varepsilon \right\}.$$

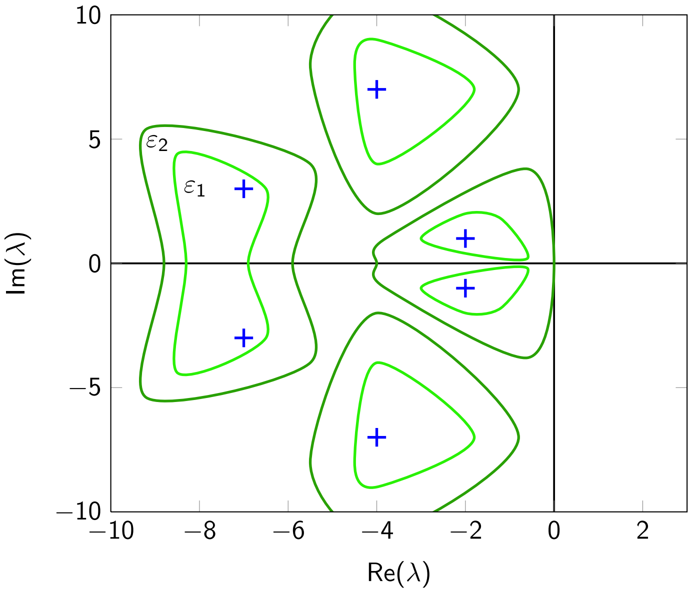
WARNING

In the pencil case, the infinite eigenvalues need special attention!

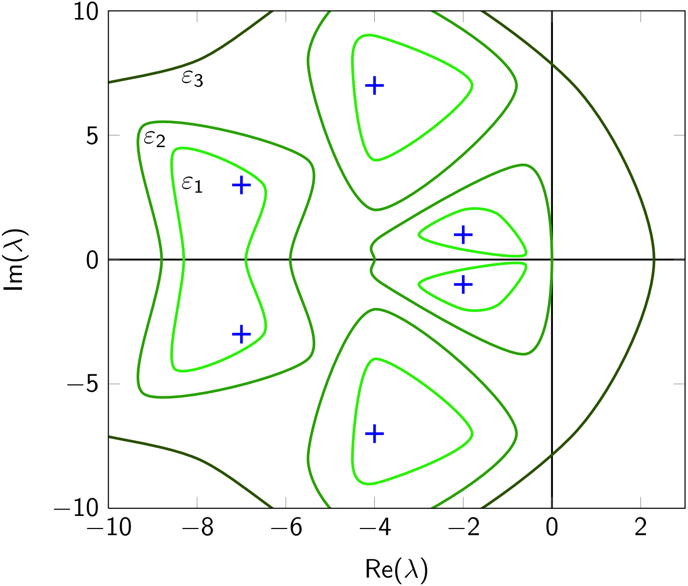
Graphical Interpretation



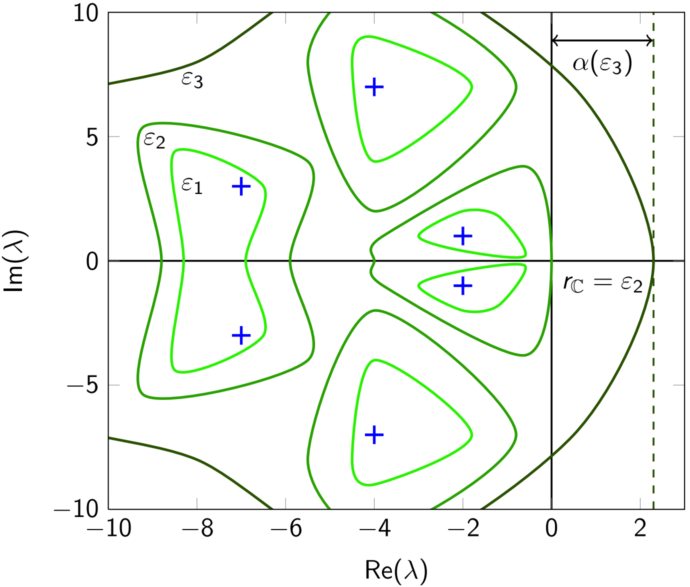
Graphical Interpretation



Graphical Interpretation



Graphical Interpretation



Algorithm Outline

Finding $r_{\mathbb{C}}$ is equivalent to finding the (unique) root of $\alpha(\varepsilon)$. Thus we apply a root-finding algorithm. We

