$H_\infty$-Norm Computation for Large-Scale Descriptor Systems

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1 Introduction

2 $\mathcal{H}_\infty$-Norm and Structured Complex Stability Radius

3 Computation of the Structured Pseudospectral Abscissa

4 Numerical Examples

5 Conclusions and Open Problems
1 Introduction

2 $\mathcal{H}_\infty$-Norm and Structured Complex Stability Radius

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4 Numerical Examples

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Continuous-Time Descriptor Systems

Given: Continuous-time LTI descriptor system

\[ \Sigma : \begin{cases} 
E \dot{x}(t) = Ax(t) + Bu(t), \\
y(t) = Cx(t) 
\end{cases} \]

- \( E, A \in \mathbb{R}^{n \times n} \), \( B \in \mathbb{R}^{n \times m} \), \( C \in \mathbb{R}^{p \times n} \), \( m, p \ll n \),
- descriptor vector \( x(t) \in \mathbb{R}^n \), input vector \( u(t) \in \mathbb{R}^m \), output vector \( y(t) \in \mathbb{R}^p \).
- Assumptions: \( \lambda E - A \) is regular, all matrices are large and sparse.
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- descriptor vector \( x(t) \in \mathbb{R}^n \), input vector \( u(t) \in \mathbb{R}^m \), output vector \( y(t) \in \mathbb{R}^p \).
- Assumptions: \( \lambda E - A \) is regular, all matrices are large and sparse.

Frequency domain representation

Transfer function \( G(s) := C(sE - A)^{-1}B \)
**H_∞-Spaces and H_∞-Norm**

**Definition: the space H_p^m**

\[ H_p^m \] – Hardy space of \( p \times m \) functions of the form

\[ G(s) = C(sE - A)^{-1}B \]

which are analytic and bounded in the open right half-plane, i.e., they are

- **well-defined** (\( \lambda E - A \) regular);
- **stable** (all poles in open left half-plane);
- **proper** (bounded at infinity).
**H∞-Spaces and H∞-Norm**

**Definition: the space H_{p\times m}^∞**

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**Definition: H∞-norm**

Natural norm for the space \( H_{∞}^{p×m} \):

\[
\| G \|_{H_{∞}} := \sup_{s \in \mathbb{C}^+} \sigma_{\max}(G(s)) = \sup_{\omega \in \mathbb{R}} \sigma_{\max}(G(i\omega)).
\]
Applications

Model order reduction

Let $\hat{\Sigma} = (\hat{\lambda} \hat{E} - \hat{A}, \hat{B}, \hat{C})$ be a reduced order model of the system $\Sigma$. The transfer function of the error system $\Sigma^\text{err} := \Sigma - \hat{\Sigma}$ is given by

$$G^\text{err}(s) = [C - \hat{C}] \left( s \begin{bmatrix} E & 0 \\ 0 & \hat{E} \end{bmatrix} - \begin{bmatrix} A & 0 \\ 0 & \hat{A} \end{bmatrix} \right)^{-1} \begin{bmatrix} B \end{bmatrix}.$$ 

$\|G^\text{err}\|_{H_\infty}$ is the size of the worst-case approximation error.
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$\|G^{\text{err}}\|_{\mathcal{H}\infty}$ is the size of the worst-case approximation error.

$\mathcal{H}_\infty$-control [Green, Limebeer '95]

- Plant $P$, dynamic compensator $K$,
- noise $w$, estimation error $z$,
- $\mathcal{H}_\infty$-norm of the transfer function from $w$ to $z$ displays the worst-case influence of the disturbances $w$ on the output $z$. 
1 Introduction

2 $\mathcal{H}_\infty$-Norm and Structured Complex Stability Radius

3 Computation of the Structured Pseudospectral Abscissa

4 Numerical Examples

5 Conclusions and Open Problems
Structured Complex Stability Radius

What happens to stability/properness if we perturb $G$?

Consider the perturbed transfer function

$$G_\Delta(s) := C(sE - (A + B\Delta C))^{-1}B$$

with $\Delta \in \mathbb{C}^{m \times p}$.

