Compact Course: Optimization with Differential Equations
OvGU Magdeburg
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$\mathcal{H}_\infty$-Norm Computation for Large-Scale Descriptor Systems Via Optimization Over Structured Pseudospectra

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Descriptor Systems

Model Equations

$$\Sigma : \begin{cases} E\dot{x}(t) = Ax(t) + Bu(t), \quad x(t_0) = x_0 \\ y(t) = Cx(t), \end{cases}$$

with

- $E,\ A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{p \times n}$, E is singular!
- descriptor vector $x(t) \in \mathbb{R}^n$,
- input vector $u(t) \in \mathbb{R}^m$,
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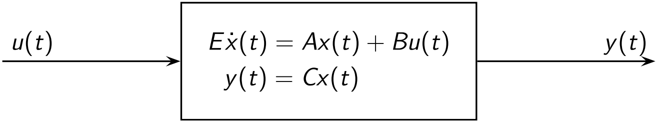

Figure: Graphical interpretation of a linear descriptor system

Matrix Pencils

Definition

A linear matrix polynomial of the form $\lambda E - A$ is called a matrix pencil.

Terminology:

- eigenvalues: roots of the characteristic polynomial $\det(\lambda E - A)$,
- regular pencil: if $\det(\lambda E - A) \not\equiv 0$, otherwise singular pencil.

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Examples

- $\lambda \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, $\det(\lambda E - A) = (\lambda - 1)(3\lambda - 2)$, $\Lambda(E, A) = \{1, \frac{2}{3}\}$;
- $\lambda \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, $\det(\lambda E - A) = -2(\lambda - 1)$, $\Lambda(E, A) = \{1, \infty\}$;
- $\lambda \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $\det(\lambda E - A) \equiv 0$, $\Lambda(E, A) = \mathbb{C}$ (singular pencil).

Weierstraß Canonical Form

Reminder: Jordan canonical form

For every matrix $A \in \mathbb{C}^{n \times n}$ there exists a nonsingular $X \in \mathbb{C}^{n \times n}$ such that

$$X^{-1}AX = \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_r \end{bmatrix}, \quad J_k = \begin{bmatrix} \lambda_k & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_k \end{bmatrix}, \quad k = 1, \ldots, r.$$

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Weierstraß canonical form

For every regular pencil $\lambda E - A$ there exist nonsingular $W,\ T \in \mathbb{C}^{n\times n}$ such that

$$W(\lambda E - A)T = \lambda \begin{bmatrix} I_{n_f} & 0 \\ 0 & N \end{bmatrix} - \begin{bmatrix} J & 0 \\ 0 & I_{n_\infty} \end{bmatrix},$$

where J and N are in Jordan canonical form, $\lambda I_{n_f} - J$ has only finite eigenvalues and $\lambda N - I_{n_\infty}$ has only infinite eigenvalues.

Decoupling of the System

Change of variables

$$
\begin{aligned}
E \quad \dot{x}(t) &= A \quad x(t) + Bu(t),\\
y(t) &= C \quad x(t).
\end{aligned}
$$

Decoupling of the System

Change of variables

$$\begin{aligned} WETT^{-1}\dot{x}(t) &= WATT^{-1}x(t) + WBu(t), \\ y(t) &= \quad CTT^{-1}x(t). \end{aligned}$$

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$$\begin{bmatrix} I_{n_f} & 0 \\ 0 & N \end{bmatrix} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} J & 0 \\ 0 & I_{n_\infty} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(t),$$

$$y(t) = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}.$$

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Decomposition into slow/fast subsystems

- slow: $\dot{x}_1(t) = Jx_1(t) + B_1u(t), \quad y_1(t) = C_1x_1(t),$
  solution: $x_1(t) = e^{J(t-t_0)}x_1(t_0) + \int_{t_0}^{t} e^{J(t-\tau)}B_1u(\tau)\mathrm{d}\tau,$
- fast: $N\dot{x}_2(t) = x_2(t) + B_2u(t), \quad y_2(t) = C_2x_2(t),$
  solution: $x_2(t) = -\sum_{i=0}^{\nu-1} N^iB_2u^{(i)}(t).$

The Fast Subsystem

$$x_2(t) = -\sum_{i=0}^{\nu-1} N^i B_2 u^{(i)}(t)$$

Features

- need the first $\nu - 1$ derivatives of the input signal $\Longrightarrow$ inputs must be sufficiently smooth,
- $\nu =$ index of nilpotency of N ($=$ smallest k such that $N^k = 0$), ν is called algebraic index of the system,
- There are restrictions to the initial conditions (consistency):

$$x_2(t_0) = -\sum_{i=0}^{\nu-1} N^i B_2 u^{(i)}(t_0).$$

Stability

Definitions

The system Σ is called (asymptotically) stable if $\lim_{t\to\infty} x(t) = 0$ for $u \equiv 0$.

