A Greedy Subspace Method for Computing the $\mathcal{H}_\infty$-Norm

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Motivation

Consider a function

$$G : \Omega \to \mathbb{C}^{p\times m}, \quad G(s) := C(s)D(s)^{-1}B(s),$$

where $\Omega \supseteq \mathbb{C}^+ := \{s \in \mathbb{C} : \mathrm{Re}(s) > 0\}$ and

$$\begin{aligned} B : \Omega \to \mathbb{C}^{n\times m}, &\quad B(s) := f_1(s)B_1 + \cdots + f_{\kappa_B}(s)B_{\kappa_B}, \\ C : \Omega \to \mathbb{C}^{p\times n}, &\quad C(s) := g_1(s)C_1 + \cdots + g_{\kappa_C}(s)C_{\kappa_C}, \\ D : \Omega \to \mathbb{C}^{n\times n}, &\quad D(s) := h_1(s)D_1 + \cdots + h_{\kappa_D}(s)D_{\kappa_D}, \end{aligned}$$

and all f_i, g_i, and h_i are meromorphic in Ω.

Typical Examples

Transfer functions of ...

1. differential-algebraic control systems/descriptor systems:

$$G(s) = C(sE - A)^{-1}B,$$

2. higher-order systems:

$$G(s) = C\left(s^2M + sD + K\right)^{-1}B,$$

3. delay differential-algebraic equations:

$$G(s) = C\left(sE - A_0 - \mathrm{e}^{-s\tau_1}A_1 - \mathrm{e}^{-s\tau_2}A_2\right)^{-1}B,$$

4. systems with input/output delays:

$$G(s) = C(sE - A)^{-1}B\mathrm{e}^{-s\tau},$$

5. spatially discretized Maxwell's equation:

$$G(s) = sB^{\mathsf{T}}(s^2I_n - \sqrt{s}D_2 + D_3)^{-1}B.$$

Define the Hardy space

$$\mathcal{H}_\infty^{p\times m} := \left\{ G : \mathbb{C}^+ \to \mathbb{C}^{p\times m} \;\middle|\; G \text{ is analytic and } \sup_{s\in\mathbb{C}^+} \|G(s)\|_2 < \infty \right\},$$

with the $\mathcal{H}_\infty$-norm

$$\|G\|_{\mathcal{H}_\infty} := \sup_{s\in\mathbb{C}^+} \|G(s)\|_2 = \sup_{\omega\in\mathbb{R}} \sigma_{\max}(G(\mathrm{i}\omega)).$$

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Goal of this talk

- computation of the $\mathcal{H}_\infty$-norm for this general class of functions;
- focus on large-scale systems $\Longrightarrow$ use ideas from model order reduction.

History

- Methods based on Hamiltonian eigenvalue problems:
  - one of the first algorithms: [Boyd, Balakrishnan '90]
  - generalization to descriptor systems: [Benner, Sima, V. '12]
  - extension to large-scale systems: [Benner, Lowe, V. '16], [Aliyev, Benner, Mengi, V. '16]
  - implicit determinant method: [Freitag, Spence, Van Dooren '14]
- Methods based on optimization over pseudospectra:
  - first paper: [Guglielmi, Gürbüzbalaban, Overton '13]
  - improvement of the optimization strategy: [Mitchell, Overton '16]
  - generalization to descriptor systems: [Benner, V. '14]

Application 1: Assessing Robust Stability of Delay DAEs

Consider a delay DAE

$$\frac{\mathrm{d}}{\mathrm{d}t}Ex(t) = A_0x(t) + A_1x(t-\tau).$$

Introduce a delay DAE with complex perturbations Δ_i, $i = 1, 2, 3$:

$$\frac{\mathrm{d}}{\mathrm{d}t}(E + B_1\Delta_1 C)x(t) = (A_0 + B_2\Delta_2 C)x(t) + (A_1 + B_3\Delta_3 C)x(t-\tau),$$

Let $\Delta := \begin{bmatrix}\Delta_1^{\mathsf{H}} & \Delta_2^{\mathsf{H}} & \Delta_3^{\mathsf{H}}\end{bmatrix}^{\mathsf{H}}$.