**Question:** What is the smallest $\varepsilon$ such that there exists a $\Delta$ with $\|\Delta\|_2 < \varepsilon$ and $G_\Delta \notin \mathcal{H}^{p \times m}_\infty$? (= structured complex stability radius $r_{\mathbb{C}}(E, A, B, C)$)
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**Question:** What is the smallest $\varepsilon$ such that there exists a $\Delta$ with $\|\Delta\|_2 < \varepsilon$ and $G_\Delta \not\in \mathcal{H}_\infty^{p \times m}$? (=$\text{structured complex stability radius } r_C(E, A, B, C)$)

**Connection to $\mathcal{H}_\infty$-norm**

$$r_C(E, A, B, C) = \begin{cases} \|G\|_{\mathcal{H}_\infty}^{-1} & \text{if } G \not\equiv 0, \\ \infty & \text{if } G \equiv 0. \end{cases}$$

standard: [Hinrichsen, Pritchard ’86], descriptor: [B., V. ’12]
Distinction of Cases

**Reminder**

\[ G_\Delta(s) := C(sE - (A + B\Delta C))^{-1}B, \]
\[ r_C(E, A, B, C) := \min \{ \|\Delta\|_2 : G_\Delta \notin \mathcal{H}_\infty^{p \times m} \}. \]

**Behavior at \( \infty \)**

\[ r_\infty^C(E, A, B, C) := \inf \{ \|\Delta\|_2 : G_\Delta \text{ is improper or} \]
\[ \lambda E - (A + B\Delta C) \text{ is a singular pencil} \}. \]
Distinction of Cases

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\[ G_\Delta(s) := C(sE - (A + B\Delta C))^{-1}B, \]
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Behavior at \( \infty \)

\[ r_C^\infty(E, A, B, C) := \inf \left\{ \|\Delta\|_2 : G_\Delta \text{ is improper or} \right. \]
\[ \lambda E - (A + B\Delta C) \text{ is a singular pencil} \}. \]

Lemma

\[ r_C^\infty(E, A, B, C) = \begin{cases} 
1/ \lim_{\omega \to \infty} \sigma_{\max}(G(i\omega)) & \text{if } G \not\equiv 0, \\
\infty & \text{if } G \equiv 0. 
\end{cases} \]
Distinction of Cases

Reminder

\[ G_\Delta(s) : = C(sE - (A + B\Delta C))^{-1}B, \]
\[ r_C(E, A, B, C) : = \min \{ \|\Delta\|_2 : G_\Delta \not\in \mathcal{H}_\infty^{p \times m} \} . \]

Behavior at (finite) poles

\[ r_f(E, A, B, C) : = \inf \{ \|\Delta\|_2 : \Pi_f(E, A + B\Delta C, B, C) \cap i\mathbb{R} \neq \emptyset \} , \]
where \( \Pi_f(E, A, B, C) \) denotes the set of poles of \( G(s) = C(sE - A)^{-1}B. \)
## Definitions

**Definition: structured pseudospectrum of \( G \)**

\[
\Psi_{\varepsilon}(E, A, B, C) = \{s \in \mathbb{C} : s \in \Psi_f(E, A + B\Delta C, B, C) \text{ for some } \\
\Delta \in \mathbb{C}^{m \times p} \text{ with } \|\Delta\|_2 < \varepsilon \}.
\]

**Definition: structured pseudospectral abscissa**

\[
\alpha_{\varepsilon}(E, A, B, C) := \max \{\text{Re } s : s \in \Psi_{\varepsilon}(E, A, B, C)\}.
\]

(= the real part of the rightmost pseudopole)
### Definitions

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\((= \text{the real part of the rightmost pseudopole})\)

**Consequence**

\[
\alpha_{rf}^C(E, A, B, C) = 0. 
\]
Definitions

**Definition: structured pseudospectrum of $G$**

$$\Pi_\varepsilon(E, A, B, C) = \{s \in \mathbb{C} : s \in \Pi_f(E, A + B\Delta C, B, C) \text{ for some } \Delta \in \mathbb{C}^{m \times p} \text{ with } \|\Delta\|_2 < \varepsilon\}.$$ 

