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On the Computation of Particular Eigenvectors of Hamiltonian Matrix Pencils for Passivity Enforcement of Descriptor Systems

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(joint work with Peter Benner)

NETWORK THEORY

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Motivation

Given: Continuous-time LTI descriptor system

$$\Sigma : \quad \begin{cases} E\dot{x}(t) = Ax(t) + Bu(t), \\ \quad y(t) = Cx(t) + Du(t), \end{cases}$$

- $E, A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{m \times n}$, $D \in \mathbb{R}^{m \times m}$,
- descriptor vector $x(t) \in \mathbb{R}^n$, input vector $u(t) \in \mathbb{R}^m$, output vector $y(t) \in \mathbb{R}^m$.

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Transfer function

Alternative representation: Laplace transforming the signals x, u, y yields

$$Y(s) = G(s)U(s)$$

with transfer function $G(s) := C(sE - A)^{-1}B + D$.

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Passivity Enforcement

Find a passive system

$$\tilde{\Sigma}: \quad \begin{cases} \tilde{E}\dot{x}(t) = \tilde{A}x(t) + \tilde{B}u(t), \\ y(t) = \tilde{C}x(t) + \tilde{D}u(t), \end{cases}$$

with transfer function $\tilde{G}$ such that $\|\tilde{G} - G\|$ is small in some system norm!

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Connection to Hamiltonian eigenvalue problems

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Passivity of descriptor systems is related to bounded/positive realness of the proper parts of their transfer functions. Under some conditions, a proper stable transfer function is strictly bounded real, if and only if

$$\lambda \left[\begin{array}{cc|cc} E & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & E^T & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] - \left[\begin{array}{cc|cc} A & B & 0 & 0 \\ C & D & 0 & I_m \\ \hline 0 & 0 & -A^T & -C^T \\ 0 & -I_m & -B^T & -D^T \end{array}\right]$$

has no finite, purely imaginary eigenvalues.

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has no finite, purely imaginary eigenvalues.

Strategy: If the matrix pencils have finite, purely imaginary eigenvalues, we perturb these away from the imaginary axis! To compute the optimal perturbation, we need the corresponding eigenvectors!

Definition

Let $\mathcal{J} := \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$. A matrix pencil $\lambda\mathcal{N} - \mathcal{H} \in \mathbb{R}^{2n\times 2n}$ is called skew-Hamiltonian/Hamiltonian (sH/H) if $(\mathcal{N}\mathcal{J})^T = -\mathcal{N}\mathcal{J}$ (skew-Hamiltonian) and $(\mathcal{H}\mathcal{J})^T = \mathcal{H}\mathcal{J}$ (Hamiltonian).

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Properties

- block structure: $\lambda\mathcal{N} - \mathcal{H} = \lambda \begin{bmatrix} F & G \\ H & F^T \end{bmatrix} - \begin{bmatrix} R & S \\ T & -R^T \end{bmatrix}$ with skew-symmetric G, H, and symmetric S, T,

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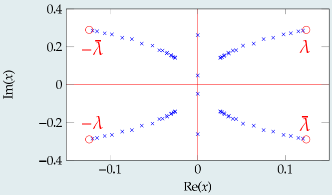
- Hamiltonian eigensymmetry (symmetry with respect to real and imaginary axis),



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Properties

- $\mathcal{J}$-congruence transformations preserve structure, i.e., $\lambda\tilde{\mathcal{N}} - \tilde{\mathcal{H}} := \mathcal{J}\mathcal{P}^T\mathcal{J}^T(\lambda\mathcal{N} - \mathcal{H})\mathcal{P}$ is again sH/H,

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- structured Schur form:

$$\mathcal{J}\mathcal{Q}^T\mathcal{J}^T(\lambda\mathcal{N} - \mathcal{H})\mathcal{Q} = \lambda\begin{bmatrix} N_1 & N_2 \\ 0 & N_1^T \end{bmatrix} - \begin{bmatrix} H_1 & H_2 \\ 0 & -H_1^T \end{bmatrix},$$

  with orthogonal $\mathcal{Q}$, upper triangular N_1, and upper quasi-triangular H_1; however existence cannot be guaranteed.

Generalized Symplectic URV Decomposition

Theorem [Benner, Byers, Mehrmann, Xu '99]

Let $\lambda\mathcal{N} - \mathcal{H}$ be a real skew-Hamiltonian/Hamiltonian matrix pencil. Then there exist orthogonal matrices $\mathcal{Q}_1$, $\mathcal{Q}_2$ such that

$$\mathcal{Q}_1^T \mathcal{N} \left(\mathcal{J}\mathcal{Q}_1\mathcal{J}^T\right) = \begin{bmatrix} N_1 & N_2 \\ 0 & N_1^T \end{bmatrix},$$

$$\left(\mathcal{J}\mathcal{Q}_2\mathcal{J}^T\right)^T \mathcal{N}\mathcal{Q}_2 = \begin{bmatrix} M_1 & M_2 \\ 0 & M_1^T \end{bmatrix} := \mathcal{M}, \qquad \text{(URV)}$$

$$\mathcal{Q}_1^T \mathcal{H}\mathcal{Q}_2 = \begin{bmatrix} H_{11} & H_{12} \\ 0 & H_{22} \end{bmatrix},$$

where N_1, M_1, H_{11} are upper triangular, H_{22}^T is upper quasi triangular, and N_2, M_2 are skew-symmetric.

