Linear-Quadratic Optimal Control, Lur'e Equations, and Structured Matrix Pencils

Daniel Bankmann[1] Timo Reis[2] Olaf Rendel[2] **Matthias Voigt[1]**

[1]Technische Universität Berlin
Institut für Mathematik

[2]Universität Hamburg
Fachbereich Mathematik

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The Linear-Quadratic Control Problem for ODEs

Classical linear-quadratic optimal control problem: Minimize

$$\mathcal{J}(x,u) := \int_0^\infty \begin{pmatrix} x(\tau) \\ u(\tau) \end{pmatrix}^{\mathsf{T}} \begin{bmatrix} Q & S \\ S^{\mathsf{T}} & R \end{bmatrix} \begin{pmatrix} x(\tau) \\ u(\tau) \end{pmatrix} \mathrm{d}\tau$$

with $Q = Q^{\mathsf{T}} \in \mathbb{R}^{n \times n}$, $S \in \mathbb{R}^{n \times m}$, and $R = R^{\mathsf{T}} \in \mathbb{R}^{m \times m}$ (invertible) subject to

$$\tfrac{\mathrm{d}}{\mathrm{d}t} x(t) = Ax(t) + Bu(t), \quad x(0) = x^0, \quad \lim_{t \to \infty} x(t) = 0.$$

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Problems:

- Feasibility, i. e., is

$$V^+(x^0) := \inf \left\{ \mathcal{J}(x,u) : \tfrac{\mathrm{d}}{\mathrm{d}t} x = Ax + Bu,\ x(0) = x^0,\ \lim_{t \to \infty} x(t) = 0 \right\}$$

  finite for all x^0?
- Optimal value $V^+(x^0)$?

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- Feasibility:
  - $V^+(x^0) < \infty$ for all $x^0 \in \mathbb{R}^n$ if and only if $[A, B]$ is stabilizable. Algebraically: $\operatorname{rank} \begin{bmatrix} -\lambda I_n + A & B \end{bmatrix} = n$ for all $\lambda \in \overline{\mathbb{C}^+}$.
  - $V^+(x^0) > -\infty$ for all $x^0 \in \mathbb{R}^n$ if the KYP inequality

$$\begin{bmatrix} A^\mathsf{T}P + PA + Q & PB + S \\ B^\mathsf{T}P + S^\mathsf{T} & R \end{bmatrix} \geq 0, \quad P = P^\mathsf{T}$$

has a solution for $P \in \mathbb{R}^{n\times n}$. Then $(x^0)^\mathsf{T} P x^0 \leq V^+(x^0)$.

The Linear-Quadratic Control Problem for ODEs

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- Optimal value:
  - $V^+(x^0) = (x^0)^\mathsf{T} P^+ x^0$, where P^+ is the stabilizing and maximal solution of the algebraic Riccati equation

    $$A^\mathsf{T} P + PA + Q - \left(PB + S\right) R^{-1} \left(B^\mathsf{T} P + S^\mathsf{T}\right) = 0, \quad P = P^\mathsf{T}.$$

  - Optimal state trajectories need not exist, but in case they do, they are given by

    $$\dot{x}(t) = \left(A - BR^{-1}\left(B^\mathsf{T} P^+ + S^\mathsf{T}\right)\right) x(t), \quad x(0) = x^0.$$

  - Alternatively solve the two-point boundary value problem

    $$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x(t) \\ \lambda(t) \end{pmatrix} = \underbrace{\begin{bmatrix} A - BR^{-1}S^\mathsf{T} & -BR^{-1}B^\mathsf{T} \\ SR^{-1}S^\mathsf{T} - Q & -\left(A - BR^{-1}S^\mathsf{T}\right)^\mathsf{T} \end{bmatrix}}_{=:\mathcal{H}} \begin{pmatrix} x(t) \\ \lambda(t) \end{pmatrix},$$

    $$x(0) = x^0, \quad \lim_{t\to\infty} \lambda(t) = 0.$$

  - Determine P^+ by the stable invariant subspaces of the Hamiltonian matrix $\mathcal{H}$.