**Definition: structured pseudospectral abscissa**

$$\alpha_\varepsilon(E, A, B, C) := \max \{\Re s : s \in \Pi_\varepsilon(E, A, B, C)\}.$$ 

($= \text{the real part of the rightmost pseudopole}$)

**Consequence**

$$\alpha_{rf}^f(E, A, B, C) = 0.$$ 

$\implies$ We have to find the value $\varepsilon$ for which the corresponding structured pseudospectrum $\Pi_\varepsilon(E, A, B, C)$ touches the imaginary axis!
Graphical Interpretation
Graphical Interpretation

The figure shows a graphical interpretation of the real and imaginary parts of \( \lambda \) for different values of \( \varepsilon \). The contours represent the regions where \( \lambda \) lies for given \( \varepsilon \) values.

- \( \varepsilon_1 \) and \( \varepsilon_2 \) are different parameter values that define the regions.

The plus signs (+) indicate specific points in the complex plane that are of interest for the analysis.

The graph helps in visualizing the stability properties of the system by showing how \( \lambda \) behaves under different perturbations.
Graphical Interpretation
Graphical Interpretation

![Graphical Interpretation]

- \( \varepsilon_3 \)
- \( \varepsilon_2 \)
- \( \varepsilon_1 \)
- \( r_{fc}^f = \varepsilon_2 \)
- \( \alpha(\varepsilon_3) \)
Finding $r_C^f(E,A,B,C)$ is equivalent to finding the (unique) root of $\alpha(\varepsilon) := \alpha_{\varepsilon}(E,A,B,C)$. Thus we apply a root-finding algorithm. We

- do not have derivative information $\implies$ Newton-like method not possible,
- can evaluate $\alpha(\varepsilon)$ relatively cheaply $\implies$ secant method.
Algorithm Outline

Finding $r_f^c(E, A, B, C)$ is equivalent to finding the (unique) root of $\alpha(\varepsilon) := \alpha_\varepsilon(E, A, B, C)$. Thus we apply a root-finding algorithm. We

- do not have derivative information $\implies$ Newton-like method not possible,
- can evaluate $\alpha(\varepsilon)$ relatively cheaply $\implies$ secant method.
Algorithm Outline

Finding $r_f^\xi(E, A, B, C)$ is equivalent to finding the (unique) root of $\alpha(\varepsilon) := \alpha_\varepsilon(E, A, B, C)$. Thus we apply a root-finding algorithm. We
- do not have derivative information $\implies$ Newton-like method not possible,
- can evaluate $\alpha(\varepsilon)$ relatively cheaply $\implies$ secant method.
Algorithm Outline

Finding $r^f_{\mathbb{C}}(E, A, B, C)$ is equivalent to finding the (unique) root of $\alpha(\varepsilon) := \alpha_\varepsilon(E, A, B, C)$. Thus we apply a root-finding algorithm. We

- do not have derivative information $\Rightarrow$ Newton-like method not possible,
- can evaluate $\alpha(\varepsilon)$ relatively cheaply $\Rightarrow$ secant method.
Algorithm Outline

Finding $r_f^E(E, A, B, C)$ is equivalent to finding the (unique) root of $\alpha(\varepsilon) := \alpha_\varepsilon(E, A, B, C)$. Thus we apply a root-finding algorithm. We
- do not have derivative information $\implies$ Newton-like method not possible,
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![Graph showing root-finding process]

Max Planck Institute Magdeburg

Matthias Voigt, $\mathcal{H}_\infty$-Norm Computation for Large-Scale Descriptor Systems
Algorithm Outline