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$$r := \inf\left\{\|\Delta\|_2 \;\middle|\; \text{perturbed system is not exponentially stable}\right\} = ??$$

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Answer: Define

$$G(s) := C(sE - A_0 - \mathrm{e}^{-s\tau}A_1)^{-1}\begin{bmatrix}-sB_1 & B_2 & \mathrm{e}^{-s\tau}B_3\end{bmatrix}.$$

Under some technical conditions, $r := \|G\|_{\mathcal{H}_\infty}^{-1}$.

Application 2: Robust Control

Consider a DAE (a plant P) with inputs and outputs

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} Ex(t) &= Ax(t) + B_1 u(t) + B_2 w(t), \\ y(t) &= C_1 x(t), \\ z(t) &= C_2 x(t), \end{aligned}$$

with $B_i \in \mathbb{R}^{n \times m_i}$, $C_i \in \mathbb{R}^{p_i \times n}$.

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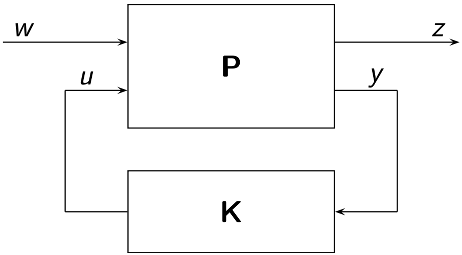
with $B_i \in \mathbb{R}^{n\times m_i}$, $C_i \in \mathbb{R}^{p_i\times n}$.
Task: design a controller $K \in \mathbb{R}^{m\times n}$ such that $u(t) = Ky(t)$.



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with $B_i \in \mathbb{R}^{n \times m_i}$, $C_i \in \mathbb{R}^{p_i \times n}$.
Closed-loop system:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}Ex(t) &= (A + B_1KC_1)x(t) + B_2w(t), \\ z(t) &= C_2x(t). \end{aligned}$$

Want: controller K such that the influence on stability of the closed loop system by disturbances of the form $w(t) = \Delta z(t)$ is as small as possible $\Longrightarrow$ find K that minimizes the $\mathcal{H}_\infty$-norm of the closed loop transfer function

$$G_K(s) := C_2(sE - (A + B_1KC_1))^{-1}B_2.$$

Algorithm

We have

$$G(s) = C(s)D(s)^{-1}B(s). \qquad \big((p \times n) \cdot (n \times n) \cdot (n \times m)\big)$$

Often, $n \gg p$, m, i. e., the large-scale part of the problem is the middle factor. Do model reduction to make the middle factor small, i. e., determine $V,\ W \in \mathbb{R}^{n \times k}$ such that

$$\widetilde{G}(s) = \widetilde{C}(s)\widetilde{D}(s)^{-1}\widetilde{B}(s),$$

with

$$\widetilde{C}(s) = C(s)V, \quad \widetilde{D}(s) = W^{\mathsf{H}}D(s)V, \quad \widetilde{B}(s) = W^{\mathsf{H}}B(s).$$

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Question: How to choose the right subspaces V and W?

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Question: How to choose the right subspaces V and W?
Answer: Do interpolation of the maximum singular values and update interpolation points using information of the current reduced order model.

Interpolatory Model Reduction

Theorem: Tangential Interpolation [Beattie, Gugercin '08]

Let $\mu \in \mathbb{C}$ be such that $C(\mu)$, $D(\mu)$, and $B(\mu)$ are analytic and both $D(\mu)$ and $\widetilde{D}(\mu)$ are invertible. Suppose also that $b \in \mathbb{C}^m \setminus \{0\}$ and $c \in \mathbb{C}^p \setminus \{0\}$ are given. Let $\mathcal{V} = \mathrm{Col}(V)$ and $\mathcal{W} = \mathrm{Col}(W)$. Then the following statements hold:

a) $D(\mu)^{-1}B(\mu)b \in \mathcal{V} \Longrightarrow G(\mu)b = \widetilde{G}(\mu)b$,

b) $\left(c^{\mathsf{H}}C(\mu)D(\mu)^{-1}\right)^{\mathsf{H}} \in \mathcal{W} \Longrightarrow c^{\mathsf{H}}G(\mu) = c^{\mathsf{H}}\widetilde{G}(\mu)$,

c) $D(\mu)^{-1}B(\mu)b \in \mathcal{V}$, $\left(c^{\mathsf{H}}C(\mu)D(\mu)^{-1}\right)^{\mathsf{H}} \in \mathcal{W} \Longrightarrow c^{\mathsf{H}}G'(\mu)\,b = c^{\mathsf{H}}\widetilde{G}'(\mu)\,b$.