- do not have derivative information $\Longrightarrow$ Newton-like method not possible,
- can evaluate $\alpha(\varepsilon)$ relatively cheaply $\Longrightarrow$ secant method.

Algorithm Outline

Finding $r_{\mathbb{C}}$ is equivalent to finding the (unique) root of $\alpha(\varepsilon)$. Thus we apply a root-finding algorithm. We

- do not have derivative information $\Longrightarrow$ Newton-like method not possible,
- can evaluate $\alpha(\varepsilon)$ relatively cheaply $\Longrightarrow$ secant method.

Sketch of the algorithm

1. Choose initial ε.
2. Compute rightmost pseudoeigenvalue to get $\alpha(\varepsilon)$.
3. Update ε and repeat Steps 2 and 3.

Algorithm Outline

Finding $r_{\mathbb{C}}$ is equivalent to finding the (unique) root of $\alpha(\varepsilon)$. Thus we apply a root-finding algorithm. We

- do not have derivative information $\Longrightarrow$ Newton-like method not possible,
- can evaluate $\alpha(\varepsilon)$ relatively cheaply $\Longrightarrow$ secant method.

Sketch of the algorithm

1. Choose initial ε.
2. Compute rightmost pseudoeigenvalue to get $\alpha(\varepsilon)$.
3. Update ε and repeat Steps 2 and 3.

Perturbation Strategy

"Theorem"

The whole structured pseudospectrum can be obtained by using only rank-1 perturbations, i.e. $\Delta = uv^H$ with vectors u, v.

Perturbation Strategy

"Theorem"

The whole structured pseudospectrum can be obtained by using only rank-1 perturbations, i.e. $\Delta = uv^H$ with vectors u, v.

Strategy [Guglielmi, Overton '11]

Compute a sequence of suitable structured rank-1 perturbed pencils $\lambda E - (A + \varepsilon B u v^H C)$ such that one of the perturbed eigenvalues converges to the rightmost pseudoeigenvalue of $\lambda E - A$!

First-Order Perturbation Theory

Lemma [Stewart, Sun '90]

Let x, y be right and left eigenvectors corresponding to a simple finite eigenvalue λ of the pencil $\lambda E - A$. Let $\lambda E - (A + tBuv^H C)$ be a perturbed matrix pencil with eigenvalue $\tilde{\lambda}$. Then it holds

$$\tilde{\lambda} = \lambda + t\frac{y^H Buv^H Cx}{y^H Ex} + \mathcal{O}\left(t^2\right).$$

First-Order Perturbation Theory

Lemma [Stewart, Sun '90]

Let x, y be right and left eigenvectors corresponding to a simple finite eigenvalue λ of the pencil $\lambda E - A$. Let $\lambda E - (A + tBuv^H C)$ be a perturbed matrix pencil with eigenvalue $\tilde{\lambda}$. Then it holds

$$\tilde{\lambda} = \lambda + t\frac{y^H Buv^H Cx}{y^H Ex} + \mathcal{O}\left(t^2\right).$$

Corollary

$$\left.\frac{\mathrm{d}\tilde{\lambda}(t)}{\mathrm{d}t}\right|_{t=0} = \frac{y^H Buv^H Cx}{y^H Ex}.$$

Construction of Structured Rank-1 Perturbations

Given:

- Pencil $\lambda E - A$ with simple eigenvalue λ, right/left eigenvectors x, y, $y^H E x > 0$;
- vectors $u \in \mathbb{C}^m$, $v \in \mathbb{C}^p$ with $\|u\|_2 = \|v\|_2 = 1$.

