$\mathcal{L}_\infty$-Norm Computation for Descriptor Systems

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1. Introduction
2. Testing Properness of a Transfer Function
3. The $\mathcal{L}_\infty$-Norm Algorithm
4. Numerical Examples
5. Outlook

Considered Descriptor Systems

We start with a continuous linear time-invariant descriptor system

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t), \end{aligned} \tag{1}$$

with $E, A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{p \times n}$, $D \in \mathbb{R}^{p \times m}$, descriptor vector $x(t) \in \mathbb{R}^n$, control vector $u(t) \in \mathbb{R}^m$, and output vector $y(t) \in \mathbb{R}^p$.

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In the sequel we assume that (1) is regular, i.e., the pencil $A - \lambda E$ is regular ($\exists \lambda \in \mathbb{C}$ such that $\det(A - \lambda E) \neq 0$).

$\mathcal{L}_\infty$-Norm and Properness

Definition: $\mathcal{L}_\infty$-Norm

For systems (1) with no poles on the imaginary axis we define the $\mathcal{L}_\infty$-norm of the transfer function (2) by

$$\|G\|_{\mathcal{L}_\infty} := \sup_{\omega \in \mathbb{R}} \sigma_{max}\left(G\left(\mathrm{i}\omega\right)\right). \tag{3}$$

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A transfer function G is called proper if $\lim_{\omega\to\infty} \|G(\mathrm{i}\omega)\| < \infty$ and strictly proper if $\lim_{\omega\to\infty} \|G(\mathrm{i}\omega)\| = 0$. Otherwise it is called improper.

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When calculating the $\mathcal{L}_\infty$-norm we have to ensure that G is proper, so we want to state an algorithm that efficiently tests if this property is fulfilled!

Solution of a Descriptor System

Theorem: Reduction to Weierstraß Canonical Form

Every regular matrix pencil $A - \lambda E$ can be reduced to Weierstraß canonical form, i.e., $\exists$ nonsingular $W,\ T \in \mathbb{R}^{n \times n}$ such that

$$E = W \begin{bmatrix} I_{n_f} & 0 \\ 0 & N \end{bmatrix} T, \quad A = W \begin{bmatrix} J & 0 \\ 0 & I_{n_\infty} \end{bmatrix} T,$$

where J and N are in Jordan canonical form and N is nilpotent with index of nilpotency ν.

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"slow" subsystem	"fast" subsystem
$\dot{x}_1(t) = Jx_1(t) + B_1 u(t),$	$N\dot{x}_2(t) = x_2(t) + B_2 u(t),$
$y_1(t) = C_1 x_1(t),$	$y_2(t) = C_2 x_2(t) + Du(t).$

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$$x_1(t) = e^{Jt} x_1(0) + \int_0^t e^{J(t-\tau)} B_1 u(\tau) \mathrm{d}\tau,$$

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$$\begin{aligned} N\dot{x}_2(t) &= x_2(t) + B_2 u(t), \\ y_2(t) &= C_2 x_2(t) + Du(t). \end{aligned}$$

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$$N\dot{x}_2(t) = x_2(t) + B_2 u(t), \quad y_2(t) = C_2 x_2(t) + Du(t).$$

$$x_2(t) = -\sum_{i=0}^{\nu-1} N^i B_2 u^{(i)}(t).$$

C-Controllability and C-Observability

Controllability Conditions

1. The slow subsystem (fast subsystem) of (1) is C-controllable.
2. $\operatorname{rank}[sE - A, B] = n \; \forall s \in \mathbb{C}$ ($\operatorname{rank}[E, B] = n$).

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Basic Theorem and Outline of the Testing Routine

Theorem

Let $(E; A, B, C, D)$ be a descriptor system with both C-controllable and C-observable fast subsystem. Let furthermore $U^T EV = \begin{bmatrix} T & 0 \\ 0 & 0 \end{bmatrix}$ be a factorization of E with full-rank matrix $T \in \mathbb{R}^{r \times r}$ and $U^T AV = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ with the same block partitioning. Then the transfer function of the system is proper if and only if A_{22} is invertible.