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Let $\sigma := \sigma_{\max}(G(\mu))$ be the largest singular value of $G(\mu)$ with right and left singular vectors $v \in \mathbb{C}^m$ and $w \in \mathbb{C}^p$. Then we have

$$G(\mu)v = \sigma w$$
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Let $\sigma := \sigma_{\max}(G(\mu))$ be the largest singular value of $G(\mu)$ with right and left singular vectors $v \in \mathbb{C}^m$ and $w \in \mathbb{C}^p$. Then we have

$$G(\mu)v = \sigma w = \widetilde{G}(\mu)v,$$
$$w^{\mathsf{H}}G(\mu) = v^{\mathsf{H}}\sigma = w^{\mathsf{H}}\widetilde{G}(\mu),$$

but we do not know, whether $\sigma = \sigma_{\max}\left(\widetilde{G}(\mu)\right)$.

Interpolatory Model Reduction

Solution: Do full interpolation, i. e., instead of one pair singular vectors, put all of them into the projection spaces:

$$\begin{aligned}
\operatorname{im}\left(D(\mu)^{-1}B(\mu)\right) \subseteq \mathcal{V} &\Longrightarrow G(\mu) = \widetilde{G}(\mu), \\
\operatorname{im}\left(C(\mu)D(\mu)^{-1}\right)^{\mathsf{H}} \subseteq \mathcal{W} &\Longrightarrow G(\mu) = \widetilde{G}(\mu), \\
\operatorname{im}\left(D(\mu)^{-1}B(\mu)\right) \subseteq \mathcal{V},\ \operatorname{im}\left(C(\mu)D(\mu)^{-1}\right)^{\mathsf{H}} \subseteq \mathcal{W} &\Longrightarrow G'(\mu) = \widetilde{G}'(\mu).
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What Next? Use a greedy method to collect more interpolation data!

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What Next? Use a greedy method to collect more interpolation data!

- Compute the maximum singular value of the "low-dimensional" transfer function $\widetilde{G}$ on the imaginary axis and the optimal frequency ω_* using established methods for the small-scale case:
  - Boyd-Balakrishnan algorithm for linear systems: based on eigenvalue computations for Hamiltonian matrices/even matrix pencils
    [Boyd, Balakrishnan '90], [Benner, Sima, V. '12]
  - eigenvalue optimization methods for the general problem
    [Mengi, Yildirim, Kiliç '14]
- add subspaces from the current optimizer ω_* to get better local information.

Algorithm

Main steps of the algorithm:

a) create an initial reduced-order model with transfer function $\widetilde{G}$ and projection matrices V, W,

b) compute $\sup_{\omega \in \mathbb{R}} \left\| \widetilde{G}(\mathrm{i}\omega) \right\|_2$ with global optimizer ω_*,

c) update V and W with subspaces at $\mathrm{i}\omega_*$:

- if $m = p$: set $\widetilde{V} := D(\mathrm{i}\omega_*)^{-1} B(\mathrm{i}\omega_*)$, $\quad \widetilde{W} := \left(C(\mathrm{i}\omega_*) D(\mathrm{i}\omega_*)^{-1} \right)^{\mathsf{H}}$,
- if $m < p$: set $\widetilde{V} := D(\mathrm{i}\omega_*)^{-1} B(\mathrm{i}\omega_*)$, $\quad \widetilde{W} := \left(C(\mathrm{i}\omega_*) D(\mathrm{i}\omega_*)^{-1} \right)^{\mathsf{H}} G(\mathrm{i}\omega_*)$,
- if $m > p$: set $\widetilde{V} := D(\mathrm{i}\omega_*)^{-1} B(\mathrm{i}\omega_*) G(\mathrm{i}\omega_*)^{\mathsf{H}}$, $\quad \widetilde{W} := \left(C(\mathrm{i}\omega_*) D(\mathrm{i}\omega_*)^{-1} \right)^{\mathsf{H}}$,
- set $V := \begin{bmatrix} V & \widetilde{V} \end{bmatrix}$ and $W := \begin{bmatrix} W & \widetilde{W} \end{bmatrix}$ and update $\widetilde{G}$,

d) if not converged, repeat b), c), and d).