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Equivalent Condition

The system Σ is (asymptotically) stable $\iff$ all finite eigenvalues of $\lambda E - A$ are in the open left half-plane.

Frequency Domain Analysis

Laplace transform

$$\mathcal{L}\{f\}(s) := \int_0^\infty e^{-st} f(t) \mathrm{d}t$$

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Transfer function

Assume $t_0 = 0$ and $Ex(0) = 0$. Then

$$\mathcal{L}(\Sigma) : \begin{cases} E\mathcal{L}\{\dot{x}\}(s) = A\mathcal{L}\{x\}(s) + B\mathcal{L}\{u\}(s), \\ \mathcal{L}\{y\}(s) = C\mathcal{L}\{x\}(s). \end{cases}$$

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Then

$$\mathcal{L}\{y\}(s) = \underbrace{C(sE - A)^{-1}B}_{=:G(s)} \mathcal{L}\{u\}(s).$$

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The transfer function $G(s)$ maps inputs to outputs in the frequency domain.

$\mathcal{H}_\infty$-Spaces and $\mathcal{H}_\infty$-Norm

Definition: the space $\mathcal{H}_\infty^{p\times m}$

$\mathcal{H}_\infty^{p\times m}$ – Hardy space of $p \times m$ functions of the form

$$G(s) = C(sE - A)^{-1}B$$

which are analytic and bounded in the open right half-plane, i.e. they are

- stable (all poles in open left half-plane);
- proper (bounded at infinity).

Remark: λ is pole of $G \Longrightarrow \lambda$ is eigenvalue of $\lambda E - A$ (not vice versa).

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Definition: $\mathcal{H}_\infty$-norm

Natural norm for the space $\mathcal{H}_\infty^{p\times m}$:

$$\|G\|_{\mathcal{H}_\infty} := \sup_{s\in\mathbb{C}^+} \sigma_{\max}\left(G\left(s\right)\right) = \sup_{\omega\in\mathbb{R}} \sigma_{\max}\left(G\left(\mathrm{i}\omega\right)\right).$$

Objective and Applications

Objective

Compute the $\mathcal{H}_\infty$-norm for systems with large and sparse matrices E, A, B, C!

Applications

- measure the distance between dynamical systems, e.g., for model order reduction;
- measure how robust a system is with respect to certain perturbations.

1 Descriptor Systems – Preliminaries

2 $\mathcal{H}_\infty$-Norm and Structured Complex Stability Radius

3 Computation of the Structured Pseudospectral Abscissa

4 Numerical Examples

5 Conclusions and Open Problems

Structured Complex Stability Radius

What happens to stability/properness if we perturb G?

Consider the perturbed transfer function

$$G_\Delta(s) := C(sE - (A + B\Delta C))^{-1}B$$

with $\Delta \in \mathbb{C}^{m \times p}$.

Question: What is the smallest ε such that there exists a Δ with $\|\Delta\|_2 < \varepsilon$ and $G_\Delta \notin \mathcal{H}_\infty^{p\times m}$? (= structured complex stability radius $r_\mathbb{C}(E, A, B, C)$)

Connection to $\mathcal{H}_\infty$-norm

$$r_\mathbb{C}(E, A, B, C) = \begin{cases} \|G\|_{\mathcal{H}_\infty}^{-1} & \text{if } G \not\equiv 0, \\ \infty & \text{if } G \equiv 0. \end{cases}$$

standard: [Hinrichsen, Pritchard '86], descriptor: [V. '11]

Distinction of Cases

Reminder

$$G_\Delta(s) := C(sE - (A + B\Delta C))^{-1}B$$

$$r_{\mathbb{C}}(E, A, B, C) := \min\left\{\|\Delta\|_2 : G_\Delta \notin \mathcal{H}_\infty^{p \times m}\right\}$$

Behavior at ∞

$$r_{\mathbb{C}}^\infty(E, A, B, C) := \inf\{\|\Delta\|_2 : G_\Delta \text{ is improper or } \lambda E - (A + B\Delta C) \text{ is a singular pencil}\}$$

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Lemma

$$r_\mathbb{C}^\infty(E, A, B, C) = \begin{cases} 1/\lim\limits_{\omega\to\infty} \sigma_{\max}(G(\mathrm{i}\omega)) & \text{if } G \not\equiv 0, \\ \infty & \text{if } G \equiv 0. \end{cases}$$

Distinction of Cases

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$$G_\Delta(s) := C(sE - (A + B\Delta C))^{-1}B$$
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Behavior at (finite) poles

$$r_{\mathbb{C}}^{\mathrm{f}}(E, A, B, C) := \inf\left\{\|\Delta\|_2 : \Pi_{\mathrm{f}}(E, A + B\Delta C, B, C) \cap \mathrm{i}\mathbb{R} \neq \emptyset\right\},$$

where $\Pi_{\mathrm{f}}(E, A, B, C)$ denotes the set of poles of $G(s) = C(sE - A)^{-1}B$.