Construction of Structured Rank-1 Perturbations

Given:

- Pencil $\lambda E - A$ with simple eigenvalue λ, right/left eigenvectors x, y, $y^H E x > 0$;
- vectors $u \in \mathbb{C}^m$, $v \in \mathbb{C}^p$ with $\|u\|_2 = \|v\|_2 = 1$.

$$\operatorname{Re}\left(\left. \frac{\mathrm{d}\tilde{\lambda}(t)}{\mathrm{d}t} \right|_{t=0} \right) = \frac{\operatorname{Re}\left(y^H B u v^H C x\right)}{y^H E x} \leq \frac{\left\|y^H B\right\|_2 \|Cx\|_2}{y^H E x}.$$

Construction of Structured Rank-1 Perturbations

Given:

- Pencil $\lambda E - A$ with simple eigenvalue λ, right/left eigenvectors x, y, $y^H E x > 0$;
- vectors $u \in \mathbb{C}^m$, $v \in \mathbb{C}^p$ with $\|u\|_2 = \|v\|_2 = 1$.

$$\operatorname{Re}\left(\left. \frac{\mathrm{d}\tilde{\lambda}(t)}{\mathrm{d}t} \right|_{t=0} \right) = \frac{\operatorname{Re}\left(y^H B u v^H C x\right)}{y^H E x} \leq \frac{\left\|y^H B\right\|_2 \|Cx\|_2}{y^H E x}.$$

Equality holds for

$$u = \frac{B^T y}{\|B^T y\|_2}, \quad v = \frac{Cx}{\|Cx\|_2}.$$

$\Longrightarrow$ This choice of u, v yields locally maximal growth in $\operatorname{Re}(\tilde{\lambda}(t))$ as t increases from 0.

Subsequent Steps

Given:

- Perturbed pencil $\lambda E - \hat{A} = \lambda E - (A + \varepsilon B \hat{u} \hat{v}^H C)$ with simple eigenvalue $\hat{\lambda}$, left/right eigenvectors $\hat{x}$, $\hat{y}$, $\hat{y}^H E \hat{x} > 0$;
- vectors $u \in \mathbb{C}^m$, $v \in \mathbb{C}^p$ with $\|u\|_2 = \|v\|_2 = 1$.

Subsequent Steps

Given:

- Perturbed pencil $\lambda E - \hat{A} = \lambda E - (A + \varepsilon B \hat{u} \hat{v}^H C)$ with simple eigenvalue $\hat{\lambda}$, left/right eigenvectors $\hat{x}$, $\hat{y}$, $\hat{y}^H E \hat{x} > 0$;
- vectors $u \in \mathbb{C}^m$, $v \in \mathbb{C}^p$ with $\|u\|_2 = \|v\|_2 = 1$.

Consider

$$\lambda E - \left(\hat{A} + tB \left(uv^H - \hat{u}\hat{v}^H \right) C \right),$$

which is an ε-norm rank-1 perturbation of $\lambda E - A$ for $t = 0$, $t = \varepsilon$.

Subsequent Steps

Given:

- Perturbed pencil $\lambda E - \hat{A} = \lambda E - (A + \varepsilon B \hat{u} \hat{v}^H C)$ with simple eigenvalue $\hat{\lambda}$, left/right eigenvectors $\hat{x}$, $\hat{y}$, $\hat{y}^H E \hat{x} > 0$;
- vectors $u \in \mathbb{C}^m$, $v \in \mathbb{C}^p$ with $\|u\|_2 = \|v\|_2 = 1$.