Analysis

Lemma

Let V, W be constructed as above at interpolation points $\mathrm{i}\omega_1, \ldots, \mathrm{i}\omega_k$. Then the following statements hold:

a) $\sigma_{\max}(G(\mathrm{i}\omega_j)) = \sigma_{\max}(\widetilde{G}(\mathrm{i}\omega_j)), \quad j = 1, \ldots, k.$

b) If $\sigma_{\max}(G(\mathrm{i}\cdot))$ is simple at $\omega_1, \ldots, \omega_k$, then $\sigma_{\max}(G(\mathrm{i}\cdot))$, $\sigma_{\max}(\widetilde{G}(\mathrm{i}\cdot))$ are differentiable at $\omega_1, \ldots, \omega_k$ and

$$\sigma'_{\max}(G(\mathrm{i}\omega_j)) = \sigma'_{\max}(\widetilde{G}(\mathrm{i}\omega_j)), \quad j = 1, \ldots, k.$$

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Superlinear Convergence

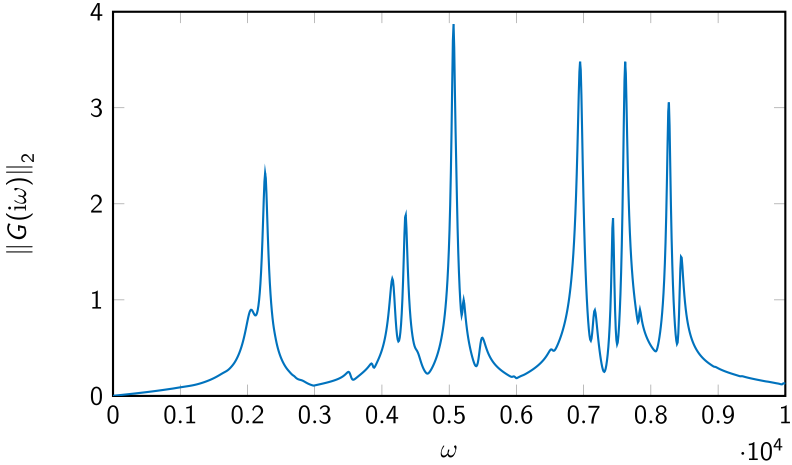
[Aliyev, Benner, Mengi, V. '16]

Let ω_* be a local maximizer of $\sigma_{\max}(G(\mathrm{i}\cdot))$ such that $\sigma_{\max}(G(\mathrm{i}\omega_*))$ is simple. If for some $k \geq 2$, ω_{k-1}, ω_k, and ω_{k+1} are sufficiently close to ω_*, then it holds

$$|\omega_{k+1} - \omega_*| \leq \nu \cdot \max\left\{|\omega_{k-1} - \omega_*|, |\omega_k - \omega_*|\right\} \cdot |\omega_k - \omega_*|.$$

Example 1: M20PI_n

Model with $n = 1182$, $m = p = 3$.



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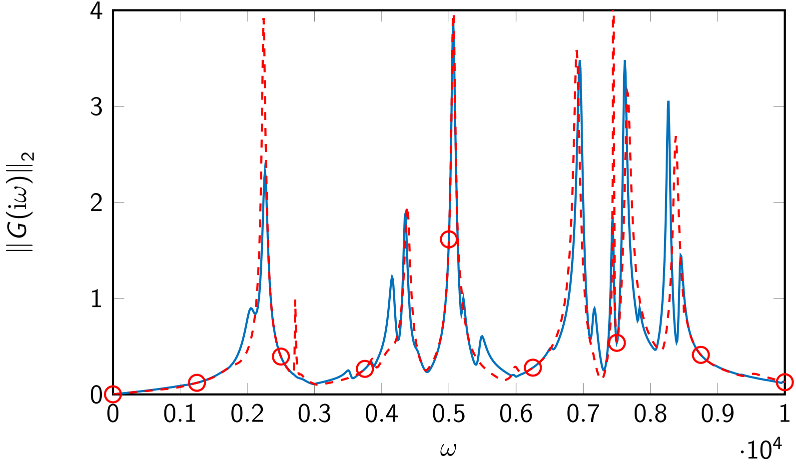

