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The Singular Linear-Quadratic Optimal Control Problem for Differential-Algebraic Systems

Timo Reis[1] **Matthias Voigt[2]**

[1]Department of Mathematics
University of Hamburg
Hamburg, Germany

[2]Computational Methods in Systems and Control Theory
Max Planck Institute for Dynamics of Complex Technical Systems
Magdeburg, Germany

Linear-Quadratic Optimal Control

Problem Formulation (ODE Case)

$$\mathcal{J}(x_0, u) = \int_0^\infty \begin{pmatrix} x(t) \\ u(t) \end{pmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{pmatrix} x(t) \\ u(t) \end{pmatrix} \mathrm{d}t \longrightarrow \text{inf!}$$

subject to

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0, \quad \lim_{t\to\infty} x(t) = 0.$$

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subject to

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0, \quad \lim_{t \to \infty} x(t) = 0.$$

Definitions

Minimizer: A function $\hat{u} \in \mathcal{L}^2_{\mathrm{loc}}(\mathbb{R}^+, \mathbb{R}^m)$ such that

$$\mathcal{J}(x_0, \hat{u}) = \inf \left\{ \mathcal{J}(x_0, u) : u \in \mathcal{L}^2_{\mathrm{loc}}(\mathbb{R}^+, \mathbb{R}^m) \right\}.$$

Optimal value:

$$\mathcal{V}(x_0) = \mathcal{J}(x_0, \hat{u}) = \inf \left\{ \mathcal{J}(x_0, u) : u \in \mathcal{L}^2_{\mathrm{loc}}(\mathbb{R}^+, \mathbb{R}^m) \right\}.$$

Feasibility and Solution

Questions

1. Feasibility: Does a minimizer exist for all initial values x_0?
2. Construction: If yes, how can we construct such a minimizer?

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Answers

If $R > 0$ and some further assumptions hold, then:

1. The optimal control problem is feasible if and only if the algebraic Riccati equation

$$A^T X + XA + Q - (XB + S)R^{-1}(B^T X + S^T) = 0, \quad X = X^T$$

has a solution X.

Feasibility and Solution

Questions

1. Feasibility: Does a minimizer exist for all initial values x_0?
2. Construction: If yes, how can we construct such a minimizer?

Answers

If $R > 0$ and some further assumptions hold, then:

2. The optimal solution is constructed by the unique maximal solution X^+ of the ARE, i.e.,

$$\hat{u}(t) = -R^{-1}(B^T X^+ + S^T)x(t) \quad \text{and} \quad \mathcal{V}(x_0) = x_0^T X^+ x_0.$$

Properties of the solution:

a) **Maximality:** $X \leq X^+$ for all other solutions X.
b) **Stabilization:** Closed-loop matrix $A - BR^{-1}(B^T X^+ + S^T)$ has only eigenvalues in the closed left half-plane.

Construction of the Solution

Construction of the Solution [Lancaster, Rodman '95]

Let

$$\begin{bmatrix} A - BR^{-1}S & -BR^{-1}B^T \\ S^TR^{-1}S - Q & -\left(A - BR^{-1}S\right)^T \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} T,$$

with $Y := \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} \in \mathbb{R}^{2n \times n}$ and $T \in \mathbb{R}^{n \times n}$. If Y spans a semi-stable Lagrangian invariant subspace, then

1. $X^+ = Y_2 Y_1^{-1}$;
2. $T = A - BR^{-1}(B^TX^+ + S^T)$ with $\Lambda(T) \subset \mathbb{C}^- \cup \mathrm{i}\mathbb{R}$.

Singular Linear-Quadratic Optimal Control

Problem

All the preceeding statements only hold under the assumption that $R > 0$. But what happens, if R is singular, i.e., $R \geq 0$?
Note that neither the algebraic Riccati equation

$$A^T X + XA + Q - (XB + S)R^{-1}(B^T X + S^T) = 0, \quad X = X^T,$$

nor the Hamiltonian matrix

$$\begin{bmatrix} A - BR^{-1}S & -BR^{-1}B^T \\ S^T R^{-1} S - Q & -\left(A - BR^{-1}S\right)^T \end{bmatrix}$$

can be formulated.