Structured Doubling of the Matrix Pencil

Define the double-sized matrix pencil

$$\lambda \mathcal{B}_{\mathcal{N}} - \mathcal{B}_{\mathcal{H}} := \lambda \begin{bmatrix} \mathcal{N} & 0 \\ 0 & \mathcal{N} \end{bmatrix} - \begin{bmatrix} \mathcal{H} & 0 \\ 0 & -\mathcal{H} \end{bmatrix}.$$

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There exists an orthogonal matrix $\mathcal{X}$ such that

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Using (URV), compute orthogonal $\mathcal{J}$-congruence transformation such that

$$\begin{aligned} \lambda \bar{\mathcal{B}}_{\mathcal{N}} - \bar{\mathcal{B}}_{\mathcal{H}} :&= \mathcal{J} \mathcal{Q}^T \mathcal{J}^T \left(\lambda \tilde{\mathcal{B}}_{\mathcal{N}} - \tilde{\mathcal{B}}_{\mathcal{H}} \right) \mathcal{Q} \\ &= \lambda \left[\begin{array}{cc|cc} N_1 & 0 & N_2 & 0 \\ 0 & M_1 & 0 & M_2 \\ \hline 0 & 0 & N_1^T & 0 \\ 0 & 0 & 0 & M_1^T \end{array} \right] - \left[\begin{array}{cc|cc} 0 & H_{11} & 0 & H_{12} \\ -H_{22}^T & 0 & H_{12}^T & 0 \\ \hline 0 & 0 & 0 & H_{22} \\ 0 & 0 & -H_{11}^T & 0 \end{array} \right]. \end{aligned}$$

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eigenvalues: $\Lambda(\mathcal{H}, \mathcal{N}) = \pm \mathrm{i} \sqrt{\Lambda(N_1^{-1} H_{11} M_1^{-1} H_{22}^T)}$.

Selection of Required Eigenvalues (1)

Intermediate Step: Compute $\mathrm{Eig}_{\mathbb{R}^+}\left(\begin{bmatrix} 0 & H_{11} \\ H_{22}^T & 0 \end{bmatrix}, \begin{bmatrix} N_1 & 0 \\ 0 & M_1 \end{bmatrix}\right)$.

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- Perform eigenvalue reordering by computing orthogonal matrices U_i with

$$U_2^T N_1 U_1 = \begin{bmatrix} N_1^{(11)} & N_1^{(12)} \\ 0 & N_1^{(22)} \end{bmatrix}, \quad U_2^T H_{11} U_3 = \begin{bmatrix} H_{11}^{(11)} & H_{11}^{(12)} \\ 0 & H_{11}^{(22)} \end{bmatrix},$$

$$U_4^T M_1 U_3 = \begin{bmatrix} M_1^{(11)} & M_1^{(12)} \\ 0 & M_1^{(22)} \end{bmatrix}, \quad U_4^T H_{22}^T U_1 = \begin{bmatrix} H_{22}^{(11)} & H_{22}^{(12)} \\ 0 & H_{22}^{(22)} \end{bmatrix}.$$

[Granat, Kågström, Kressner '03]

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- $P^{(11)} := \left(N_1^{(11)}\right)^{-1} H_{11}^{(11)} \left(M_1^{(11)}\right)^{-1} H_{22}^{(11)}$ has only positive real eigenvalues. Also:

$$\Lambda\left(\begin{bmatrix} 0 & H_{11}^{(11)} \\ H_{22}^{(11)} & 0 \end{bmatrix}, \begin{bmatrix} N_1^{(11)} & 0 \\ 0 & M_1^{(11)} \end{bmatrix}\right) = \pm\sqrt{\Lambda\left(P^{(11)}\right)}.$$

Selection of Required Eigenvalues (2)

- $\lambda \begin{bmatrix} N_1^{(11)} & 0 \\ 0 & M_1^{(11)} \end{bmatrix} - \begin{bmatrix} 0 & H_{11}^{(11)} \\ H_{22}^{(11)} & 0 \end{bmatrix}$ still has negative real eigenvalues.