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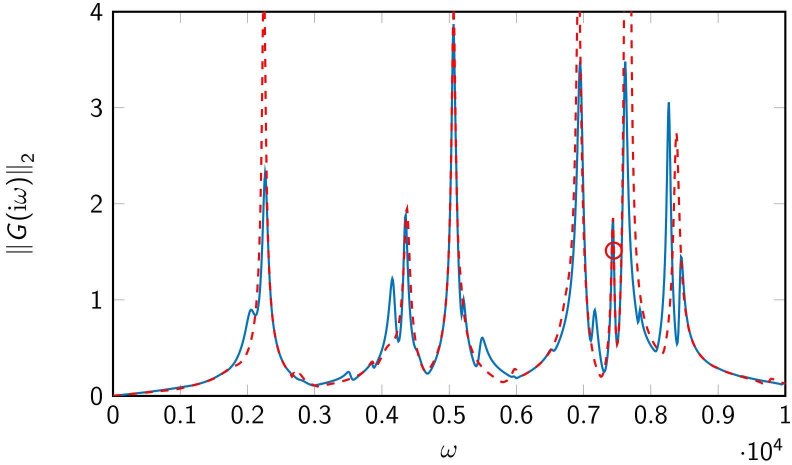

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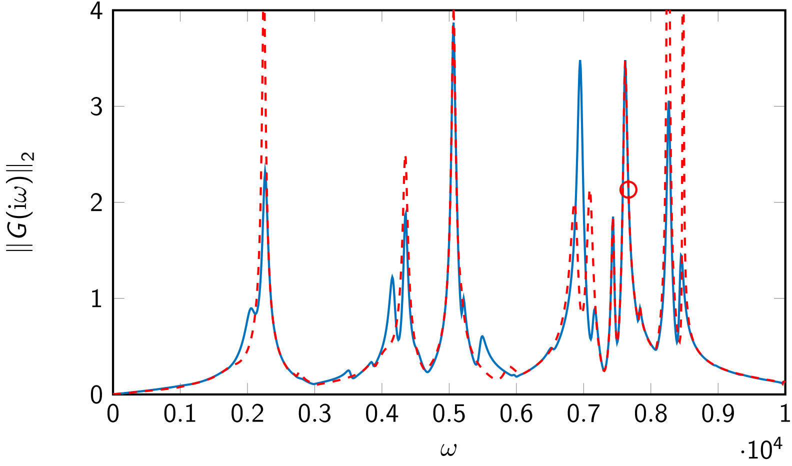

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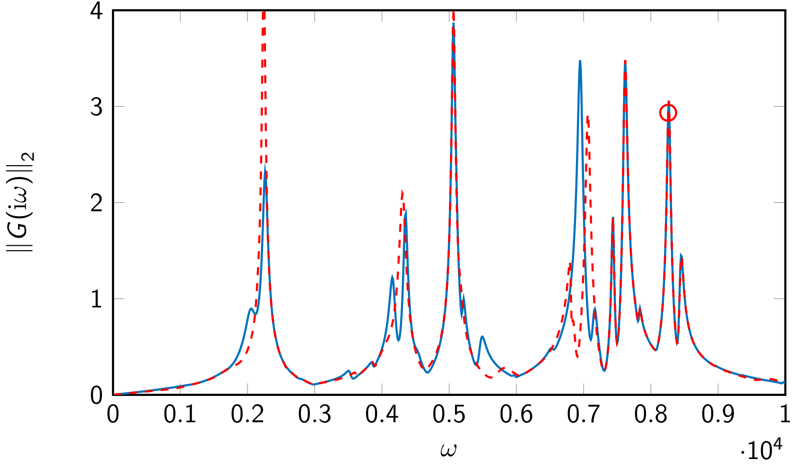