Singular Linear-Quadratic Control Problem for ODEs

Up to now $R = R^{\mathsf{T}}$ is invertible. The situation is more involved, if R is not invertible:

- Feasibility check with the KYP inequality still works.
- The algebraic Riccati equation does not exist, instead resort to the Lur'e equation [Reis '11]

$$\begin{bmatrix} A^{\mathsf{T}}P + PA + Q & PB + S \\ B^{\mathsf{T}}P + S^{\mathsf{T}} & R \end{bmatrix} = \begin{bmatrix} K^{\mathsf{T}} \\ L^{\mathsf{T}} \end{bmatrix} \begin{bmatrix} K & L \end{bmatrix}, \quad P = P^{\mathsf{T}}$$

to be solved for $(P, K, L) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{q \times n} \times \mathbb{R}^{q \times m}$ with

$$\operatorname{rank}_{\mathbb{R}(s)} \begin{bmatrix} -sI_n + A & B \\ K & L \end{bmatrix} = n + q.$$

If the problem is feasible and $[A, B]$ is stabilizable, then there exists a stabilizing solution (P^+, K^+, L^+) satisfying

$$\operatorname{rank} \begin{bmatrix} -\lambda I_n + A & B \\ K^+ & L^+ \end{bmatrix} = n + q \quad \forall\, \lambda \in \mathbb{C}^+.$$

Singular Linear-Quadratic Control Problem for ODEs

- Optimal controls are not unique even if they exist, they solve the optimality DAE
$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix}\begin{pmatrix} x(t) \\ u(t) \end{pmatrix} = \begin{bmatrix} A & B \\ K^+ & L^+ \end{bmatrix}\begin{pmatrix} x(t) \\ u(t) \end{pmatrix}, \quad x(0) = x^0,$$
where (P^+, K^+, L^+) is a stabilizing solution.
- The Hamiltonian BVP does not exist, instead resort to the even BVP
$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} 0 & I_n & 0 \\ -I_n & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{pmatrix} \mu(t) \\ x(t) \\ u(t) \end{pmatrix} = \begin{bmatrix} 0 & A & B \\ A^{\mathsf{T}} & Q & S \\ B^{\mathsf{T}} & S^{\mathsf{T}} & R \end{bmatrix}\begin{pmatrix} \mu(t) \\ x(t) \\ u(t) \end{pmatrix},$$
$$x(0) = x^0, \quad \lim_{t\to\infty} \mu(t) = 0.$$
- Construction of (P^+, K^+, L^+) using the stable deflating subspaces of the even matrix pencil
$$\begin{bmatrix} 0 & -sI_n + A & B \\ sI_n + A^{\mathsf{T}} & Q & S \\ B^{\mathsf{T}} & S^{\mathsf{T}} & R \end{bmatrix} \in \mathbb{R}[s]^{(2n+m)\times(2n+m)}.$$

Differential-Algebraic Systems/Descriptor Systems

Linear time-invariant differential-algebraic systems/descriptor systems

$$\frac{\mathrm{d}}{\mathrm{d}t} Ex(t) = Ax(t) + Bu(t)$$

where

- $E, A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$,
- assume $sE - A$ is regular, i. e., $\det(sE - A) \not\equiv 0$,
- state $x \in \mathcal{L}^2_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^n)$, input $u \in \mathcal{L}^2_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^m)$.

Typical applications

- network models: electrical circuits, gas networks, constrained multi-body systems, . . .
- semi-discretization of PDEs (e. g., Navier-Stokes),
- linearization of non-linear DAEs,
- . . . .

Geometric Concepts

a) behavior:

$$\mathfrak{B}_{[E,A,B]} := \left\{ (x,u) \in \mathcal{L}^2_{\mathrm{loc}}(\mathbb{R},\mathbb{R}^n) \times \mathcal{L}^2_{\mathrm{loc}}(\mathbb{R},\mathbb{R}^m) : \tfrac{\mathrm{d}}{\mathrm{d}t} Ex = Ax + Bu \right\}.$$

b) system space: smallest subspace in $\mathbb{R}^{n+m}$ such that

$$\begin{pmatrix} x(t) \\ u(t) \end{pmatrix} \in \mathcal{V}_{\mathrm{sys}} \quad \forall (x,u) \in \mathfrak{B}_{[E,A,B]} \text{ and almost all } t \in \mathbb{R}.$$

c) space of consistent initial differential variables:

$$\mathcal{V}_{\mathrm{diff}} := \left\{ x^0 \in \mathbb{R}^n : \exists (x,u) \in \mathfrak{B}_{[E,A,B]} \text{ with } Ex(0) = Ex^0 \right\}.$$

A system is called impulse controllable, if $\mathcal{V}_{\mathrm{diff}} = \mathbb{R}^n$.