Finding $r_C^f(E, A, B, C)$ is equivalent to finding the (unique) root of $\alpha(\varepsilon) := \alpha_\varepsilon(E, A, B, C)$. Thus we apply a root-finding algorithm. We
- do not have derivative information $\implies$ Newton-like method not possible,
- can evaluate $\alpha(\varepsilon)$ relatively cheaply $\implies$ secant method.
Finding \( r^f_C(E, A, B, C) \) is equivalent to finding the (unique) root of \( \alpha(\varepsilon) := \alpha_\varepsilon(E, A, B, C) \). Thus we apply a root-finding algorithm. We
- do not have derivative information \( \implies \) Newton-like method not possible,
- can evaluate \( \alpha(\varepsilon) \) relatively cheaply \( \implies \) secant method.
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3. Computation of the Structured Pseudospectral Abscissa

4. Numerical Examples

5. Conclusions and Open Problems
Theorem

Let \( s \in \mathbb{C} \setminus \Pi_f(E, A, B, C) \) be given and \( \varepsilon > 0 \). Then the following statements are equivalent:

(a) \( s \in \Pi_\varepsilon(E, A, B, C) \).

(b) \( \sigma_{\max}(G(s)) > \varepsilon^{-1} \).

(c) There exist vectors \( u \in \mathbb{C}^m \) and \( v \in \mathbb{C}^p \) with \( \|u\|_2 < 1 \) and \( \|v\|_2 < 1 \) such that \( s \in \Pi_f(E, A + \varepsilon Bu v^H C, B, C) \).

unstructured: [RIEDEL ’94], structured: [B., V. ’12]
Preliminaries

Theorem

Let $s \in \mathbb{C} \setminus \Pi_f(E, A, B, C)$ be given and $\varepsilon > 0$. Then the following statements are equivalent:

(a) $s \in \Pi_\varepsilon(E, A, B, C)$.
(b) $\sigma_{\text{max}}(G(s)) > \varepsilon^{-1}$.
(c) There exist vectors $u \in \mathbb{C}^m$ and $v \in \mathbb{C}^p$ with $\|u\|_2 < 1$ and $\|v\|_2 < 1$ such that $s \in \Pi_f(E, A + \varepsilon Buv^H C, B, C)$.

unstructured: [RIEDEL ’94], structured: [B., V. ’12]

Corollary

$$\Pi_\varepsilon(E, A, B, C) = \Pi_f(E, A, B, C) \cup \{ s \in \mathbb{C} : \sigma_{\text{max}}(G(s)) > \varepsilon^{-1} \}$$

with boundary

$$\partial \Pi_\varepsilon(E, A, B, C) = \{ s \in \mathbb{C} : \sigma_{\text{max}}(G(s)) = \varepsilon^{-1} \}.$$
First-Order Perturbation Theory

**Strategy:** Compute a sequence of suitable structured rank-1 perturbed pencils \( \lambda E - (A + \varepsilon Buv^H C) \) such that one of the perturbed eigenvalues converges to the rightmost pseudopole of \( G \! \]

\[ \text{[GUGLIELMI, OVERTON '11]} \]
First-Order Perturbation Theory

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[GUGLIELMI, OVERTON '11]

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**Lemma** [STEWART, SUN '90]

Let $x, y \in \mathbb{C}^n$ be right and left eigenvectors corresponding to a simple finite eigenvalue $\lambda = \frac{y^H Ax}{y^H Ex}$ of the pencil $\lambda E - A$. Let $\lambda E - (A + tBuv^H C)$ be a perturbed matrix pencil with eigenvalue $\tilde{\lambda}$. Then it holds

$$\tilde{\lambda} = \lambda + t \frac{y^H Buv^H Cx}{y^H Ex} + O(t^2) .$$
First-Order Perturbation Theory