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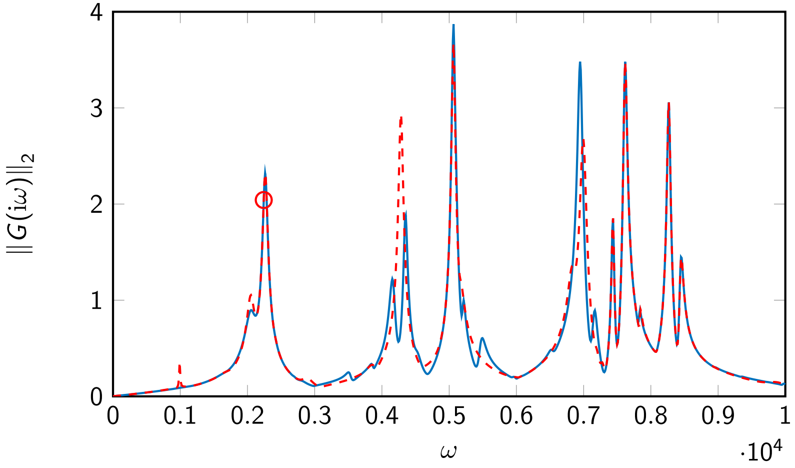

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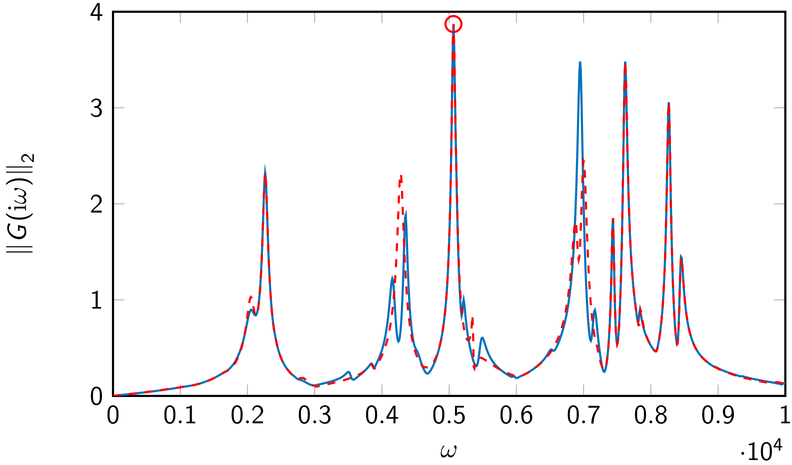

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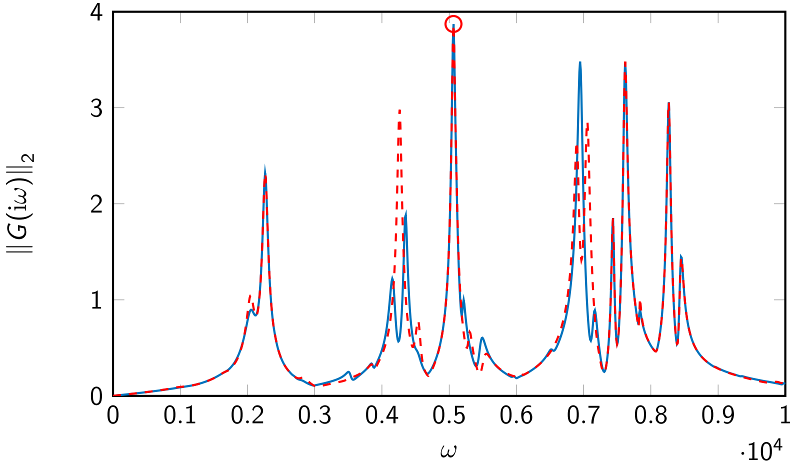
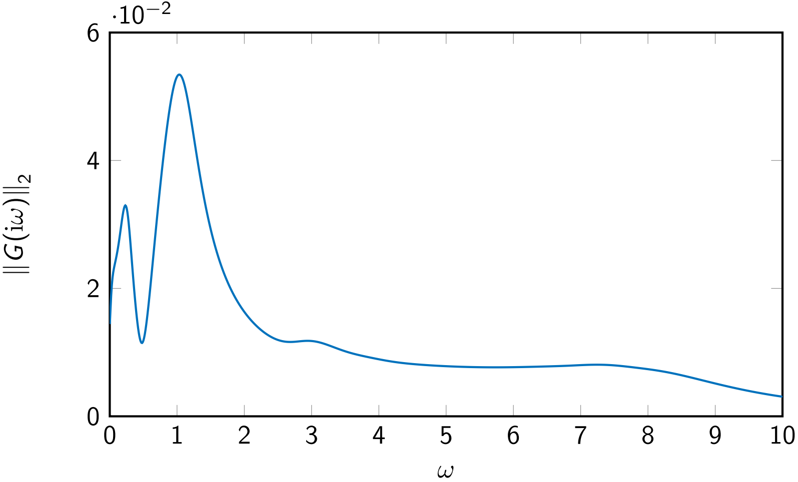