Consider

$$\lambda E - \left(\hat{A} + tB\left(uv^H - \hat{u}\hat{v}^H\right)C\right),$$

which is an ε-norm rank-1 perturbation of $\lambda E - A$ for $t = 0$, $t = \varepsilon$.

$$\operatorname{Re}\left(\left.\frac{\mathrm{d}\tilde{\lambda}(t)}{\mathrm{d}t}\right|_{t=0}\right) = \frac{\operatorname{Re}\left(\hat{y}^H B\left(uv^H - \hat{u}\hat{v}^H\right)C\hat{x}\right)}{\hat{y}^H E \hat{x}} \leq \frac{\left\|\hat{y}^H B\right\|_2 \left\|C\hat{x}\right\|_2 - \operatorname{Re}\left(\hat{y}^H B \hat{u}\hat{v}^H C \hat{x}\right)}{\hat{y}^H E \hat{x}}.$$

Subsequent Steps

Given:

- Perturbed pencil $\lambda E - \hat{A} = \lambda E - (A + \varepsilon B \hat{u} \hat{v}^H C)$ with simple eigenvalue $\hat{\lambda}$, left/right eigenvectors $\hat{x}$, $\hat{y}$, $\hat{y}^H E \hat{x} > 0$;
- vectors $u \in \mathbb{C}^m$, $v \in \mathbb{C}^p$ with $\|u\|_2 = \|v\|_2 = 1$.

Consider

$$\lambda E - \left(\hat{A} + tB \left(uv^H - \hat{u}\hat{v}^H \right) C \right),$$

which is an ε-norm rank-1 perturbation of $\lambda E - A$ for $t = 0$, $t = \varepsilon$.

$$\operatorname{Re}\left(\left. \frac{\mathrm{d}\tilde{\lambda}(t)}{\mathrm{d}t} \right|_{t=0} \right) = \frac{\operatorname{Re}\left(\hat{y}^H B \left(uv^H - \hat{u}\hat{v}^H \right) C \hat{x} \right)}{\hat{y}^H E \hat{x}} \leq \frac{\left\| \hat{y}^H B \right\|_2 \left\| C \hat{x} \right\|_2 - \operatorname{Re}\left(\hat{y}^H B \hat{u} \hat{v}^H C \hat{x} \right)}{\hat{y}^H E \hat{x}}.$$

Again, equality holds for

$$u = \frac{B^T \hat{y}}{\|B^T \hat{y}\|_2}, \quad v = \frac{C\hat{x}}{\|C\hat{x}\|_2}.$$

Choice of the Eigenvalues

We showed how to optimally perturb a chosen eigenvalue!
But: Which one is the best choice?

Choice of the Eigenvalues

We showed how to optimally perturb a chosen eigenvalue!
But: Which one is the best choice? It should

- have a sufficiently large real part,
- be sufficiently controllable and observable, i.e., $\left\|B^T y\right\|_2$ and $\|Cx\|_2$ are "large"

to get fast convergence (to the correct maximizer).

Choice of the Eigenvalues

We showed how to optimally perturb a chosen eigenvalue!
But: Which one is the best choice? It should

- have a sufficiently large real part,
- be sufficiently controllable and observable, i.e., $\left\|B^T y\right\|_2$ and $\|Cx\|_2$ are "large"

to get fast convergence (to the correct maximizer).
Idea: Use an iterative eigensolver which converges to the eigenvalues which have highest "dominance" with respect to some predefined measure!

Choice of the Eigenvalues

We showed how to optimally perturb a chosen eigenvalue!
But: Which one is the best choice? It should

- have a sufficiently large real part,
- be sufficiently controllable and observable, i.e., $\left\|B^T y\right\|_2$ and $\|Cx\|_2$ are "large"

to get fast convergence (to the correct maximizer).
Idea: Use an iterative eigensolver which converges to the eigenvalues which have highest "dominance" with respect to some predefined measure!
$\Longrightarrow$ Subspace Accelerated MIMO Dominant Pole Algorithm (SAMDP)

[Rommes, Martins '06]

Sketch of the Algorithm

Sketch of the algorithm

1. Choose dominant eigenvalue λ with right/left eigenvectors x, y.
2. Construct the perturbed pencil $\lambda E - \left(A + \varepsilon \frac{BB^T y x^H C^T C}{\|B^T y\|_2 \|Cx\|_2}\right)$.
3. Repeat Steps 1 and 2 until convergence.