Explicit Representation of the Spaces

Feedback equivalence form

There exist nonsingular matrices $W,\ T \in \mathbb{R}^{n \times n}$ and $F \in \mathbb{R}^{m \times n}$ such that

$$W \begin{bmatrix} -sE + A & B \end{bmatrix} \begin{bmatrix} T & 0 \\ FT & I_m \end{bmatrix} = \begin{bmatrix} -sI_{n_1} + A_{11} & 0 & 0 & B_1 \\ 0 & I_{n_2} & -sE_{23} & B_2 \\ 0 & 0 & -sE_{33} + I_{n_3} & 0 \end{bmatrix},$$

where E_{33} is nilpotent.

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Explicit representations of the spaces:

$$\mathcal{V}_{\mathrm{sys},F} = \left\{ \begin{pmatrix} x_1 \\ -B_2 u \\ 0 \\ u \end{pmatrix} \in \mathbb{R}^{n+m} \ : \ x_1 \in \mathbb{R}^{n_1},\ u \in \mathbb{R}^m \right\},$$

$$\mathcal{V}_{\mathrm{diff},F} = \mathbb{R}^{n_1+n_2} \times \ker \begin{bmatrix} E_{23} \\ E_{33} \end{bmatrix}.$$

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Notation

For symmetric M, $N \in \mathbb{R}^{n \times n}$ and a subspace $\mathcal{V} \subseteq \mathbb{R}^n$ we write

- $M \geq_{\mathcal{V}} N \quad :\Longleftrightarrow \quad v^\mathsf{T} M v \geq v^\mathsf{T} N v \quad \forall\, v \in \mathcal{V}$,
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has a solution for $P \in \mathbb{R}^{n\times n}$. Then $(x^0)^{\mathsf{T}} E^{\mathsf{T}} P E x^0 \leq V^+(Ex^0)$.

Lur'e Equations for DAEs

Lur'e equation for DAEs:

$$\begin{bmatrix} A^{\mathsf{T}}PE + E^{\mathsf{T}}PA + Q & E^{\mathsf{T}}PB + S \\ B^{\mathsf{T}}PE + S^{\mathsf{T}} & R \end{bmatrix} =_{\mathcal{V}_{\mathrm{sys}}} \begin{bmatrix} K^{\mathsf{T}} \\ L^{\mathsf{T}} \end{bmatrix} \begin{bmatrix} K & L \end{bmatrix}, \quad P = P^{\mathsf{T}}$$

to be solved for $(P, K, L) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{q \times n} \times \mathbb{R}^{q \times m}$ with

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Stabilizing solutions: Assume that the KYP inequality is solvable. Then we have:

- **Existence:** $[E, A, B]$ behaviorally stabilizable $\Rightarrow \exists$ stabilizing solution (P^+, K^+, L^+), i. e.,

$$\operatorname{rank} \begin{bmatrix} -\lambda E + A & B \\ K^+ & L^+ \end{bmatrix} = n + q \quad \forall\, \lambda \in \mathbb{C}^+.$$

- **Uniqueness:** (P_1^+, K_1^+, L_1^+), (P_2^+, K_2^+, L_2^+) stabilizing solutions $\Rightarrow$ $E^{\mathsf{T}}P_1^+E =_{\mathcal{V}_{\mathrm{diff}}} E^{\mathsf{T}}P_2^+E$.
- **Extremality:** (P^+, K^+, L^+) is stabilizing solution of the Lur'e equation $\Rightarrow$ $E^{\mathsf{T}}P^+E \geq_{\mathcal{V}_{\mathrm{diff}}} E^{\mathsf{T}}PE$ for all solutions P of the KYP inequality.

Optimal Value and Optimal Controls

- Optimal value: If (P^+, K^+, L^+) is a stabilizing solution of the Lur'e equation, then
$$V^+(Ex^0) = (x^0)^\mathsf{T} E^\mathsf{T} P^+ Ex^0 \quad \forall\, x^0 \in \mathcal{V}_{\mathrm{diff}}.$$
- If an optimal control $(x, u) \in \mathfrak{B}_{[E,A,B]}$ with $Ex(0) = Ex^0 \in E\mathcal{V}_{\mathrm{diff}}$ exists, it satisfies the optimality DAE
$$\tfrac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} x(t) \\ u(t) \end{pmatrix} = \begin{bmatrix} A & B \\ K^+ & L^+ \end{bmatrix} \begin{pmatrix} x(t) \\ u(t) \end{pmatrix}, \quad Ex(0) = Ex^0.$$

