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SLICOT Software for Structured Matrix Pencils

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NETWORK THEORY

MAX PLANCK INSTITUTE
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TECHNICAL SYSTEMS
MAGDEBURG

1. Structure-Preserving Solvers for Generalized Hamiltonian Eigenvalue Problems
2. Under Construction: Structured Staircase Algorithm for (Skew-)Symmetric/(Skew-)Symmetric Matrix Pencils
3. Upcoming Things

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Skew-Hamiltonian/Hamiltonian Matrix Pencils

Definition: Hamiltonian matrices and pencils

Let $\mathcal{J} := \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$. We call

- the matrix $\mathcal{S} \in \mathbb{C}^{2n \times 2n}$ skew-Hamiltonian if $(\mathcal{S}\mathcal{J})^H = -\mathcal{S}\mathcal{J}$,
- the matrix $\mathcal{H} \in \mathbb{C}^{2n \times 2n}$ Hamiltonian if $(\mathcal{H}\mathcal{J})^H = \mathcal{H}\mathcal{J}$,
- the matrix pencil $\lambda\mathcal{S} - \mathcal{H} \in \mathbb{C}^{2n \times 2n}$ skew-Hamiltonian/Hamiltonian if $\mathcal{S}$ is skew-Hamiltonian and $\mathcal{H}$ is Hamiltonian.

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for descriptor systems.

Some Properties

Properties

- block structure: $\lambda\mathcal{S} - \mathcal{H} = \lambda \begin{bmatrix} F & G \\ H & F^H \end{bmatrix} - \begin{bmatrix} R & S \\ T & -R^H \end{bmatrix}$ with skew-Hermitian $G,\ H$, and Hermitian $S,\ T$,

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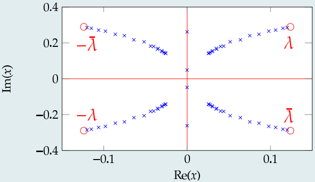

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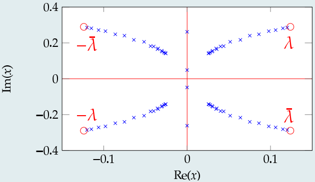

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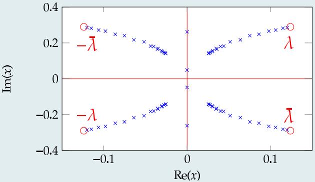

- structured Schur form:

$$\mathcal{J}\mathcal{Q}^H\mathcal{J}^T(\lambda\mathcal{S} - \mathcal{H})\mathcal{Q} = \lambda \begin{bmatrix} S_1 & S_2 \\ 0 & S_1^H \end{bmatrix} - \begin{bmatrix} H_1 & H_2 \\ 0 & -H_1^H \end{bmatrix},$$

with orthogonal $\mathcal{Q}$, upper triangular S_1, and H_1; however existence cannot be guaranteed, can be solved by structured embedding.

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- Accurate and reliable computation of the eigenvalues of $\lambda\mathcal{S} - \mathcal{H}$,

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What has been implemented:

- 14 driver routines to solve the above problems (different versions for complex and real problems and factored and unfactored matrices $\mathcal{S}$ (the factorization $\mathcal{S} = \mathcal{J}\mathcal{Z}^H\mathcal{J}^T\mathcal{Z}$ exists for skew-Hamiltonian $\mathcal{S}$)),

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In progress:

- Apply blocking technique for higher efficiency with larger problems,
- inclusion of sH/H eigensolvers and robust controller formulas into the Robust Control Toolbox of MATLAB®.