Intermezzo: Matrix Pencils

Definition

A linear matrix polynomial of the form $sE - A$ is called a matrix pencil.

Terminology:

a) eigenvalues: roots of the characteristic polynomial $\det(sE - A)$,

b) regular pencil: if $\det(sE - A) \not\equiv 0$, otherwise singular pencil,

c) deflating subspace $\operatorname{im} X$: fulfills $(sE - A)X = Y(s\tilde{E} - \tilde{A})$ for some $Y, \tilde{E}, \tilde{A}$, and all $s \in \mathbb{C}$.

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Examples

1. $s\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} - \begin{bmatrix}1 & 0\\0 & 2\end{bmatrix}$, $\det(sE - A) = (s-1)(s-2)$, $\Lambda(E, A) = \{1, 2\}$;
2. $s\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} - \begin{bmatrix}1 & 0\\0 & 2\end{bmatrix}$, $\det(sE - A) = -2(s-1)$, $\Lambda(E, A) = \{1, \infty\}$;
3. $s\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} - \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}$, $\det(sE - A) \equiv 0$, $\Lambda(E, A) = \mathbb{C}$ (singular pencil).

Lur'e Equations and Even Matrix Pencils

Solution [Reis '11]

1. Replace the algebraic Riccati equation by the Lur'e equation:

$$\begin{aligned} A^T X + XA + Q &= K^T K, \\ XB + S &= K^T L, \quad X = X^T, \\ R &= L^T L, \end{aligned}$$

   to be solved for $(X, K, L) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{p \times n} \times \mathbb{R}^{p \times m}$ with p as small as possible.

2. Replace the Hamiltonian matrix by an even matrix pencil:

$$s\mathcal{E} - \mathcal{A} = \begin{bmatrix} 0 & -sI_n + A & B \\ sI_n + A^T & Q & S \\ B^T & S^T & R \end{bmatrix}.$$

   (even $\Leftrightarrow \mathcal{E} = -\mathcal{E}^T$, $\mathcal{A} = \mathcal{A}^T$)

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Answers [Reis '11]

If $R \geq 0$ and some further assumptions hold, then:

1. The optimal control problem is feasible if and only if the Lur'e equation

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has a solution (X, K, L).

Feasibility and Solution

Questions

1. Feasibility: Does a minimizer exist for all initial values x_0?
2. Construction: If yes, how can we construct such a minimizer?

Answers

[Reis '11]

If $R \geq 0$ and some further assumptions hold, then:

2. The optimal solution is constructed by the unique maximal solution (X^+, K^+, L^+) of the Lur'e equation, i.e., $\hat{u}(\cdot)$ satisfies

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B\hat{u}(t), \quad x(0) = x_0, \\ 0 &= K^+x(t) + L^+\hat{u}(t), \quad \text{and} \quad \mathcal{V}(x_0) = x_0^T X^+ x_0. \end{aligned}$$

Properties of the solution:

a) **Maximality:** $X \leq X^+$ for all other solutions (X, K, L).

b) **Stabilization:** Closed-loop matrix pencil $\begin{bmatrix} -sI_n + A & B \\ K^+ & L^+ \end{bmatrix}$ has finite eigenvalues only in the closed left half-plane.

Construction of the Solution

Construction of the Solution [Reis '11]

Let

$$\begin{bmatrix} 0 & -sI_n + A & B \\ sI_n + A^T & Q & S \\ B^T & S^T & R \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix} = \begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \end{bmatrix} \left(s\tilde{\mathcal{E}} - \tilde{\mathcal{A}}\right)$$

with $Y := \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix} \in \mathbb{R}^{2n+m \times n+m}$ and $Z := \begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \end{bmatrix} \in \mathbb{R}^{2n+m \times n+p}$ and $s\tilde{\mathcal{E}} - \tilde{\mathcal{A}} \in \mathbb{R}[s]^{n+p \times n+m}$. If Y spans a semi-stable maximally $\mathcal{E}$-neutral right deflating subspace, then