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Testing Routine

1. Remove uncontrollable and unobservable infinite poles of system (1). [Varga '89] [SLICOT: TG01JD]
2. use URV decomposition to factorize E as in the Theorem.
3. use RRQR decompostion to determine rank(A_{22}). [SLICOT: TG01FD]

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Basics

Computation of the $\mathcal{L}_\infty$-norm for proper G is connected to the computation of the eigenvalues of certain skew-Hamiltonian/Hamiltonian matrix pencils $M_\gamma - \lambda N$ with

$$M_\gamma = \begin{bmatrix} A & 0 \\ 0 & -A^T \end{bmatrix} + \begin{bmatrix} B & 0 \\ 0 & -C^T \end{bmatrix} \begin{bmatrix} -D & \gamma I \\ \gamma I & -D^T \end{bmatrix} \begin{bmatrix} C & 0 \\ 0 & B^T \end{bmatrix}, \; N = \begin{bmatrix} E & 0 \\ 0 & E^T \end{bmatrix}.$$

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Theorem [Boyd, Balakrishnan, Kabamba '89, V. '09]

Assume the pencil $A - \lambda E$ is regular and has no finite eigenvalues on the imaginary axis, $\gamma > 0$ is not a singular value of D and $\omega_0 \in \mathbb{R}$. Then, γ is a singular value of $G(\mathrm{i}\omega_0)$ if and only if $M_\gamma - \mathrm{i}\omega_0 N$ is singular.

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Theorem

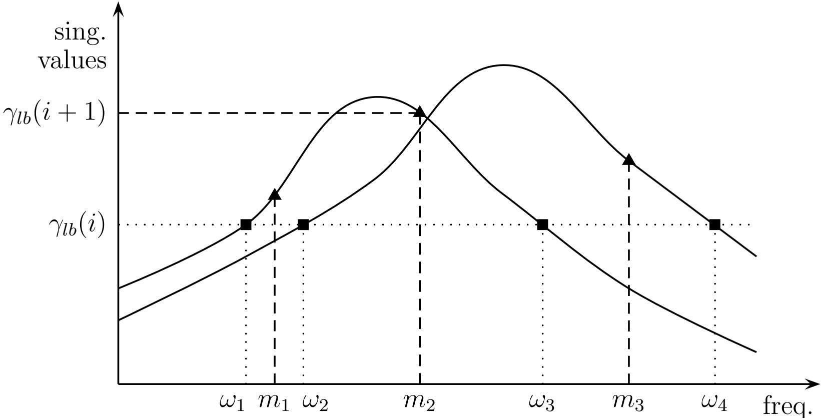
Let $\gamma > \min_{\omega \in \mathbb{R}} \sigma_{max}(G(\mathrm{i}\omega))$ be not a singular value of D. Then $\|G\|_{\mathcal{L}_\infty} \geq \gamma$ if and only if $M_\gamma - \lambda N$ has imaginary eigenvalues (i.e., at least one).

Two-Step Algorithm for Computing the $\mathcal{L}_\infty$-Norm

Compute an initial value $\gamma_{lb} < \|G\|_{\mathcal{L}_\infty}$,
repeat
  $\gamma := (1 + 2\epsilon)\gamma_{lb}$,
  Form the pencil $M_\gamma - \lambda N$ and compute its eigenvalues,
  if no imaginary eigenvalues **then**
    $\gamma_{ub} = \gamma$, break,
  else
    $\{\mathrm{i}\omega_1, \ldots, \mathrm{i}\omega_k\}$ = finite imaginary eigenvalues,
    $m_j = \frac{1}{2}(\omega_j + \omega_{j+1}),\ j = 1, \ldots, k-1$,
    compute the largest singular value of $G(\mathrm{i}m_j),\ j = 1, \ldots, k-1$,
    $\gamma_{lb} = \max_j \left(\sigma_{max}\left(G\left(\mathrm{i}m_j\right)\right)\right)$,
  end if
until break
$\|G\|_{\mathcal{L}_\infty} = \frac{1}{2}(\gamma_{lb} + \gamma_{ub})$.