Definitions

Definition: structured pseudospectrum of $G(s)$

$$\Pi_\varepsilon(E, A, B, C) = \{s \in \mathbb{C} : s \in \Pi_f(E, A + B\Delta C, B, C) \text{ for some } \Delta \in \mathbb{C}^{m\times p} \text{ with } \|\Delta\|_2 < \varepsilon\}.$$

Definition: structured pseudospectral abscissa

$$\alpha_\varepsilon(E, A, B, C) := \max\{\operatorname{Re} z : z \in \Pi_\varepsilon(E, A, B, C)\}.$$

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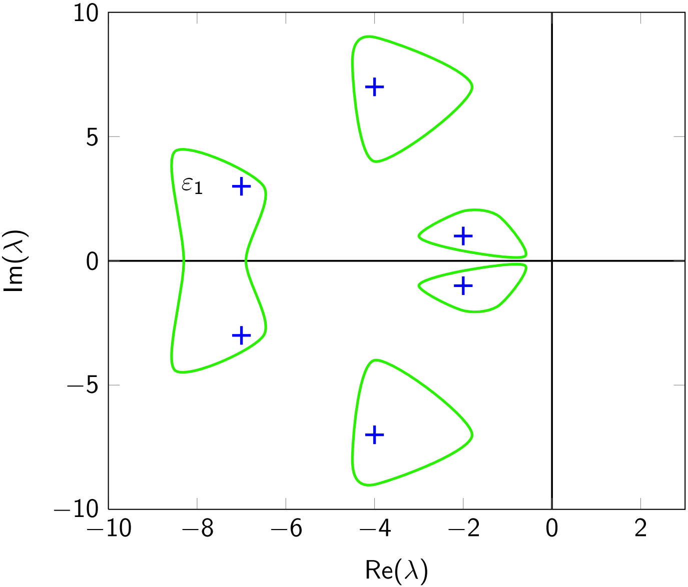
Consequence

$$\alpha_{r_{\mathbb{C}}^{\mathrm{f}}}(E, A, B, C) = 0.$$

$\Longrightarrow$ We have to find the value ε for which the corresponding structured pseudospectrum $\Pi_\varepsilon(E, A, B, C)$ touches the imaginary axis!

Graphical Interpretation



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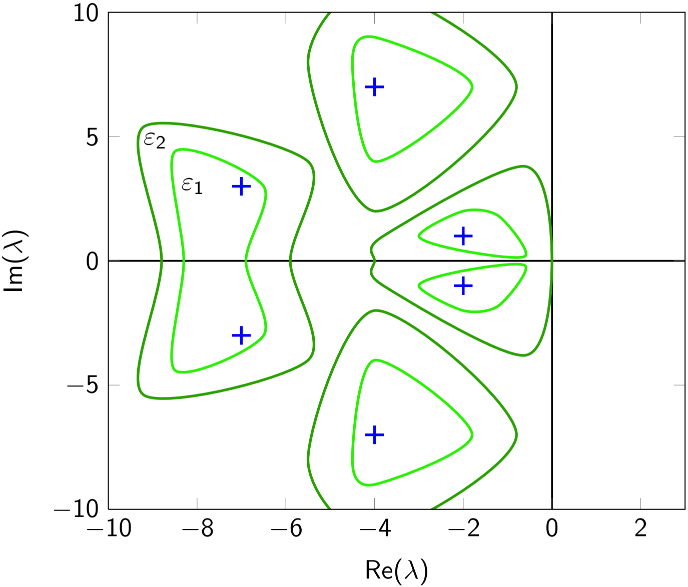

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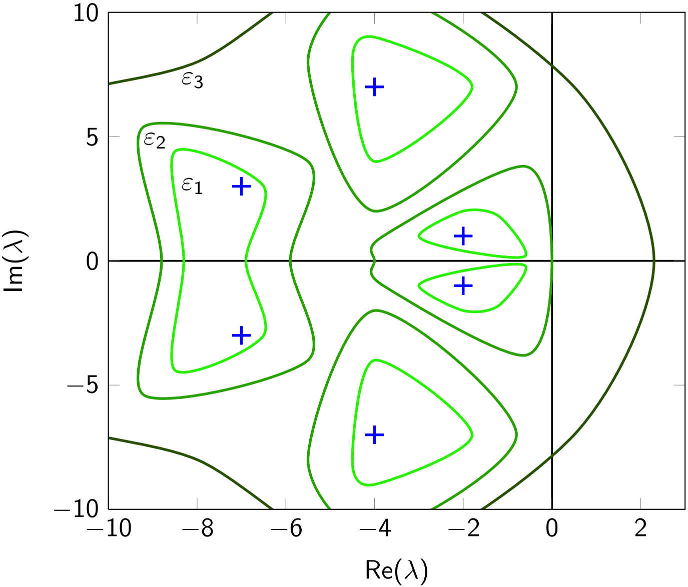