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- Triangularize and reorder eigenvalues, i.e., find orthogonal V_1, V_2 such that

$$V_1^T \left(\lambda \begin{bmatrix} N_1^{(11)} & 0 \\ 0 & M_1^{(11)} \end{bmatrix} - \begin{bmatrix} 0 & H_{11}^{(11)} \\ H_{22}^{(11)} & 0 \end{bmatrix} \right) V_2 = \lambda \begin{bmatrix} R_{11} & R_{12} \\ 0 & R_{22} \end{bmatrix} - \begin{bmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{bmatrix}$$

with $\Lambda(S_{11}, R_{11}) \subset \mathbb{R}^+$. [Benner, Byers, Mehrmann, Xu '99]

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- Compute the right eigenvectors of $\lambda R_{11} - S_{11}$, i.e., $W = \mathrm{Eig}_{\mathbb{R}^+}(S_{11}, R_{11})$.
- Collect all information: $X := \begin{bmatrix} X^{(1)} \\ X^{(2)} \end{bmatrix} := \begin{bmatrix} U_1^{(1)} & 0 \\ 0 & U_3^{(1)} \end{bmatrix} V_2^{(1)} W$.

Extraction of Eigenvectors

$$X := \begin{bmatrix} X^{(1)} \\ X^{(2)} \end{bmatrix} = \mathrm{Eig}_{\mathbb{R}^+} \left(\begin{bmatrix} 0 & H_{11} \\ H_{22}^T & 0 \end{bmatrix}, \begin{bmatrix} N_1 & 0 \\ 0 & M_1 \end{bmatrix} \right),$$

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$$Y := \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \mathcal{X}\mathcal{Q} \begin{bmatrix} \tilde{X} \\ 0 \end{bmatrix} = \mathrm{Eig}_{\mathrm{i}\mathbb{R}^+}\left(\mathcal{B}_{\mathcal{H}}, \mathcal{B}_{\mathcal{N}}\right),$$

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$$\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} -\mathrm{i}\mathcal{Q}_1^{(22)}X^{(1)} + \mathcal{Q}_2^{(11)}X^{(2)} \\ \mathrm{i}\mathcal{Q}_1^{(12)}X^{(1)} + \mathcal{Q}_2^{(21)}X^{(2)} \\ \mathrm{i}\mathcal{Q}_1^{(22)}X^{(1)} + \mathcal{Q}_2^{(11)}X^{(2)} \\ -\mathrm{i}\mathcal{Q}_1^{(12)}X^{(1)} + \mathcal{Q}_2^{(21)}X^{(2)} \end{bmatrix} = \mathrm{Eig}_{\mathrm{i}\mathbb{R}^+}\left(\begin{bmatrix} \mathcal{H} & 0 \\ 0 & -\mathcal{H} \end{bmatrix}, \begin{bmatrix} \mathcal{N} & 0 \\ 0 & \mathcal{N} \end{bmatrix} \right),$$

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$$\begin{bmatrix} \tilde{X} \\ 0 \end{bmatrix} = \mathrm{Eig}_{\mathrm{i}\mathbb{R}^+}(\bar{\mathcal{B}}_{\mathcal{H}}, \bar{\mathcal{B}}_{\mathcal{N}}),$$

$$\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} -\mathrm{i}\mathcal{Q}_1^{(22)}X^{(1)} + \mathcal{Q}_2^{(11)}X^{(2)} \\ \mathrm{i}\mathcal{Q}_1^{(12)}X^{(1)} + \mathcal{Q}_2^{(21)}X^{(2)} \\ \mathrm{i}\mathcal{Q}_1^{(22)}X^{(1)} + \mathcal{Q}_2^{(11)}X^{(2)} \\ -\mathrm{i}\mathcal{Q}_1^{(12)}X^{(1)} + \mathcal{Q}_2^{(21)}X^{(2)} \end{bmatrix} = \mathrm{Eig}_{\mathrm{i}\mathbb{R}^+}\left(\begin{bmatrix} \mathcal{H} & 0 \\ 0 & -\mathcal{H} \end{bmatrix}, \begin{bmatrix} \mathcal{N} & 0 \\ 0 & \mathcal{N} \end{bmatrix} \right),$$

$$Y_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} -\mathrm{i}\mathcal{Q}_1^{(22)}X^{(1)} + \mathcal{Q}_2^{(11)}X^{(2)} \\ \mathrm{i}\mathcal{Q}_1^{(12)}X^{(1)} + \mathcal{Q}_2^{(21)}X^{(2)} \end{bmatrix} = \mathrm{Eig}_{\mathrm{i}\mathbb{R}^+}(\mathcal{H}, \mathcal{N}).$$

Numerical Example

Test example:

- Test system with $n = 12$ descriptor variables, $m = 2$ inputs/outputs,
- build sH/H matrix pencil corresponding to bounded realness,
- dimension $= 28$,
- 4 purely imaginary eigenvalues with positive imaginary part.

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eigenvalues	residuals new method	residuals MATLAB eig
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Discussion of the Method

- More reliable detection of purely imaginary eigenvalues (no errors in real parts),
- slightly more accurate computation of eigenvalues and eigenvectors,
- slightly faster than standard methods for matrix pencils.

Conclusion

What we have done

- Structure-preserving algorithm for the computation of purely imaginary eigenvalues of sH/H matrix pencils,
- condensed form which can be used to reorder the needed eigenvalues and to compute corresponding eigenvectors by exploiting the structure.

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