[SLICOT: AB13DD]

Graphical Interpretation



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- relative error at most ε.
- Computing time is affected by number of frequency points in each step $\Rightarrow$ generally last step takes less time than the first.
- Care must be taken of the accuracy of the eigenvalue computation $\Rightarrow$ we use a structure-preserving approach which exploits the skew-Hamiltonian/Hamiltonian structure of $M_\gamma - \lambda N$ and computes imaginary eigenvalues very well. `[SLICOT: new routine MB04BD]`
  [Benner, Byers, Mehrmann, Xu '99]

Improving the Accuracy of the Eigenvalue Computation

The blocks of M_γ contain lots of matrix products and inverses $\Rightarrow$ naively building M_γ could produce a large error in the matrix and consequently in the eigenvalues.

Improving the Accuracy of the Eigenvalue Computation

$$\mathcal{M}_\gamma - \lambda\mathcal{N} = \left[\begin{array}{cc|cc} A & 0 & B & 0 \\ 0 & -A^T & 0 & -C^T \\ \hline C & 0 & D & -\gamma I_p \\ 0 & B^T & -\gamma I_m & D^T \end{array}\right] - \lambda \left[\begin{array}{cc|cc} E & 0 & 0 & 0 \\ 0 & E^T & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

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$$\mathcal{M}_\gamma - \lambda\mathcal{N} = \left[\begin{array}{cc|cc} 0 & A^T & R_{11} & R_{12} \\ A & 0 & R_{21} & R_{22} \\ \hline R_{11}^T & R_{21}^T & S_{11} & S_{12} \\ R_{12}^T & R_{22}^T & S_{12}^T & S_{22} \end{array}\right] - \lambda \left[\begin{array}{cc|cc} 0 & -E^T & 0 & 0 \\ E & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

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4. permute blocks to get skew-Hamiltonian/Hamiltonian pencil.

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$$\omega_p = |\lambda_i| \text{ where } \lambda_i \text{ is selected to}$$

- maximize $\left|\frac{\mathrm{Im}(\lambda_i)}{\mathrm{Re}(\lambda_i)} \frac{1}{|\lambda_i|}\right|$, if G has poles with $\mathrm{Im}(\lambda_i) \neq 0$,
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$$\omega_i = \sqrt{\max\left\{\frac{1}{4}\left|\lambda_i\right|^2, \mathrm{Im}\left(\lambda_i\right)^2 - \mathrm{Re}\left(\lambda_i\right)^2\right\}} \text{ for } \mathrm{Im}(\lambda_i) > 0.$$

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$$\omega_p = \mathrm{argmax}\left(\sigma_{\max}\left(G(\mathrm{i}\omega_i)\right)\right) \text{ with}$$

$$\omega_i = |\lambda_i|\sqrt{\max\left\{\frac{1}{4}, 1 - 2r^2\right\}} \text{ with } r := \frac{\mathrm{Re}(\lambda_i)}{|\lambda_i|} \text{ for } \mathrm{Im}(\lambda_i) > 0.$$

Choice of the Initial Value γ_{lb}

Question: How to choose γ_{lb}?
Many systems reach their $\mathcal{L}_\infty$-norm at the frequencies $\omega = 0$ or $\omega = \infty$ (low-pass/high-pass filter) $\Rightarrow$ evaluate G at $\omega = 0,\ \infty$. Additionally evaluate G at more test frequencies (heuristic).

Method 1: [Bruinsma, Steinbuch '90]

$$\omega_p = |\lambda_i| \text{ where } \lambda_i \text{ is selected to}$$

- maximize $\left|\frac{\mathrm{Im}(\lambda_i)}{\mathrm{Re}(\lambda_i)} \frac{1}{|\lambda_i|}\right|$, if G has poles with $\mathrm{Im}(\lambda_i) \neq 0$,
- minimize $|\lambda_i|$, if G has only real poles.

Method 2: [Sima '06]

$$\omega_p = \mathrm{argmax}\left(\sigma_{\max}\left(G(\mathrm{i}\omega_i)\right)\right) \text{ with}$$

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Computation of $G(\infty)$ (1)

Another problem is the computation of $G(\infty)$ since generally $\lim_{s \to \infty} G(s) \neq D$ for descriptor systems.