Even Matrix Pencils

Assume that $[E, A, B]$ is impulse controllable. Lur'e equations have a close relationship to matrix pencils

$$s\mathcal{E} - \mathcal{A} := \begin{bmatrix} 0 & -sE + A & B \\ sE^{\mathsf{T}} + A^{\mathsf{T}} & Q & S \\ B^{\mathsf{T}} & S^{\mathsf{T}} & R \end{bmatrix} \in \mathbb{R}[s]^{(2n+m)\times(2n+m)}.$$

The pencil $s\mathcal{E} - \mathcal{A}$ is even, since $\mathcal{E} = -\mathcal{E}^{\mathsf{T}}$ and $\mathcal{A} = \mathcal{A}^{\mathsf{T}}$.

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Deflating Subspaces [Van Dooren '83]

- A subspace $\mathcal{Y} \subseteq \mathbb{R}^N$ with basis matrix $Y \in \mathbb{R}^{N\times k}$ is called deflating subspace of $s\mathcal{E} - \mathcal{A} \in \mathbb{R}[s]^{N\times N}$ if there exist $Z \in \mathbb{R}^{N\times \ell}$ and $s\tilde{\mathcal{E}} - \tilde{\mathcal{A}} \in \mathbb{R}[s]^{\ell\times k}$ of full row rank such that
$$(s\mathcal{E} - \mathcal{A})Y = Z(s\tilde{\mathcal{E}} - \tilde{\mathcal{A}}).$$
- A subspace $\mathcal{Y}$ is called $\mathcal{E}$-neutral if $y_1^{\mathsf{T}}\mathcal{E}y_2 = 0$ for all $y_1,\ y_2 \in \mathcal{Y}$.

Construction of Solutions of the Lur'e Equation

Theorem [Reis, Rendel, V. '15]

If there exists a solution (P, K, L) of the Lur'e equation, then there exist

$$Y = \begin{bmatrix} Y_\mu \\ Y_x \\ Y_u \end{bmatrix} \in \mathbb{R}^{(2n+m)\times(n+m)}, \quad Z = \begin{bmatrix} Z_\mu \\ Z_x \\ Z_u \end{bmatrix} \in \mathbb{R}^{(2n+m)\times(n+q)} \text{ such that}$$

(a) the space $\mathcal{Y} = \operatorname{im} Y$ is $\mathcal{E}$-neutral and of dimension $n + m$;

(b) $\mathcal{V}_{\mathrm{sys}} \subseteq \operatorname{im} \begin{bmatrix} Y_x \\ Y_u \end{bmatrix}$;

(c) $\operatorname{rank} EY_x = \operatorname{rank} E$;

(d) there exist $s\tilde{\mathcal{E}} - \tilde{\mathcal{A}} \in \mathbb{R}[s]^{(n+q)\times(n+m)}$ with $\operatorname{rank}_{\mathbb{R}(s)} \left(s\tilde{\mathcal{E}} - \tilde{\mathcal{A}}\right) = n + q$ such that

$$\begin{bmatrix} 0 & -sE + A & B \\ sE^\mathsf{T} + A^\mathsf{T} & Q & S \\ B^\mathsf{T} & S^\mathsf{T} & R \end{bmatrix} \begin{bmatrix} Y_\mu \\ Y_x \\ Y_u \end{bmatrix} = \begin{bmatrix} Z_\mu \\ Z_x \\ Z_u \end{bmatrix} \left(s\tilde{\mathcal{E}} - \tilde{\mathcal{A}}\right).$$

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Remark: The converse holds true under some additional assumptions.

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Remarks

- Y_x, Y_u can be chosen such that $\begin{bmatrix} Y_x \\ Y_u \end{bmatrix}$ is invertible. With $\begin{bmatrix} Y_x^- & Y_u^- \end{bmatrix} := \begin{bmatrix} Y_x \\ Y_u \end{bmatrix}^{-1}$, we obtain $P = Y_\mu Y_x^-$.
- $s\tilde{\mathcal{E}} - \tilde{\mathcal{A}}$ is equivalent to $\begin{bmatrix} -sE + A & B \\ K & L \end{bmatrix}$.
- $[E, A, B]$ behaviorally stabilizable $\Rightarrow$ construct stabilizing solutions by choosing a semi-stable deflating subspace.
- Existence and uniqueness of optimal controls can be directly read off the spectral structure of $s\mathcal{E} - \mathcal{A}$.