Timings: 0.25s (pseudopole method: 4.17s, even pencil method: 1.74s)

Example 2: mimo8x8_system

Model with $n = 13309$, $m = p = 8$.



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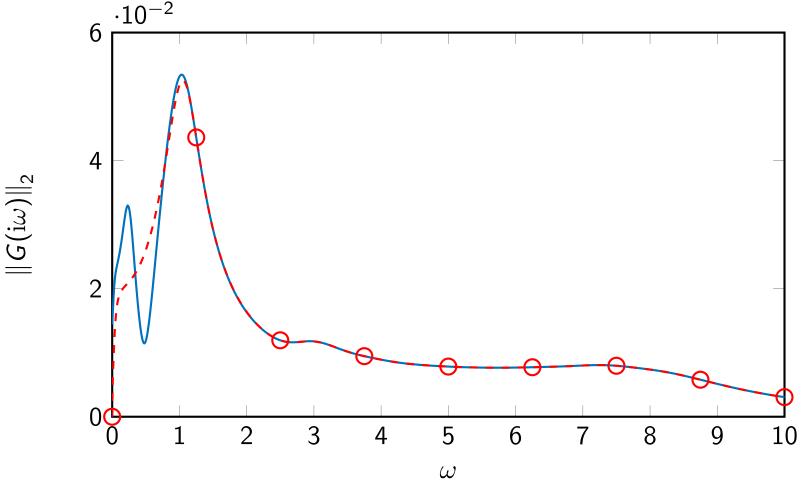

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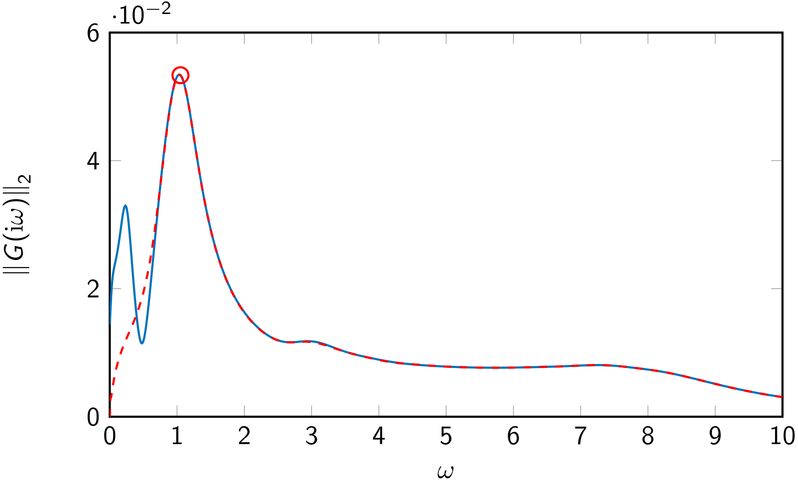