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**Lemma**

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Let $x, y \in \mathbb{C}^n$ be right and left eigenvectors corresponding to a simple finite eigenvalue $\lambda = \frac{y^H A x}{y^H E x}$ of the pencil $\lambda E - A$. Let $\lambda E - (A + tBuv^H C)$ be a perturbed matrix pencil with eigenvalue $\tilde{\lambda}$. Then it holds

$$\tilde{\lambda} = \lambda + t \frac{y^H Buv^H C x}{y^H E x} + O(t^2).$$

**Corollary**

$$\frac{d\tilde{\lambda}(t)}{dt} \bigg|_{t=0} = \frac{y^H Buv^H C x}{y^H E x}.$$
Given: Matrix pencil $\lambda E - A$ with simple eigenvalue $\lambda$ and right and left eigenvectors $x, y$, normalized such that $y^HEx > 0$; vectors $u \in \mathbb{C}^m$, $v \in \mathbb{C}^p$ with $\|u\|_2 = \|v\|_2 = 1$. 
Construction of Structured Rank-1 Perturbations

**Given:** Matrix pencil $\lambda E - A$ with simple eigenvalue $\lambda$ and right and left eigenvectors $x, y$, normalized such that $y^H Ex > 0$; vectors $u \in \mathbb{C}^m$, $v \in \mathbb{C}^p$ with $\|u\|_2 = \|v\|_2 = 1$.

\[
\text{Re} \left( \frac{d \tilde{\lambda}(t)}{dt} \bigg|_{t=0} \right) = \frac{\text{Re} \left( y^H Buv^H Cx \right)}{y^H Ex} \\
\leq \frac{\|y^H B\|_2 \|Cx\|_2}{y^H Ex}.
\]
Construction of Structured Rank-1 Perturbations

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\text{Re} \left( \frac{d\tilde{\lambda}(t)}{dt} \right) \bigg|_{t=0} = \frac{\text{Re} \left( y^H B u v^H C x \right)}{y^H Ex} \leq \frac{\|y^H B\|_2 \|C x\|_2}{y^H Ex}.
\]

Equality holds for

\[
u = \frac{C x}{\|C x\|_2}, \quad u = \frac{B^T y}{\|B^T y\|_2}.
\]

\[\Rightarrow\] This choice of $u, v$ yields locally maximal growth in $\text{Re}(\tilde{\lambda}(t))$ as $t$ increases from 0.
Subsequent Steps

**Given:** Perturbed matrix pencil \( \lambda E - \hat{A} = \lambda E - (A + \varepsilon B \hat{u} \hat{v}^H C) \) with simple eigenvalue \( \hat{\lambda} \), and right and left eigenvectors \( \hat{x}, \hat{y} \), normalized such that \( \hat{y}^H E \hat{x} > 0 \); vectors \( u \in \mathbb{C}^m \), \( v \in \mathbb{C}^p \) with \( \| u \|_2 = \| v \|_2 = 1 \).
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Given: Perturbed matrix pencil $\lambda E - \hat{A} = \lambda E - (A + \varepsilon B\hat{u}\hat{v}^H C)$ with simple eigenvalue $\hat{\lambda}$, and right and left eigenvectors $\hat{x}$, $\hat{y}$, normalized such that $\hat{y}^H E\hat{x} > 0$; vectors $u \in \mathbb{C}^m$, $v \in \mathbb{C}^p$ with $\|u\|_2 = \|v\|_2 = 1$. Consider

$$\lambda E - \left(\hat{A} + tB\left(uv^H - \hat{u}\hat{v}^H\right)C\right),$$

which is an $\varepsilon$-norm rank-1 perturbation of $\lambda E - A$ for $t = 0$, $t = \varepsilon$. 
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**Given:** Perturbed matrix pencil $\lambda E - \hat{A} = \lambda E - (A + \varepsilon B\hat{u}\hat{v}^H C)$ with simple eigenvalue $\hat{\lambda}$, and right and left eigenvectors $\hat{x}$, $\hat{y}$, normalized such that $\hat{y}^H E \hat{x} > 0$; vectors $u \in \mathbb{C}^m$, $v \in \mathbb{C}^p$ with $\|u\|_2 = \|v\|_2 = 1$. Consider

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$$
\text{Re} \left( \frac{d\tilde{\lambda}(t)}{dt} \bigg|_{t=0} \right) = \frac{\text{Re} (\hat{y}^H B (uv^H - \hat{u}\hat{v}^H) C \hat{x})}{\hat{y}^H E \hat{x}} 
\leq \frac{\|\hat{y}^H B\|_2 \|C \hat{x}\|_2 - \text{Re} (\hat{y}^H B\hat{u}\hat{v}^H C \hat{x})}{\hat{y}^H E \hat{x}}.
$$
Subsequent Steps