The Linear-Quadratic Control Problem for IDEs

Linear-quadratic optimal control problem: Minimize

$$\mathcal{J}(x,u) := \sum_{k=0}^{\infty} \begin{pmatrix} x_k \\ u_k \end{pmatrix}^{\mathsf{T}} \begin{bmatrix} Q & S \\ S^{\mathsf{T}} & R \end{bmatrix} \begin{pmatrix} x_k \\ u_k \end{pmatrix}$$

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- Existence and uniqueness of optimal controls ($\rightsquigarrow$ regularity)?

The Linear-Quadratic Control Problem for IDEs

$$V^{+}(Ex^0) := \inf \left\{ \mathcal{J}(x,u) : Ex_{k+1} = Ax_k + Bu_k,\ Ex(0) = Ex^0,\ \lim_{k\to\infty} Ex_k = 0 \right\}$$

- Feasibility:
  - $V^{+}(Ex^0) < \infty$ for all $x^0 \in \mathcal{V}_{\mathrm{shift}} (= \mathcal{V}_{\mathrm{diff}})$ if and only if $[E, A, B]$ is behaviorally (d-)stabilizable. Algebraically: rank $\begin{bmatrix} -\lambda E + A & B \end{bmatrix} = n$ for all $\lambda \in \mathbb{C} \setminus \mathbb{D}$.
  - $V^{+}(Ex^0) > -\infty$ for all $x^0 \in \mathcal{V}_{\mathrm{shift}}$ if the KYP inequality

$$\begin{bmatrix} A^{\mathsf{T}}PA - E^{\mathsf{T}}PE + Q & A^{\mathsf{T}}PB + S \\ B^{\mathsf{T}}PA + S^{\mathsf{T}} & B^{\mathsf{T}}PB + R \end{bmatrix} \geq_{\mathcal{V}_{\mathrm{sys}}} 0, \quad P = P^{\mathsf{T}}$$

has a solution for $P \in \mathbb{R}^{n\times n}$. Then $(x^0)^{\mathsf{T}} E^{\mathsf{T}} PEx^0 \leq V^{+}(Ex^0)$.

Lur'e Equations for IDEs

Lur'e equation for IDEs:

$$\begin{bmatrix} A^\mathsf{T}PA - E^\mathsf{T}PE + Q & A^\mathsf{T}PB + S \\ B^\mathsf{T}PA + S^\mathsf{T} & B^\mathsf{T}PB + R \end{bmatrix} =_{\mathcal{V}_{\mathrm{sys}}} \begin{bmatrix} K^\mathsf{T} \\ L^\mathsf{T} \end{bmatrix} \begin{bmatrix} K & L \end{bmatrix}, \quad P = P^\mathsf{T}$$

to be solved for $(P, K, L) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{q \times n} \times \mathbb{R}^{q \times m}$ with

$$\mathrm{rank}_{\mathbb{R}(s)} \begin{bmatrix} -sE + A & B \\ K & L \end{bmatrix} = n + q.$$

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Stabilizing solutions: Assume that the KYP inequality is solvable. Then we have:

- **Existence:** $[E, A, B]$ behaviorally stabilizable $\Rightarrow \exists$ stabilizing solution (P^+, K^+, L^+), i. e.,

$$\mathrm{rank} \begin{bmatrix} -\lambda E + A & B \\ K^+ & L^+ \end{bmatrix} = n + q \quad \forall\, \lambda \in \mathbb{C} \setminus \overline{\mathbb{D}}.$$

- **Uniqueness:** (P_1^+, K_1^+, L_1^+), (P_2^+, K_2^+, L_2^+) stabilizing solutions $\Rightarrow$ $E^\mathsf{T}P_1^+E =_{\mathcal{V}_{\mathrm{diff}}} E^\mathsf{T}P_2^+E$.
- **Extremality:** (P^+, K^+, L^+) is stabilizing solution of the Lur'e equation $\Rightarrow$ $E^\mathsf{T}P^+E \geq_{\mathcal{V}_{\mathrm{diff}}} E^\mathsf{T}PE$ for all solutions P of the KYP inequality.