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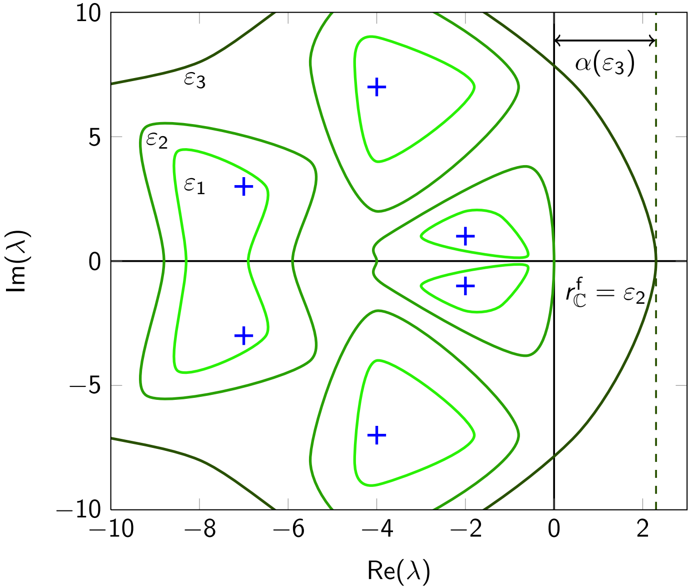

Algorithm Outline

Finding $r_{\mathbb{C}}^{\mathrm{f}}(E, A, B, C)$ is equivalent to finding the (unique) root of $\alpha(\varepsilon) := \alpha_\varepsilon(E, A, B, C)$. Thus we apply a root-finding algorithm. We

- do not have derivative information $\Longrightarrow$ Newton-like method not possible,
- can evaluate $\alpha(\varepsilon)$ relatively cheaply $\Longrightarrow$ secant method.

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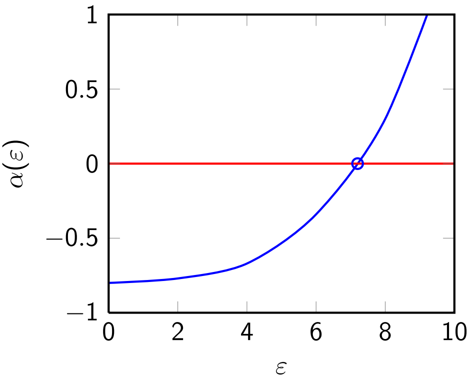

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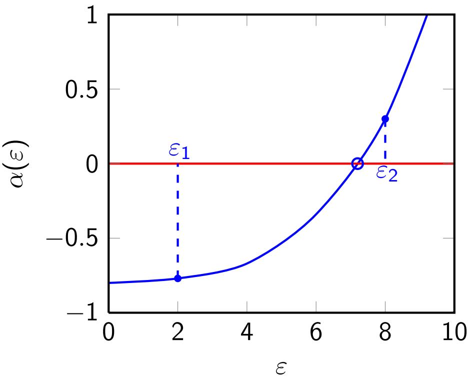

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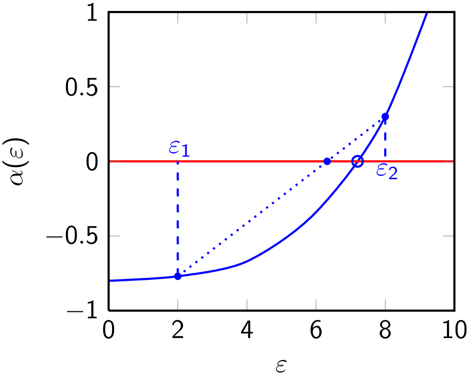

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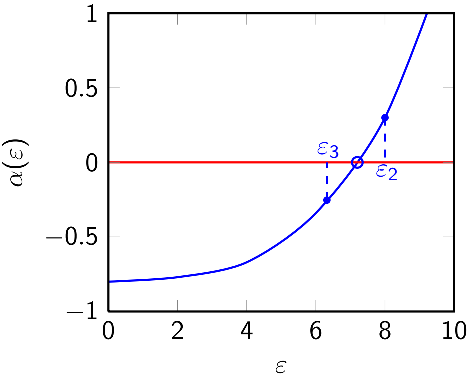

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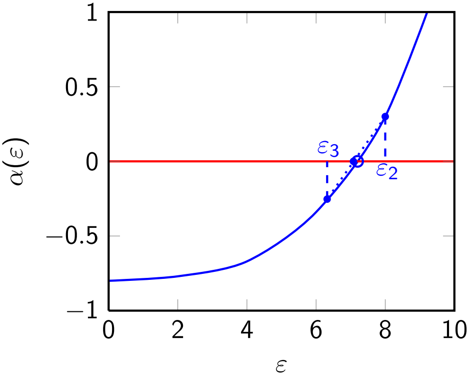