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Solution: Additively decompose G into a strictly proper part G_{sp} and a polynomial part P.

Major Steps:

1. Perform QZ algorithm with eigenvalue reordering such that $E := Q^T EZ = \begin{bmatrix} E_f & W_E \\ 0 & E_\infty \end{bmatrix}, \quad A := Q^T AZ = \begin{bmatrix} A_f & W_A \\ 0 & A_\infty \end{bmatrix}$ and $\Lambda(A_f, E_f) \subset \mathbb{C}$ and $\Lambda(A_\infty, E_\infty) = \{\infty\}$, update $B := Q^T B = \begin{bmatrix} B_f \\ B_\infty \end{bmatrix}, \quad C := CZ = \begin{bmatrix} C_f & C_\infty \end{bmatrix}.$ `[LAPACK: DGGES]`

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2. Solve the generalized Sylvester equation
$A_f Y + ZA_\infty + W_A = 0,\ E_f Y + ZE_\infty + W_E = 0$ in order to block-diagonalize the pencil $A - \lambda E$, update
$B = \begin{bmatrix} B_f \\ B_\infty \end{bmatrix} := \begin{bmatrix} B_f + ZB_\infty \\ B_\infty \end{bmatrix}, \ C = \begin{bmatrix} C_f & C_\infty \end{bmatrix} := \begin{bmatrix} C_f & C_f Y + C_\infty \end{bmatrix}.$
[SLICOT: SB04OD]

Computation of $G(\infty)$ (2)

3. Decompose the transfer function G as follows:

$$\begin{aligned} G(s) &= \begin{bmatrix} C_f & C_\infty \end{bmatrix} \left(s \begin{bmatrix} E_f & 0 \\ 0 & E_\infty \end{bmatrix} - \begin{bmatrix} A_f & 0 \\ 0 & A_\infty \end{bmatrix} \right)^{-1} \begin{bmatrix} B_f \\ B_\infty \end{bmatrix} + D \\ &= C_f(sE_f - A_f)^{-1}B_f + C_\infty(sE_\infty - A_\infty)^{-1}B_\infty + D. \end{aligned}$$

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$$= \underbrace{C_f(sE_f - A_f)^{-1}B_f}_{:=G_{sp}(s)} + \underbrace{C_\infty(sE_\infty - A_\infty)^{-1}B_\infty + D}_{:=P(s)}.$$

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Result: $G(\infty) = D - C_\infty A_\infty^{-1} B_\infty$.

1. Introduction
2. Testing Properness of a Transfer Function
3. The $\mathcal{L}_\infty$-Norm Algorithm
4. Numerical Examples
5. Outlook

Mass Spring Damper System

Example:

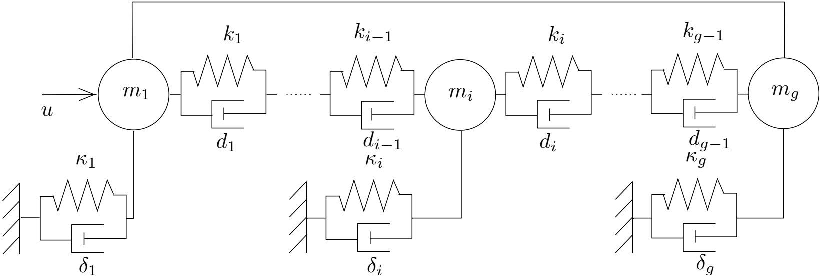


Figure: Mass spring damper system with holonomic constraints

Equations and Test Configuration

Descriptor system:

$$\begin{aligned}\dot{\mathbf{p}}(t) &= \mathbf{v}(t),\\ M\dot{\mathbf{v}}(t) &= -K\mathbf{p}(t) - D\mathbf{v}(t) + F^T\lambda(t) + B_2u(t),\\ 0 &= F\mathbf{p}(t),\\ y(t) &= C_1\mathbf{p}(t).\end{aligned}$$

Test configuration:

$$\begin{aligned}&m_1 = \ldots = m_g = 100,\\ &k_1 = \ldots k_{g-1} = \kappa_2 = \ldots = \kappa_{g-1} = 2, \quad \kappa_1 = \kappa_g = 4,\\ &d_1 = \ldots d_{g-1} = \delta_2 = \ldots = \delta_{g-1} = 5, \quad \delta_1 = \delta_g = 10.\\ &B_2 = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}^T, \quad C_1 = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}.\end{aligned}$$

$\Rightarrow n = 2g + 1$ descriptor variables, $p = m = 1$ inputs and outputs, algebraic index $= 3$.