Optimal Value and Optimal Controls

- Optimal value: If (P^+, K^+, L^+) is a stabilizing solution of the Lur'e equation, then
$$V^+(Ex^0) = (x^0)^\mathsf{T} E^\mathsf{T} P^+ Ex^0 \quad \forall\, x^0 \in \mathcal{V}_{\mathrm{shift}}.$$
- If an optimal control $(x, u) \in \mathfrak{B}^{\mathrm{d}}_{[E,A,B]} := \left\{(x,u) \in (\ell^2(\mathbb{R}))^n \times (\ell^2(\mathbb{R}))^m : Ex_{k+1} = Ax_k + Bu_k\right\}$ with $Ex_0 = Ex^0 \in E\mathcal{V}_{\mathrm{shift}}$ exists, it satisfies the optimality IDE
$$\begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} x_{k+1} \\ u_{k+1} \end{pmatrix} = \begin{bmatrix} A & B \\ K^+ & L^+ \end{bmatrix} \begin{pmatrix} x_k \\ u_k \end{pmatrix}, \quad Ex_0 = Ex^0.$$

Structured Matrix Pencils

Assume that $[E, A, B]$ is impulse controllable.

Optimality BVP for IDEs:

$$\begin{bmatrix} 0 & E & 0 \\ A^{\mathsf{T}} & 0 & 0 \\ B^{\mathsf{T}} & 0 & 0 \end{bmatrix} \begin{pmatrix} \mu_{k+1} \\ x_{k+1} \\ u_{k+1} \end{pmatrix} = \begin{bmatrix} 0 & A & B \\ E^{\mathsf{T}} & Q & S \\ 0 & S^{\mathsf{T}} & R \end{bmatrix} \begin{pmatrix} \mu_k \\ x_k \\ u_k \end{pmatrix},$$

$$Ex_0 = Ex^0, \quad \lim_{k\to\infty} E^{\mathsf{T}}\mu_k = 0.$$

Solutions of the Lur'e equation can be constructed by the deflating subspaces of the BVD matrix pencil

$$\begin{bmatrix} 0 & -zE + A & B \\ -zA^{\mathsf{T}} + E^{\mathsf{T}} & Q & S \\ -zB^{\mathsf{T}} & S^{\mathsf{T}} & R \end{bmatrix} \in \mathbb{R}[z]^{(2n+m)\times(2n+m)}.$$

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But: This pencil has no special structure that can be exploited for numerical computations.

Structured Matrix Pencils

Reformulate BVP as palindromic BVP:

$$\begin{bmatrix} 0 & E & 0 \\ A^{\mathsf{T}} & Q & S \\ B^{\mathsf{T}} & S^{\mathsf{T}} & R \end{bmatrix} \begin{pmatrix} m_{k+1} \\ x_{k+1} \\ u_{k+1} \end{pmatrix} = \begin{bmatrix} 0 & A & B \\ E^{\mathsf{T}} & Q & S \\ 0 & S^{\mathsf{T}} & R \end{bmatrix} \begin{pmatrix} m_k \\ x_k \\ u_k \end{pmatrix},$$

$$Ex_0 = Ex^0, \quad \sum_{k=0}^{\infty} E^{\mathsf{T}} m_k = E^{\mathsf{T}} \mu_0.$$

Construction of (P^+, K^+, L^+) using the stable deflating subspaces of the palindromic matrix pencil

$$z\mathcal{A}^{\mathsf{T}} - \mathcal{A} := \begin{bmatrix} 0 & -zE + A & B \\ -zA^{\mathsf{T}} + E^{\mathsf{T}} & (1-z)Q & (1-z)S \\ -zB^{\mathsf{T}} & (1-z)S^{\mathsf{T}} & (1-z)R \end{bmatrix} \in \mathbb{R}[z]^{(2n+m)\times(2n+m)}.$$

Construction of Solutions of the Lur'e Equation

Theorem [Bankmann, V. '17]

Assume that $\operatorname{rank}\begin{bmatrix}-E+A & B\end{bmatrix} = n$. If there exists a solution (P, K, L) of the Lur'e equation, then there exist

$$Y = \begin{bmatrix} Y_\mu \\ Y_x \\ Y_u \end{bmatrix} \in \mathbb{R}^{(2n+m)\times(n+m)}, \quad Z = \begin{bmatrix} Z_\mu \\ Z_x \\ Z_u \end{bmatrix} \in \mathbb{R}^{(2n+m)\times(n+q)} \text{ such that}$$