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which is an $\varepsilon$-norm rank-1 perturbation of $\lambda E - A$ for $t = 0$, $t = \varepsilon$.

$$\Re \left(\frac{d\hat{\lambda}(t)}{dt}\right)\bigg|_{t=0} = \frac{\Re \left(\hat{y}^H B (uv^H - \hat{u} \hat{v}^H) C \hat{x}\right)}{\hat{y}^H E \hat{x}} \leq \frac{\|\hat{y}^H B\|_2 \|C \hat{x}\|_2 - \Re \left(\hat{y}^H B \hat{u} \hat{v}^H C \hat{x}\right)}{\hat{y}^H E \hat{x}}.$$

Again, equality holds for

$$u = \frac{B^T \hat{y}}{\|B^T \hat{y}\|_2}, \quad v = \frac{C \hat{x}}{\|C \hat{x}\|_2},$$

which is our next perturbation!
Choice of the Poles

We showed how to optimally perturb a chosen pole!

**But:** Which pole is the best choice?
Choice of the Poles

We showed how to optimally perturb a chosen pole!

**But:** Which pole is the best choice? It should

- have a sufficiently large real part,
- be sufficiently controllable and observable, i.e., \( \| B^T y \|_2 \) and \( \| Cx \|_2 \) are “large”

... to get fast convergence.
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**Idea:** Use an iterative eigensolver which converges to the eigenvalues which have highest “dominance” with respect to some predefined measure!
Choice of the Poles

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**Idea:** Use an iterative eigensolver which converges to the eigenvalues which have highest “dominance” with respect to some predefined measure!

$\implies$ **Subspace Accelerated MIMO Dominant Pole Algorithm (SAMDP)**

[Rommes, Martins '06]
Dominant Poles

Assume that $\lambda E - A$ has only simple eigenvalues $\lambda_k$ with left and right eigenvectors $y_k$ and $x_k$ such that $y_k^H Ex_k = 1$. If $G(s)$ is proper then

$$G(s) = C(sE - A)^{-1}B = \sum_{k=1}^{n} \frac{R_k}{s - \lambda_k} + R_\infty$$

with residues

$$R_k = Cx_k y_k^H B, \quad R_\infty = \lim_{\omega \to \infty} G(i\omega).$$
Dominant Poles

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with residues

$$R_k = C x_k y_k^H B, \quad R_{\infty} = \lim_{\omega \to \infty} G(i\omega).$$

Observation: If $\lambda_j$ is close to the imaginary axis and $\|R_j\|_2$ is large, then

$$G(i\omega) \approx \frac{R_j}{-\text{Re}(\lambda_j)} + \sum_{k=1}^{n} \frac{R_k}{i\omega - \lambda_k} + R_{\infty}$$

for $\omega \approx \text{Im}(\lambda_j)$ and therefore $\|G(i\omega)\|_2$ is large, too.
Dominant Poles

**Definition**

An eigenvalue $\lambda_j \in \Lambda(E, A)$ is called **dominant pole** of $G(s)$, if

\[
\frac{\|R_k\|_2}{|\text{Re}(\lambda_k)|} < \frac{\|R_j\|_2}{|\text{Re}(\lambda_j)|}, \quad k = 1, \ldots, n, \quad k \neq j.
\]
Dominant Poles

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An eigenvalue $\lambda_j \in \Lambda(E, A)$ is called dominant pole of $G(s)$, if

$$\frac{\|R_k\|_2}{|\text{Re}(\lambda_k)|} < \frac{\|R_j\|_2}{|\text{Re}(\lambda_j)|}, \quad k = 1, \ldots, n, \quad k \neq j.$$ 

Problem with this Approach:

Poles loose dominance when they have crossed the imaginary axis

$\implies$ need an alternative dominance measure!
Dominant Poles – New Definition