Example 1 – M20PI_n

Model with $n = 1182$, $m = p = 3$.
Results: $r_{\mathbb{C}} = 2.58224\text{e-}01$, $t = 6.03s$, $\alpha(r_{\mathbb{C}}) = -3.9700\text{e-}13$.

Table: Convergence History

	k			
	1	2	3	4
$\text{Re}(\lambda_{\text{dom}})$	-6.7945e-02	-6.0215e+00	-3.7397e-04	3.6222e-11
	2.3140e-03	-6.0212e+00	-3.4533e-05	3.9094e-11
	3.0285e-03	—	-3.2591e-05	3.8420e-11
	3.0355e-03	—	-3.2572e-05	—
	3.0356e-03	—	—	—
ε_k	2.58250e-01	2.06600e-01	2.58224e-01	2.58224e-01

Example 1 – M20PI_n

Model with $n = 1182$, $m = p = 3$.
Results: $r_{\mathbb{C}} = 2.58224\text{e-}01$, $t = 6.03s$, $\alpha(r_{\mathbb{C}}) = \text{-}3.9700\text{e-}13$.

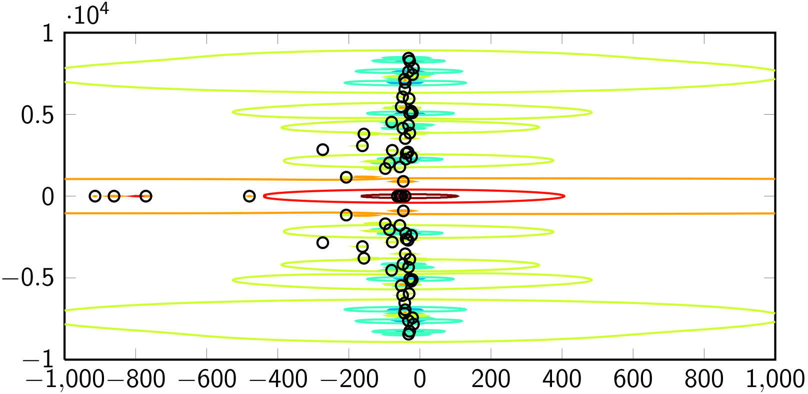


Figure: Structured pseudospectra

Example 1 – M20PI_n

Model with $n = 1182,\ m = p = 3$.
Results: $r_{\mathbb{C}} = 2.58224\text{e-}01, \quad t = 6.03s, \quad \alpha(r_{\mathbb{C}}) = \text{-}3.9700\text{e-}13$.

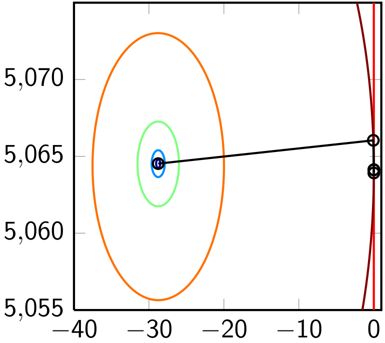


Figure: Iteration 1

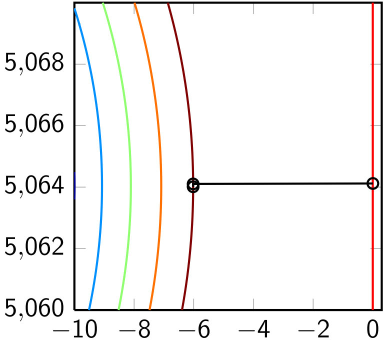


Figure: Iteration 2

Example 1 – M20PI_n

Model with $n = 1182$, $m = p = 3$.
Results: $r_{\mathbb{C}} = 2.58224\text{e-}01$, $t = 6.03s$, $\alpha(r_{\mathbb{C}}) = \text{-}3.9700\text{e-}13$.