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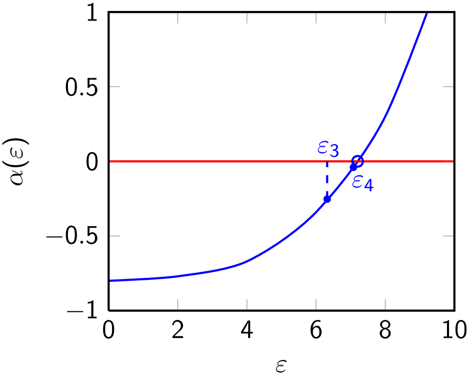

Preliminaries

Theorem

Let $s \in \mathbb{C} \backslash \Pi_{\mathrm{f}}(E, A, B, C)$ be given and $\varepsilon > 0$. Then the following statements are equivalent:

(a) $s \in \Pi_\varepsilon(E, A, B, C)$.

(b) $\sigma_{\max}(G(s)) > \varepsilon^{-1}$.

(c) There exist vectors $u \in \mathbb{C}^m$ and $v \in \mathbb{C}^p$ with $\|u\|_2 < 1$ and $\|v\|_2 < 1$ such that $s \in \Pi_{\mathrm{f}}\left(E, A + \varepsilon B u v^H C, B, C\right)$.

unstructured: [Riedel '94], structured: [V. '11]

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Corollary

$$\Pi_\varepsilon(E, A, B, C) = \Pi_\mathrm{f}(E, A, B, C) \cup \left\{s \in \mathbb{C} : \sigma_{\max}\left(G(s)\right) > \varepsilon^{-1}\right\}$$

with boundary

$$\partial\Pi_\varepsilon(E, A, B, C) = \left\{s \in \mathbb{C} : \sigma_{\max}\left(G(s)\right) = \varepsilon^{-1}\right\}.$$

First-Order Perturbation Theory

Strategy: Compute a sequence of suitable structured rank-1 perturbed pencils $\lambda E - (A + \varepsilon Buv^H C)$ such that one of the perturbed eigenvalues converges to the rightmost pseudopole of G!

[Guglielmi, Overton '11]

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[Guglielmi, Overton '11]

Lemma [Stewart, Sun '90]

Let $x, y \in \mathbb{C}^n$ be right and left eigenvectors corresponding to a simple finite eigenvalue $\lambda = \frac{y^H Ax}{y^H Ex}$ of the pencil $\lambda E - A$. Let $\lambda E - (A + tBuv^H C)$ be a perturbed matrix pencil with eigenvalue $\tilde{\lambda}$. Then it holds

$$\tilde{\lambda} = \lambda + t\frac{y^H Buv^H Cx}{y^H Ex} + \mathcal{O}\left(t^2\right).$$

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Strategy: Compute a sequence of suitable structured rank-1 perturbed pencils $\lambda E - (A + \varepsilon B u v^H C)$ such that one of the perturbed eigenvalues converges to the rightmost pseudopole of G!

[Guglielmi, Overton '11]

Lemma [Stewart, Sun '90]

Let $x, y \in \mathbb{C}^n$ be right and left eigenvectors corresponding to a simple finite eigenvalue $\lambda = \frac{y^H A x}{y^H E x}$ of the pencil $\lambda E - A$. Let $\lambda E - (A + tBuv^H C)$ be a perturbed matrix pencil with eigenvalue $\tilde{\lambda}$. Then it holds

$$\tilde{\lambda} = \lambda + t\frac{y^H B u v^H C x}{y^H E x} + \mathcal{O}\left(t^2\right).$$

Corollary

$$\left.\frac{\mathrm{d}\tilde{\lambda}(t)}{\mathrm{d}t}\right|_{t=0} = \frac{y^H B u v^H C x}{y^H E x}.$$

Construction of Structured Rank-1 Perturbations

Given: Matrix pencil $\lambda E - A$ with simple eigenvalue λ and right and left eigenvectors x, y, normalized such that $y^H E x > 0$; vectors $u \in \mathbb{C}^m$, $v \in \mathbb{C}^p$ with $\|u\|_2 = \|v\|_2 = 1$.

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Equality holds for

$$u = \frac{B^T y}{\|B^T y\|_2}, \quad v = \frac{Cx}{\|Cx\|_2}.$$

$\Longrightarrow$ This choice of u, v yields locally maximal growth in $\operatorname{Re}(\tilde{\lambda}(t))$ as t increases from 0.