Definition

An eigenvalue $\lambda_j \in \Lambda(E, A)$ is called (exponentially) dominant pole of $G(s)$, if

$$\|R_k\|_2 \exp(\beta \Re(\lambda_k)) < \|R_j\|_2 \exp(\beta \Re(\lambda_j)), \quad k = 1, \ldots, n, \quad k \neq j.$$
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An eigenvalue $\lambda_j \in \Lambda(E, A)$ is called (exponentially) dominant pole of $G(s)$, if

$$\|R_k\|_2 \exp(\beta \text{Re}(\lambda_k)) < \|R_j\|_2 \exp(\beta \text{Re}(\lambda_j)), \quad k = 1, \ldots, n, \quad k \neq j.$$

**Remarks:**

- $\beta$ is a weighting factor which can be choosen to weigh the relevance of real part with the residual.
- For our purpose: $\beta = 100$ (high weight on real part).
- SAMDP can also be used to generate very good initial estimates of the $\mathcal{H}_\infty$-norm.
The Complete Algorithm

**Input:** $\Sigma = (\lambda E - A, B, C)$, perturbation level $\varepsilon$, tolerance on relative change $\tau$.

**Output:** $\alpha_\varepsilon(E, A, B, C)$.

1. Compute the dominant pole $\lambda_0$ of $(\lambda E - A, B, C)$ with left and right eigenvectors $y_0$ and $x_0$.
2. Compute the perturbation $\hat{A} = A + \varepsilon \frac{BB^T y_0 x_0^H C^T C}{\|B^T y_0\|_2 \|Cx_0\|_2}$.
3. for $j = 1, 2, \ldots$ do
4. Compute the dominant pole $\lambda_j$ of $(\lambda E - \hat{A}, B, C)$ with left and right eigenvectors $y_j$ and $x_j$.
5. if $|\text{Re}(\lambda_j) - \text{Re}(\lambda_{j-1})| < \tau |\text{Re}(\lambda_j)|$ then
6. Set $k = j$.
8. end if
9. Compute the perturbation $\hat{A} = A + \varepsilon \frac{BB^T y_j x_j^H C^T C}{\|B^T y_j\|_2 \|Cx_j\|_2}$.
10. end for
11. $\alpha_\varepsilon(E, A, B, C) = \text{Re}(\lambda_k)$. 

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Example 1 – M20PI\_n

Model with $n = 1182$, $m = p = 3$.

**Results:**

$\|G\|_{\mathcal{H}_\infty} = 3.87260$, $t = 6.03s$, $\alpha_{rf}(E, A, B, C) = -3.9700e-13$.

**Table: Convergence History**

<table>
<thead>
<tr>
<th>$k$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{Re}(\lambda_{\text{dom}})$</td>
<td>-6.7945e-02</td>
<td>-6.0215e+00</td>
<td>-3.7397e-04</td>
<td>3.6222e-11</td>
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<tr>
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<td>2.3140e-03</td>
<td>-6.0212e+00</td>
<td>-3.4533e-05</td>
<td>3.9094e-11</td>
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<td>3.0285e-03</td>
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<td>-3.2591e-05</td>
<td>3.8420e-11</td>
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<td>3.0355e-03</td>
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<td>-3.2572e-05</td>
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<tr>
<td></td>
<td>3.0356e-03</td>
<td>—</td>
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<tr>
<td>$\varepsilon_k$</td>
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<td>2.06600e-01</td>
<td>2.58224e-01</td>
<td>2.58224e-01</td>
</tr>
</tbody>
</table>
Example 1 – M20PI\_n

Model with $n = 1182$, $m = p = 3$.

Results:

$$\|G\|_{\mathcal{H}_{\infty}} = 3.87260, \quad t = 6.03\text{s}, \quad \alpha_{\mathcal{C}}(E, A, B, C) = -3.9700\cdot10^{-13}.$$
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Model with $n = 1182$, $m = p = 3$.