(a) the space $\mathcal{Y} = \operatorname{im} Y$ is $(\mathcal{A}^\mathsf{T} - \mathcal{A})$-neutral and of dimension $n+m$;

(b) $\mathcal{V}_{\mathrm{sys}} = \operatorname{im}\begin{bmatrix} Y_x \\ Y_u \end{bmatrix}$;

(c) $\operatorname{rank}((-E+A)Y_x + BY_u) = \operatorname{rank} E$;

(d) there exist $z\tilde{\mathcal{E}} - \tilde{\mathcal{A}} \in \mathbb{R}[z]^{(n+q)\times(n+m)}$ with $\operatorname{rank}_{\mathbb{R}(z)}\left(z\tilde{\mathcal{E}} - \tilde{\mathcal{A}}\right) = n+q$ such that

$$\begin{bmatrix} 0 & -zE+A & B \\ -zA^\mathsf{T}+E^\mathsf{T} & (1-z)Q & (1-z)S \\ -zB^\mathsf{T} & (1-z)S^\mathsf{T} & (1-z)R \end{bmatrix} \begin{bmatrix} Y_\mu \\ Y_x \\ Y_u \end{bmatrix} = \begin{bmatrix} Z_\mu \\ Z_x \\ Z_u \end{bmatrix} \left(z\tilde{\mathcal{E}} - \tilde{\mathcal{A}}\right).$$

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$$\begin{bmatrix} 0 & -zE+A & B \\ -zA^\mathsf{T}+E^\mathsf{T} & (1-z)Q & (1-z)S \\ -zB^\mathsf{T} & (1-z)S^\mathsf{T} & (1-z)R \end{bmatrix} \begin{bmatrix} Y_\mu \\ Y_x \\ Y_u \end{bmatrix} = \begin{bmatrix} Z_\mu \\ Z_x \\ Z_u \end{bmatrix} \left(z\tilde{\mathcal{E}} - \tilde{\mathcal{A}}\right).$$

Remark: The converse holds true under some additional assumptions.

Construction of Solutions of the Lur'e Equation

$$\begin{bmatrix} 0 & -zE + A & B \\ -zA^{\mathsf{T}} + E^{\mathsf{T}} & (1-z)Q & (1-z)S \\ -zB^{\mathsf{T}} & (1-z)S^{\mathsf{T}} & (1-z)R \end{bmatrix} \begin{bmatrix} Y_\mu \\ Y_x \\ Y_u \end{bmatrix} = \begin{bmatrix} Z_\mu \\ Z_x \\ Z_u \end{bmatrix} \left(z\tilde{\mathcal{E}} - \tilde{\mathcal{A}} \right)$$

Construction of Solutions of the Lur'e Equation

$$\begin{bmatrix} 0 & -zE+A & B \\ -zA^{\mathsf{T}}+E^{\mathsf{T}} & (1-z)Q & (1-z)S \\ -zB^{\mathsf{T}} & (1-z)S^{\mathsf{T}} & (1-z)R \end{bmatrix} \begin{bmatrix} Y_\mu \\ Y_x \\ Y_u \end{bmatrix} = \begin{bmatrix} Z_\mu \\ Z_x \\ Z_u \end{bmatrix} \left(z\tilde{\mathcal{E}} - \tilde{\mathcal{A}}\right)$$

Remarks

- The construction of P is possible, but is implicitly hidden in Y and needs some transformations.
- $z\tilde{\mathcal{E}} - \tilde{\mathcal{A}}$ is equivalent to $\begin{bmatrix} -zE+A & B \\ (1-z)K & (1-z)L \end{bmatrix}$.
- $[E, A, B]$ behaviorally stabilizable $\Rightarrow$ construct stabilizing solutions by choosing a semi-stable deflating subspace.
- Existence and uniqueness of optimal controls can be directly read off the spectral structure of $z\mathcal{A}^{\mathsf{T}} - \mathcal{A}$.

Conclusions and Further Results

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- Lur'e equations are the proper concepts for generalizing algebraic Riccati equations for DAEs/IDEs.
- Structured matrix pencils can be employed to construct their solutions.

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Further Results

Lur'e equations appear in many more contexts such as

- for the analysis of dissipativity of dynamical systems (passivity/contractivity),
- for the construction of certain factorizations of rational matrices (spectral/normalized coprime/inner-outer factorizations),
- in model order reduction,
- many more . . . .

References

Thank your for the attention.

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