Subsequent Steps

Given: Perturbed matrix pencil $\lambda E - \hat{A} = \lambda E - (A + \varepsilon B \hat{u} \hat{v}^H C)$ with simple eigenvalue $\hat{\lambda}$, and right and left eigenvectors $\hat{x}$, $\hat{y}$, normalized such that $\hat{y}^H E \hat{x} > 0$; vectors $u \in \mathbb{C}^m$, $v \in \mathbb{C}^p$ with $\|u\|_2 = \|v\|_2 = 1$.

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$$\lambda E - \left(\hat{A} + tB\left(uv^H - \hat{u}\hat{v}^H\right)C\right),$$

which is an ε-norm rank-1 perturbation of $\lambda E - A$ for $t = 0$, $t = \varepsilon$.

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$$\operatorname{Re}\left(\left. \frac{\mathrm{d}\tilde{\lambda}(t)}{\mathrm{d}t} \right|_{t=0} \right) = \frac{\operatorname{Re}\left(\hat{y}^H B \left(uv^H - \hat{u}\hat{v}^H \right) C\hat{x} \right)}{\hat{y}^H E \hat{x}}$$

$$\leq \frac{\left\| \hat{y}^H B \right\|_2 \left\| C\hat{x} \right\|_2 - \operatorname{Re}\left(\hat{y}^H B \hat{u}\hat{v}^H C\hat{x} \right)}{\hat{y}^H E \hat{x}}.$$

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Again, equality holds for

$$u = \frac{B^T\hat{y}}{\|B^T\hat{y}\|_2}, \quad v = \frac{C\hat{x}}{\|C\hat{x}\|_2},$$

which is our next perturbation!

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$\Longrightarrow$ Subspace Accelerated MIMO Dominant Pole Algorithm (SAMDP)
[Rommes, Martins '06]

Dominant Poles

Assume that $\lambda E - A$ has only simple eigenvalues λ_k with left and right eigenvectors y_k and x_k such that $y_k^H E x_k = 1$. If $G(s)$ is proper then

$$G(s) = C(sE - A)^{-1}B = \sum_{k=1}^{n} \frac{R_k}{s - \lambda_k} + R_\infty$$

with residues

$$R_k = C x_k y_k^H B, \quad R_\infty = \lim_{\omega \to \infty} G(\mathrm{i}\omega).$$

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Observation: If λ_j is close to the imaginary axis and $\|R_j\|_2$ is large, then

$$G(\mathrm{i}\omega) \approx \frac{R_j}{-\operatorname{Re}(\lambda_j)} + \sum_{\substack{k=1 \\ k \neq j}}^{n} \frac{R_k}{\mathrm{i}\omega - \lambda_k} + R_\infty$$

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$\Longrightarrow$ Compute the dominant poles $= \lambda_j$ with largest $\dfrac{\|R_j\|_2}{|\operatorname{Re}(\lambda_j)|}$!

The Complete Algorithm

Input: $\Sigma = (\lambda E - A, B, C)$, perturbation level ε, tolerance on relative change τ.

Output: $\alpha_\varepsilon(E, A, B, C)$.

Compute the dominant pole λ_0 of $(\lambda E - A, B, C)$ with left and right eigenvectors y_0 and x_0.
Compute the perturbation $\hat{A} = A + \varepsilon \frac{BB^T y_0 x_0^H C^T C}{\|B^T y_0\|_2 \|C x_0\|_2}$.
for $j = 1, 2, \ldots$ **do**
  Compute the dominant pole λ_j of $(\lambda E - \hat{A}, B, C)$ with left and right eigenvectors y_j and x_j.
  if $|\mathrm{Re}\,(\lambda_j) - \mathrm{Re}\,(\lambda_{j-1})| < \tau\,|\mathrm{Re}\,(\lambda_j)|$ **then**
    Set $k = j$.
    Break.
  end if
  Compute the perturbation $\hat{A} = A + \varepsilon \frac{BB^T y_j x_j^H C^T C}{\|B^T y_j\|_2 \|C x_j\|_2}$.
end for
$\alpha_\varepsilon(E, A, B, C) = \mathrm{Re}\,(\lambda_k)$.

Example 1 – M20PI_n

Model with $n = 1182$, $m = p = 3$.

Results:

$\|G\|_{\mathcal{H}_\infty} = 3.87260, \quad t = 6.03s, \quad \alpha_{r_{\mathbb{C}}^{\mathrm{f}}}(E, A, B, C) =$ -3.9700e-13.