1. $X^+ = Y_2 Y_1^-$ with an arbitrary right inverse Y_1^- of Y_1;
2. $s\tilde{\mathcal{E}} - \tilde{\mathcal{A}} = \begin{bmatrix} -sI_n + A & B \\ K^+ & L^+ \end{bmatrix}$ with $\Lambda\left(\tilde{\mathcal{E}}, \tilde{\mathcal{A}}\right) \subset \mathbb{C}^- \cup \mathrm{i}\mathbb{R} \cup \{\infty\}$.

Differential-Algebraic Equations – Basics

Model Equation

$$E\dot{x}(t) = Ax(t) + Bu(t), \quad Ex(0) = Ex_0.$$

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Weierstraß canonical form

For every regular pencil $sE - A$ there exist nonsingular $W,\ T \in \mathbb{C}^{n \times n}$ such that

$$W(sE - A)T = s\begin{bmatrix} I_{n_\mathrm{f}} & 0 \\ 0 & N \end{bmatrix} - \begin{bmatrix} J & 0 \\ 0 & I_{n_\infty} \end{bmatrix},$$

where J and N are in Jordan canonical form, $sI_{n_\mathrm{f}} - J$ has only finite eigenvalues and $sN - I_{n_\infty}$ has only infinite eigenvalues.

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Change of variables

$$WETT^{-1}\dot{x}(t) = WATT^{-1}x(t) + WBu(t)$$

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where J and N are in Jordan canonical form, $sI_{n_f} - J$ has only finite eigenvalues and $sN - I_{n_\infty}$ has only infinite eigenvalues.

Change of variables

$$\begin{bmatrix} I_{n_f} & 0 \\ 0 & N \end{bmatrix} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} J & 0 \\ 0 & I_{n_\infty} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(t)$$

Differential-Algebraic Equations – Basics

Change of variables

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Decomposition into slow/fast subsystems

a) slow: $\dot{x}_1(t) = Jx_1(t) + B_1 u(t)$,

solution: $x_1(t) = e^{Jt}x_1(0) + \int_0^t e^{J(t-\tau)} B_1 u(\tau)\mathrm{d}\tau$,

b) fast: $N\dot{x}_2(t) = x_2(t) + B_2 u(t)$,

solution: $x_2(t) = -\sum_{i=0}^{\nu-1} N^i B_2 u^{(i)}(t)$.

The Fast Subsystem

$$x_2(t) = -\sum_{i=0}^{\nu-1} N^i B_2 u^{(i)}(t)$$

Features

1. need the first $\nu - 1$ derivatives of the input signal $\Longrightarrow$ inputs must be sufficiently smooth,
2. ν = index of nilpotency of N (= smallest k such that $N^k = 0$), ν is called algebraic index of the system,
3. There are restrictions to the initial conditions (consistency):

$$x_2(t_0) = -\sum_{i=0}^{\nu-1} N^i B_2 u^{(i)}(t_0).$$

Linear-Quadratic Optimal Control for DAEs

Problem Formulation (DAE Case)

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subject to

$$E\dot{x}(t) = Ax(t) + Bu(t), \quad Ex(0) = Ex_0, \quad \lim_{t\to\infty} Ex(t) = 0.$$

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Questions

1. Feasibility: Does a minimizer exist for all initial values Ex_0?
2. Construction: If yes, how can we construct such a minimizer?

Generalized Lur'e Equations and Even Matrix Pencils

Generalized Structures [Reis, V. '13]

1. Replace the Lur'e equation by the generalized Lur'e equation:

$$\begin{aligned} A^T X + X^T A + Q &= K^T K + V_\infty^T \Sigma V_\infty, \\ X^T B + S &= K^T L + V_\infty^T \Sigma W_\infty, \quad E^T X = X^T E \\ R &= L^T L + W_\infty^T \Sigma W_\infty, \end{aligned}$$

   to be solved for $(X, K, L, V_\infty, W_\infty, \Sigma) \in \mathbb{R}^{m\times m} \times \mathbb{R}^{p\times n} \times \mathbb{R}^{p\times m} \times \mathbb{R}^{n-r\times n} \times \mathbb{R}^{n-r\times m} \times \mathbb{R}^{n-r\times n-r}$ with minimal p, where Σ is a signature matrix and $\ker \begin{bmatrix} V_\infty & W_\infty \end{bmatrix} = \mathcal{V}$.