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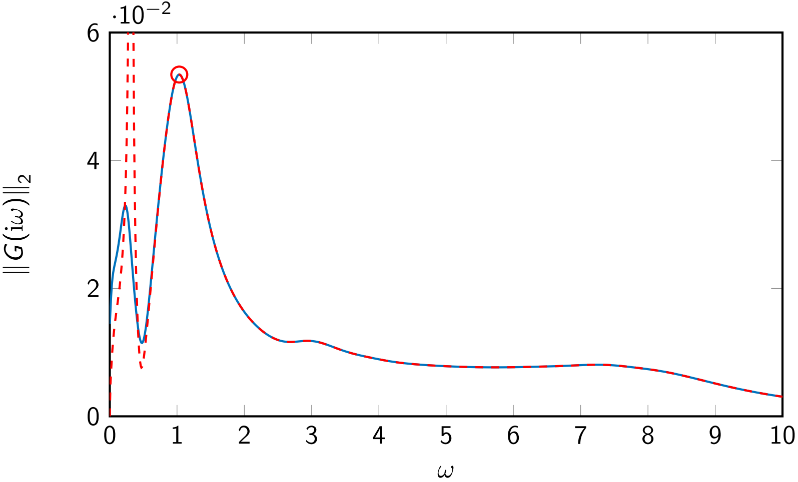

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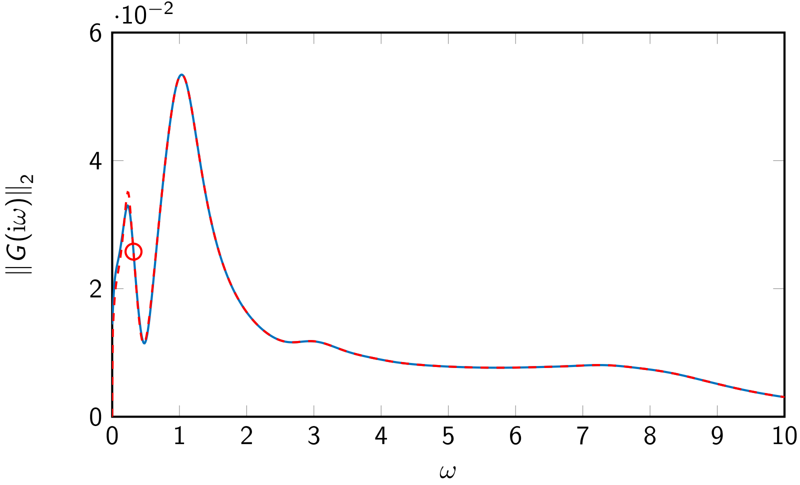

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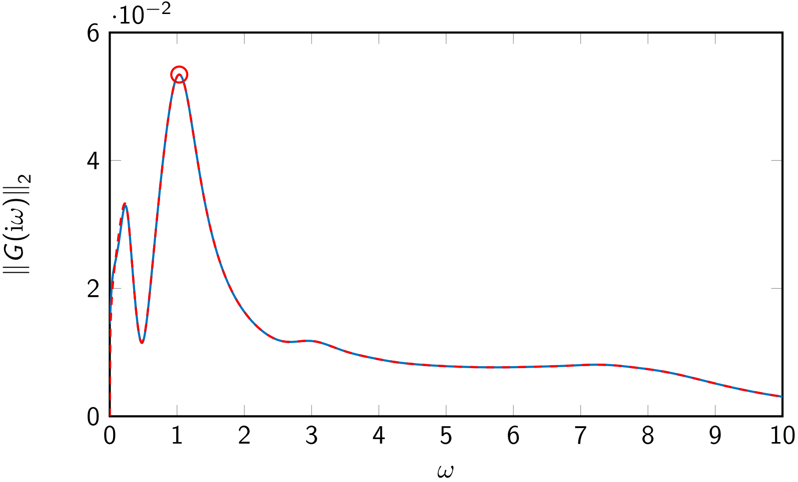

Timings: 2.09s (pseudopole method: 27.00s, even pencil method: 8.46s)

Conclusions and Outlook

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- subspace framework for computing the $\mathcal{H}_\infty$-norm for a general class of functions
$$G(s) = C(s)D(s)^{-1}B(s),$$
- local superlinear convergence.

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- subspace framework for computing the $\mathcal{H}_\infty$-norm for a general class of functions
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Outlook:

- improvement of the implementation (better choice of the initial model, better updates of the subspaces, . . .)
- apply similar techniques in the context of $\mathcal{H}_\infty$-optimization.

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