Table: Convergence History

	k = 1	2	3	4
$\mathrm{Re}(\lambda_{\mathrm{dom}})$	-6.7945e-02	-6.0215e+00	-3.7397e-04	3.6222e-11
	2.3140e-03	-6.0212e+00	-3.4533e-05	3.9094e-11
	3.0285e-03	—	-3.2591e-05	3.8420e-11
	3.0355e-03	—	-3.2572e-05	—
	3.0356e-03	—	—	—
ε_k	2.58250e-01	2.06600e-01	2.58224e-01	2.58224e-01

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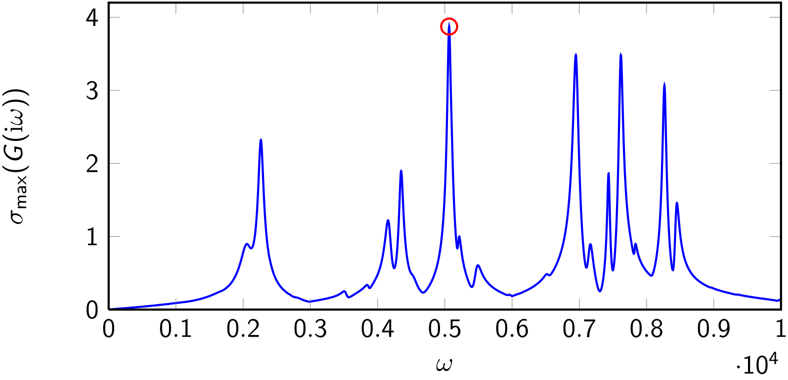

Figure: Transfer function plot with computed $\mathcal{H}_\infty$-norm

Example 2 – Mimo8x8_System

Model with $n = 13309$, $m = p = 8$.

Results:

$\|G\|_{\mathcal{H}_\infty} = 0.0534292$, $t = 106.62s$, $\alpha_{r_{\mathbb{C}}^{\mathrm{f}}}(E, A, B, C) = 2.6335\text{e-}13$.

Table: Convergence History

	k			
	1	2	3	4
$\mathrm{Re}(\lambda_{\mathrm{dom}})$	-6.2051e-03	-4.8351e-02	-9.0793e-05	-1.4183e-09
	-6.3276e-04	-4.8266e-02	-9.4865e-06	-1.3415e-09
	3.8109e-06	-4.8253e-02	3.0458e-06	-1.3273e-09
	1.1425e-04	—	5.4062e-06	-1.3245e-09
	1.3425e-04	—	5.8487e-06	-1.3241e-09
	1.3794e-04	—	5.9315e-06	—
	1.3862e-04	—	5.9470e-06	—
	1.3875e-04	—	5.9498e-06	—
ε_k	1.87276e+01	1.49821e+01	1.87168e+01	1.87164e+01

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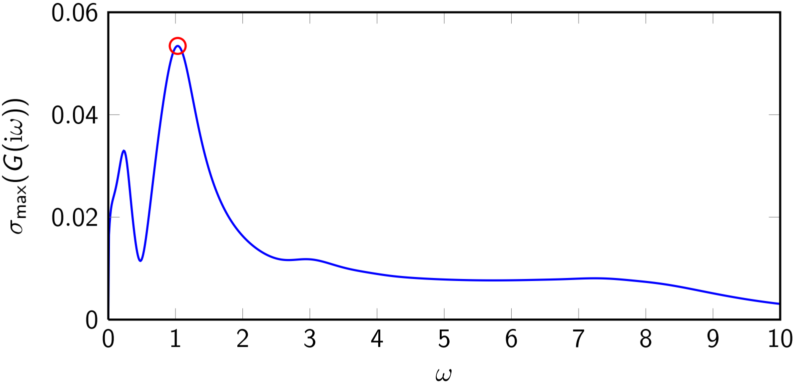

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Example 3 – Mimo46x46_System

Model with $n = 13250, \; m = p = 46$.

Results:

$\|G\|_{\mathcal{H}_\infty} = 205.631, \quad t = 167.43s, \quad \alpha_{r_{\mathbb{C}}^{\mathrm{f}}}(E, A, B, C) = 4.2864\text{e-}14.$

Table: Convergence History

	k = 1	2	3
$\mathrm{Re}(\lambda_{\mathrm{dom}})$	-9.0777e-05	-6.6047e-03	-3.6061e-06
	2.0799e-06	-6.6018e-03	2.7127e-09
	2.1973e-06	—	7.3774e-09
	2.1976e-06	—	7.3907e-09
	—	—	7.3907e-09
ε_k	4.86342e-03	3.89073e-03	4.86309e-03

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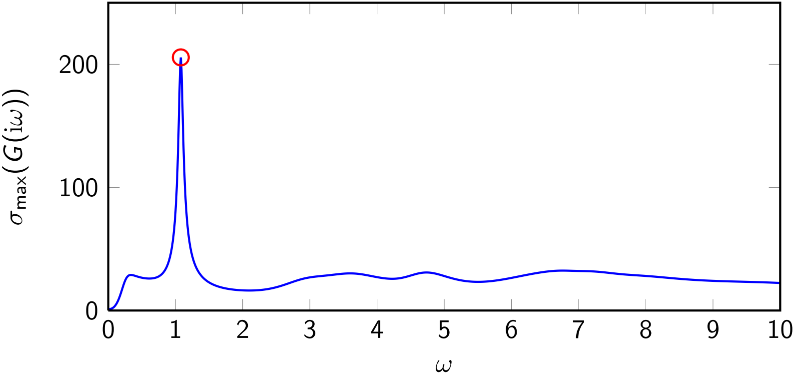