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Kronecker Canonical Form

Definition: Kronecker canonical form

Let $\lambda E - A \in \mathbb{R}^{m\times n}$. Then there exist nonsingular $P \in \mathbb{C}^{m\times m}$ and $Q \in \mathbb{C}^{n\times n}$ such that $P(\lambda E - A)Q =$ $\operatorname{diag}\left(\mathcal{O}_\eta, \mathcal{L}_{\varepsilon_1}, \ldots, \mathcal{L}_{\varepsilon_k}, \mathcal{L}_{\delta_1}^T, \ldots, \mathcal{L}_{\delta_k}^T, \mathcal{N}_{\sigma_1}, \ldots, \mathcal{N}_{\sigma_r}, \mathcal{J}_{\rho_1}, \ldots, \mathcal{J}_{\rho_s}\right)$, where

- $\mathcal{O}_\eta = \lambda \cdot 0_\eta - 0_\eta \in \mathbb{R}^\eta$,
- $\mathcal{L}_{\varepsilon_j} = \lambda \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix} \in \mathbb{R}^{\varepsilon_j \times (\varepsilon_j + 1)}$,
- $\mathcal{N}_{\sigma_j} = \lambda \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix} - \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} \in \mathbb{R}^{\sigma_j \times \sigma_j}$,
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For many structured matrix pencils there are structured Kronecker-like forms, so also for (skew-)symmetric/(skew-)symmetric matrix pencils!

Structured Staircase Form

Structured staircase form

Given: (Skew-)symmetric/(skew-)symmetric matrix pencil $\lambda\mathcal{N} - \mathcal{H} \in \mathbb{R}^{n \times n}$.

Goal: Find an orthogonal congruence transform such that

$$\lambda\tilde{\mathcal{N}} - \tilde{\mathcal{H}} := \mathcal{Q}^T \left(\lambda\mathcal{N} - \mathcal{H}\right) \mathcal{Q}$$

is in a (skew-)symmetric/(skew-)symmetric staircase form which reveals its Kronecker structure.

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Application

Information about the Kronecker structure can be used to remove singular and higher index parts of the matrix pencil which is important for many theoretical and computational issues.

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- In progress: adaption of the code to the SLICOT standard.

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Future Work

Structure-exploiting solution of palindromic and even eigenvalue problems

palindromic eigenvalue problems:

$$\mathcal{A}x = \lambda \mathcal{A}^H x,$$

even eigenvalue problems:

$$\mathcal{A}x = \lambda \mathcal{B}x \quad \text{with} \quad \mathcal{A} = \mathcal{A}^H, \quad \mathcal{B} = -\mathcal{B}^H.$$

State-of-the-art:

- included in PEPACK - a Fortran 77 software package for reducing palindromic and even matrix pencils to anti-triangular form using the Pencil Laub Trick.

Future Work

Solution of generalized projected Lyapunov equations

generalized projected continuous-time Lyapunov equations:

$$EXA^T + AXE^T = -P_l BB^T P_l^T, \quad X = P_r X P_r^T,$$
$$E^T XA + A^T XE = -P_r^T C^T C P_r, \quad X = P_l^T X P_l,$$

generalized projected discrete-time Lyapunov equations:

$$AXA^T - EXE^T = Q_l BB^T Q_l^T, \quad X = Q_r X Q_r^T,$$
$$A^T XA + E^T XE = Q_r^T C^T C Q_r, \quad X = Q_l^T X Q_l,$$

where $P_r,\ P_l,\ Q_r,\ Q_l$ are projectors onto certain deflating subspaces of $\lambda E - A$.

State-of-the-art:

- experimental Fortran code available - but needs some effort to be standardized.

Upcoming Papers & SLICOT Working Notes

- BRÜLL, MEHRMANN: *Fortran 77 Subroutines to Compute a Structured Staircase Form for a (Skew-)Symmetric/ (Skew-)Symmetric Matrix Pencil*.
- POPPE, SCHRÖDER, THIES: *Software for Computing the Numerical Solution of Palindromic and Even Eigenvalue Problems Using the Pencil Laub Trick*.
- BENNER, SIMA, VOIGT: *Fortran 77 Subroutines for the Solution of Skew-Hamiltonian/Hamiltonian Eigenproblems – Part I: Algorithms and Applications*.
- BENNER, SIMA, VOIGT: *Fortran 77 Subroutines for the Solution of Skew-Hamiltonian/Hamiltonian Eigenproblems – Part II: Implementation and Numerical Results*.

Thank you for your Attention!

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