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\[ \| G \|_{\mathcal{H}_\infty} = 3.87260, \quad t = 6.03s, \quad \alpha_f(E, A, B, C) = -3.9700e-13. \]
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Model with \( n = 1182, \ m = p = 3 \).

Results:
\[ \| G \|_{\mathcal{H}_\infty} = 3.87260, \quad t = 6.03s, \quad \alpha_{r_C}(E, A, B, C) = -3.9700e-13. \]
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Figure: Transfer function plot with computed $\mathcal{H}_{\infty}$-norm
Example 2 – Mimo8x8 System

Model with $n = 13309$, $m = p = 8$.

Results:

$\|G\|_{\mathcal{H}_\infty} = 0.0534292$, $t = 106.62s$, $\alpha_{rf}^C(E, A, B, C) = 2.6335\times10^{-13}$.

Table: Convergence History

<table>
<thead>
<tr>
<th>k</th>
<th>$\text{Re}(\lambda_{\text{dom}})$</th>
<th>$\varepsilon_k$</th>
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</thead>
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<tr>
<td>1</td>
<td>-6.2051e-03</td>
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<td>4</td>
<td>1.1425e-04</td>
<td>1.87164e+01</td>
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</tbody>
</table>
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Figure: Transfer function plot with computed $\mathcal{H}_\infty$-norm
Example 3 – Mimo46x46 System

Model with \( n = 13250, \ m = p = 46 \).

**Results:**

\[
\| G \|_{H_\infty} = 205.631, \quad t = 167.43s, \quad \alpha_{rf}(E, A, B, C) = 4.2864e-14.
\]

**Table: Convergence History**

<table>
<thead>
<tr>
<th>( \text{Re}(\lambda_{\text{dom}}) )</th>
<th>1 ( \epsilon_k )</th>
<th>2 ( \epsilon_k )</th>
<th>3 ( \epsilon_k )</th>
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<td>2.1976e-06</td>
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<td>7.3907e-09</td>
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<td>7.3907e-09</td>
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</tr>
</tbody>
</table>

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Model with $n = 13250$, $m = p = 46$.

Results:

$$\| G \|_{\mathcal{H}_\infty} = 205.631, \quad t = 167.43s, \quad \alpha_{rf}^C(E,A,B,C) = 4.2864e-14.$$ 

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Conclusions and Outlook

Conclusions

- Introduction of relations between the structured pseudospectra and the $\mathcal{H}_\infty$-norm of descriptor systems,
- development of an iterative algorithm for the computation of the $\mathcal{H}_\infty$-norm by iterating over the structured pseudospectral abscissa,

Open Problems

- Discrete-time systems $\rightarrow$ computation of the structured pseudospectral radius,
- real stability radii, passivity radius?
## Conclusions and Outlook

### Conclusions
- Introduction of relations between the structured pseudospectra and the $H_\infty$-norm of descriptor systems,
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### Open Problems
- Discrete-time systems $\Rightarrow$ computation of the structured pseudospectral radius,
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### Thank you for your Attention!
References

- **Benner, Voigt ’12**: *A structured pseudospectral method for $\mathcal{H}_\infty$-norm computation of large-scale descriptor systems*, Preprint MPIMD/12-10, Max Planck Institute Magdeburg, 2012.
Example 4 – Peec (Does Not Work)

Model with $n = 480$, $m = p = 1$, lots of peaks close to $i\mathbb{R}$!

Results:
$$\|G\|_{\mathcal{H}_\infty} = 0.0379802, \quad t = 23.20s, \quad \alpha_r(E, A, B, C) = 6.1976e-11.$$
Example 4 – Peec (Does Not Work)

Model with $n = 480$, $m = p = 1$, lots of peaks close to $i\mathbb{R}$!

Results:

$$\|G\|_{\mathcal{H}_\infty} = 0.0379802, \quad t = 23.20s, \quad \alpha_{rC}(E, A, B, C) = 6.1976\times10^{-11}.$$
Example 4 – Peec (Does Not Work)

Figure: Eigenvalues (blue pluses) and the 10 most dominant poles (red circles)