Figure: Transfer function plot with computed $\mathcal{H}_\infty$-norm

Conclusions and Outlook

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- Introduction of relations between the structured pseudospectra and the $\mathcal{H}_\infty$-norm of descriptor systems,
- development of an iterative algorithm for the computation of the $\mathcal{H}_\infty$-norm by iterating over the structured pseudospectral abscissa,

Open Problems

- Discrete-time systems $\Longrightarrow$ computation of the structured pseudospectral radius,
- real stability radii, passivity radius?

Thank you for your Attention!

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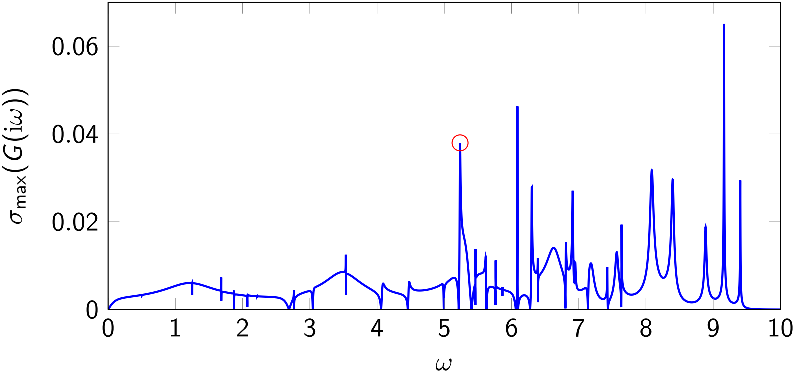
Example 4 – Peec (Does Not Work)

Model with $n = 480$, $m = p = 1$, lots of peaks close to $\mathrm{i}\mathbb{R}$!

Results:

$\|G\|_{\mathcal{H}_\infty} = 0.0379802, \quad t = 23.20s, \quad \alpha_{r_{\mathbb{C}}}(E, A, B, C) = 6.1976\text{e-}11.$



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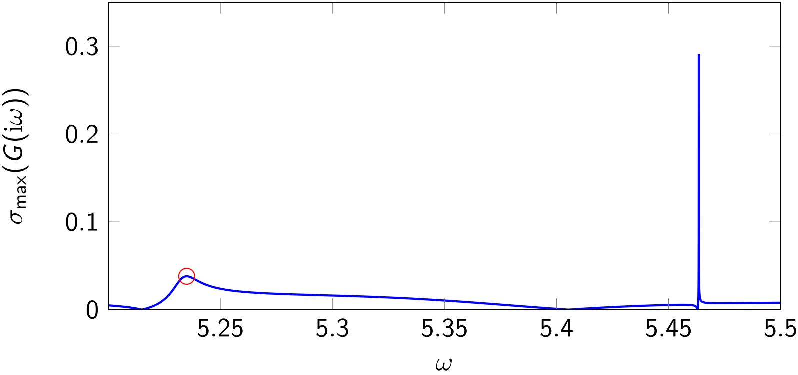

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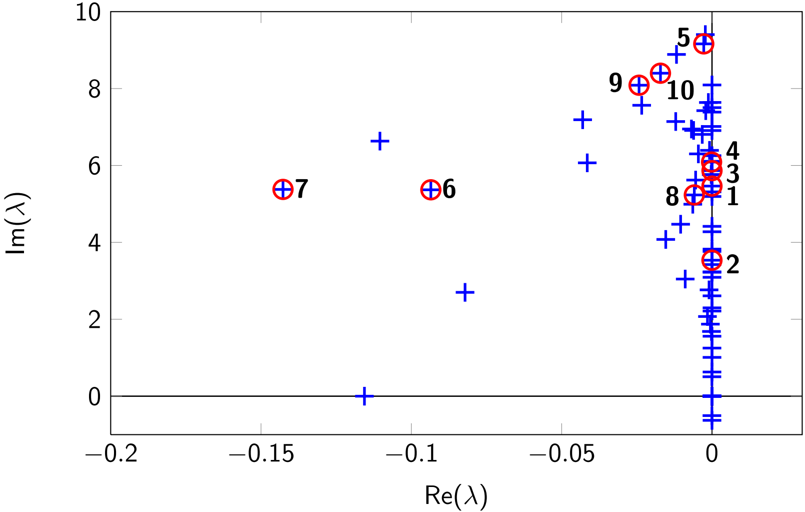

Figure: Eigenvalues (blue pluses) and the 10 most dominant poles (red circles)