2. Replace the even matrix pencil by another even matrix pencil:

$$s\mathcal{E} - \mathcal{A} = \begin{bmatrix} 0 & -sE + A & B \\ sE^T + A^T & Q & S \\ B^T & S^T & R \end{bmatrix}.$$

Feasibility and Solution

Answers [Reis, V. '13]

If $R \geq 0$ and some further assumptions hold, then:

1. The optimal control problem is feasible if and only if the generalized Lur'e equation

$$
\begin{aligned}
A^T X + X^T A + Q &= K^T K + V_\infty^T \Sigma V_\infty, \\
X^T B + S &= K^T L + V_\infty^T \Sigma W_\infty, \\
R &= L^T L + W_\infty^T \Sigma W_\infty, \\
E^T X &= X^T E
\end{aligned}
$$

has a solution $(X, K, L, V_\infty, W_\infty, \Sigma)$.

Feasibility and Solution

Answers [Reis, V. '13]

If $R \geq 0$ and some further assumptions hold, then:

2. The optimal solution is constructed by the unique maximal solution $(X^+, K^+, L^+, V_\infty^+, W_\infty^+, \Sigma^+)$ of the generalized Lur'e equation, i.e., $\hat{u}(\cdot)$ satisfies

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + B\hat{u}(t), \quad Ex(0) = Ex_0, \\ 0 &= K^+x(t) + L^+\hat{u}(t), \quad \text{and} \quad \mathcal{V}(x_0) = x_0^T E^T X^+ x_0. \end{aligned}$$

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a) **Maximality:** $E^T X \leq E^T X^+$ for all other solutions $(X, K, L, V_\infty, W_\infty, \Sigma)$.

b) **Stabilization:** Closed-loop matrix pencil $\begin{bmatrix} -sE + A & B \\ K^+ & L^+ \end{bmatrix}$ has finite eigenvalues only in the closed left half-plane.

Construction of the Solution

Construction of the Solution [Reis, V. '13]

Let

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with $Y := \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix} \in \mathbb{R}^{2n+m \times n+m}$ and $Z := \begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \end{bmatrix} \in \mathbb{R}^{2n+m \times n+p}$ and $s\tilde{\mathcal{E}} - \tilde{\mathcal{A}} \in \mathbb{R}[s]^{n+p \times n+m}$. If Y spans a semi-stable $\mathcal{E}$-neutral right deflating subspace of dimension $n+m$, then

1. $X^+ = Y_2 Y_1^-$ with an arbitrary right inverse Y_1^- of Y_1;
2. $s\tilde{\mathcal{E}} - \tilde{\mathcal{A}} = \begin{bmatrix} -sE+A & B \\ K^+ & L^+ \end{bmatrix}$ with $\Lambda\left(\tilde{\mathcal{E}}, \tilde{\mathcal{A}}\right) \subset \mathbb{C}^- \cup \mathrm{i}\mathbb{R} \cup \{\infty\}$.

Conclusion

Presented in this talk

1. Singular optimal control problems for differential-algebraic equations,
2. characterization of the optimal control via the maximal solution of a generalized Lur'e equation.

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1. Singular optimal control problems for differential-algebraic equations,
2. characterization of the optimal control via the maximal solution of a generalized Lur'e equation.

Not presented in this talk

1. Solvability criteria in terms of the eigenstructure of even pencils,
2. frequency domain analysis,
3. to do: Algorithms for solving generalized Lur'e equations.

Thanks for your Attention!

References

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3. REIS, VOIGT '13: *Spectral factorization and linear-quadratic optimal control of differential-algebraic systems*, 2013, in preparation.