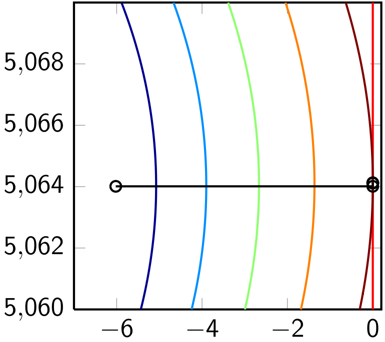


Figure: Iteration 3

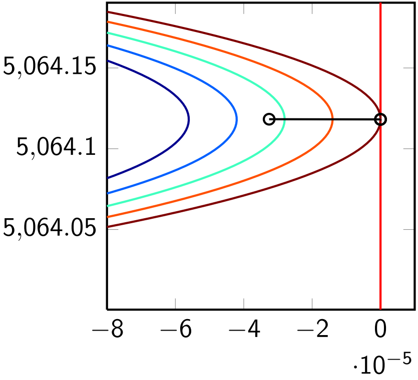


Figure: Iteration 4

Example 2 – Mimo8x8_System

Model with $n = 13309$, $m = p = 8$.
Results: $r_{\mathbb{C}} = 1.87164\text{e}{+}01$, $t = 106.62s$, $\alpha(r_{\mathbb{C}}) = 2.6335\text{e-}13$.

Table: Convergence History

	k			
	1	2	3	4
$\mathrm{Re}(\lambda_{\mathrm{dom}})$	-6.2051e-03	-4.8351e-02	-9.0793e-05	-1.4183e-09
	-6.3276e-04	-4.8266e-02	-9.4865e-06	-1.3415e-09
	3.8109e-06	-4.8253e-02	3.0458e-06	-1.3273e-09
	1.1425e-04	—	5.4062e-06	-1.3245e-09
	1.3425e-04	—	5.8487e-06	-1.3241e-09
	1.3794e-04	—	5.9315e-06	—
	1.3862e-04	—	5.9470e-06	—
	1.3875e-04	—	5.9498e-06	—
ε_k	1.87276e+01	1.49821e+01	1.87168e+01	1.87164e+01

Example 3 – Mimo46x46_System

Model with $n = 13250$, $m = p = 46$.
Results: $r_{\mathbb{C}} = 4.86309\text{e-}03$, $t = 167.43s$, $\alpha(r_{\mathbb{C}}) = 4.2864\text{e-}14$.

Table: Convergence History

	k		
	1	2	3
$\mathrm{Re}(\lambda_{\mathrm{dom}})$	-9.0777e-05	-6.6047e-03	-3.6061e-06
	2.0799e-06	-6.6018e-03	2.7127e-09
	2.1973e-06	—	7.3774e-09
	2.1976e-06	—	7.3907e-09
	—	—	7.3907e-09
ε_k	4.86342e-03	3.89073e-03	4.86309e-03

The Message

The talk in two sentences

Eigenvalue perturbations are a powerful tool for stability and robustness analysis of linear systems! Optimal perturbations can be efficiently computed!

The Message

The talk in two sentences

Eigenvalue perturbations are a powerful tool for stability and robustness analysis of linear systems! Optimal perturbations can be efficiently computed!

Thanks for Listening!

References

- Benner, Voigt '12: *Numerical computation of structured complex stability radii of large-scale matrices and pencils*, Proceedings of the 51th IEEE CDC, 2012.
- Benner, Sima, Voigt '12: *$\mathcal{L}_\infty$-norm computation for continuous-time descriptor systems using structured matrix pencils*, IEEE Trans. Automat. Control, 57(1), 2012, pp. 233–238.
- Guglielmi, Overton '11: *Fast algorithms for the approximation of the pseudospectral abscissa and pseudospectral radius of a matrix*, SIAM J. Matrix Anal. Appl., 32(4), 2011, pp. 1166–1192.
- Stewart, Sun '90: *Matrix Perturbation Theory*, Academic Press, New York, 1990.
- Rommes, Martins '06: *Efficient computation of transfer function dominant poles using subspace acceleration*, IEEE Trans. Power Syst., 21(3), Aug. 2006, pp. 1218–1226.