Performance

g	$\hat{\omega}$	$\|G\|_{\mathcal{L}_\infty}$	time in s
5	0.1475	0.1590	0.0210
10	0.1693	0.1508	0.0262
20	0.1579	0.1511	0.0822
50	0.1581	0.1511	0.4533
100	0.1581	0.1511	3.0367
200	0.1581	0.1511	25.6327
300	0.1581	0.1511	90.7589
500	0.1581	0.1511	476.0880

Table: Results of $\mathcal{L}_\infty$-norm computation for mass spring damper system

Performance

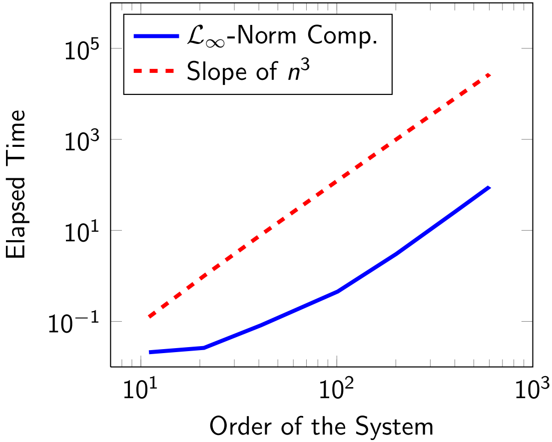


Figure: Logarithmic plot of the runtime compared to the slope of n^3

Performance

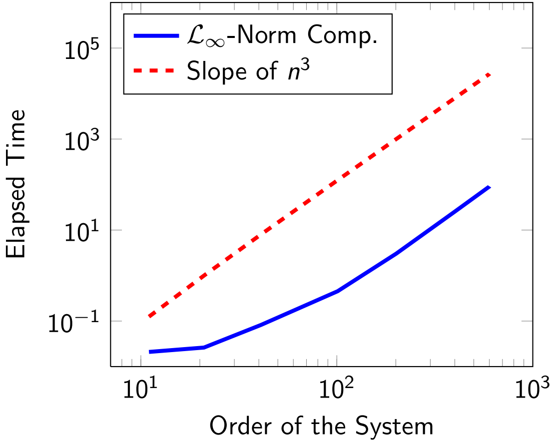


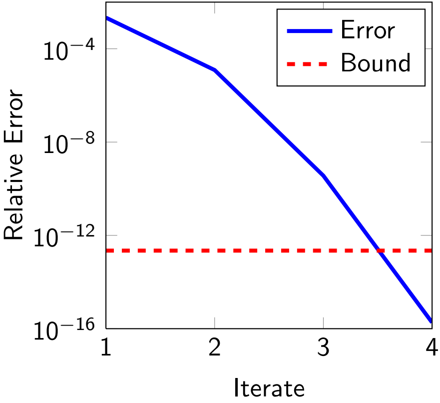
Figure: Logarithmic plot of the runtime compared to the slope of n^3

$\Rightarrow$ Cubic growth rate.

Convergence History and Relative Error

Example: Mass spring damper system with $g = 10$ masses and tolerance $\tau = 2000\varepsilon$, exact norm: $\|G\|_{\mathcal{L}_\infty} = 0.150806916481299\,51$

$\|G(0)\|_2 = 0.09550561797752823$, $\|G(\infty)\|_2 = 0.0000000000000000$,
$\|G(\mathrm{i}\omega_p)\|_2 = 0.1504803317751274$, Convergence after 4 iterations



Future Work/Open Problems

- discrete-time systems (deal with arising symplectic matrices and pencils),
- large sparse systems.

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Thank you for your Attention!