Oberwolfach Workshop "Numerical Solution of PDE Eigenvalue Problems"
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Computation of the $\mathcal{H}_\infty$-Norm for Large-Scale Systems

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Continuous-Time Descriptor Systems

Given: Continuous-time LTI descriptor system

$$\Sigma : \quad \begin{cases} E\dot{x}(t) = Ax(t) + Bu(t), \\ \quad y(t) = Cx(t) \end{cases}$$

- $E, A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{p \times n}$, $m, p \ll n$,
- descriptor vector $x(t) \in \mathbb{R}^n$, input vector $u(t) \in \mathbb{R}^m$, output vector $y(t) \in \mathbb{R}^p$.
- Assumptions: $\lambda E - A$ is regular, all matrices are large and sparse.

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Frequency domain representation

Transfer function $G(s) := C(sE - A)^{-1}B$

$\mathcal{H}_\infty$-Spaces and $\mathcal{H}_\infty$-Norm

Definition: the space $\mathcal{H}_\infty^{p \times m}$

$\mathcal{H}_\infty^{p \times m}$ – Hardy space of $p \times m$ functions of the form

$$G(s) = C(sE - A)^{-1}B$$

which are analytic and bounded in the open right half-plane, i.e., they are

- well-defined ($\lambda E - A$ regular);
- stable (all poles in open left half-plane);
- proper (bounded at infinity).

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Definition: $\mathcal{H}_\infty$-norm

Natural norm for the space $\mathcal{H}_\infty^{p\times m}$:

$$\|G\|_{\mathcal{H}_\infty} := \sup_{s\in\mathbb{C}^+} \sigma_{\max}\left(G\left(s\right)\right) = \sup_{\omega\in\mathbb{R}} \sigma_{\max}\left(G\left(\mathrm{i}\omega\right)\right).$$

$\mathcal{H}_\infty$-Norm and Structured Complex Stability Radius

What happens to stability/properness if we perturb G?

Consider the perturbed transfer function

$$G_\Delta(s) := C(sE - (A + B\Delta C))^{-1}B$$

with $\Delta \in \mathbb{C}^{m\times p}$.

Question

What is the smallest ε such that $G_\Delta \notin \mathcal{H}_\infty^{p\times m}$ for some $\|\Delta\|_2 < \varepsilon$?
$\rightsquigarrow$ structured complex stability radius $r_\mathbb{C}(E, A, B, C)$

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Connection to the $\mathcal{H}_\infty$-norm

$$r_\mathbb{C}(E, A, B, C) = \begin{cases} \|G\|_{\mathcal{H}_\infty}^{-1} & \text{if } G \not\equiv 0, \\ \infty & \text{if } G \equiv 0. \end{cases}$$

[Hinrichsen, Pritchard '86, Benner, V. '13]

How to Compute $r_{\mathbb{C}}(E, A, B, C)$?

Distinction of Cases

Let Δ be the "smallest" perturbation with $G_\Delta \notin \mathcal{H}_\infty^{p\times m}$. Two cases:

- $G_\Delta(\cdot)$ is improper or not well-defined $\rightsquigarrow$ needs special treatment;
- $G_\Delta(\cdot)$ is unstable $\rightsquigarrow$ analysis of finite poles.

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Definitions

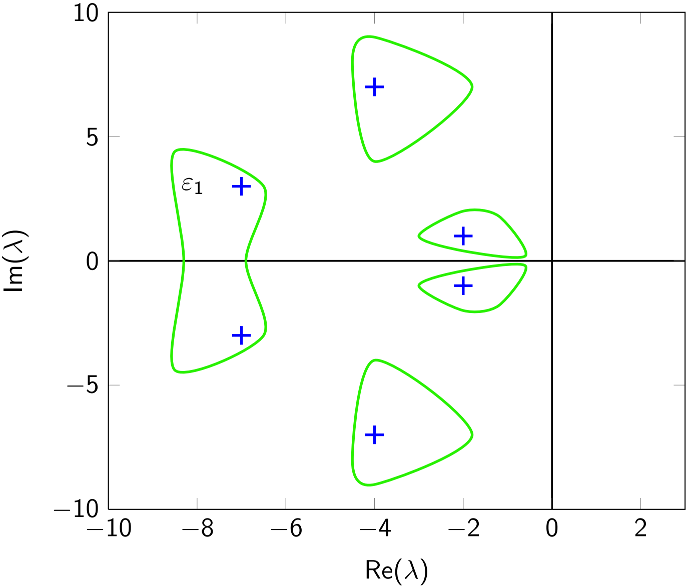
- set of finite poles of $G(\cdot)$ denoted by $\Pi_{\mathrm{f}}(E, A, B, C)$,
- structured pseudospectrum $\Pi_\varepsilon(E, A, B, C)$:

$$\Pi_\varepsilon(E, A, B, C) = \{s \in \mathbb{C} : \\ s \in \Pi_{\mathrm{f}}(E, A + B\Delta C, B, C) \text{ for } \Delta \in \mathbb{C}^{m \times p} \text{ with } \|\Delta\|_2 < \varepsilon\},$$

- structured pseudospectral abscissa $\alpha(\varepsilon)$:

$$\alpha(\varepsilon) := \max\{\operatorname{Re} s : s \in \Pi_\varepsilon(E, A, B, C)\}.$$

Graphical Interpretation



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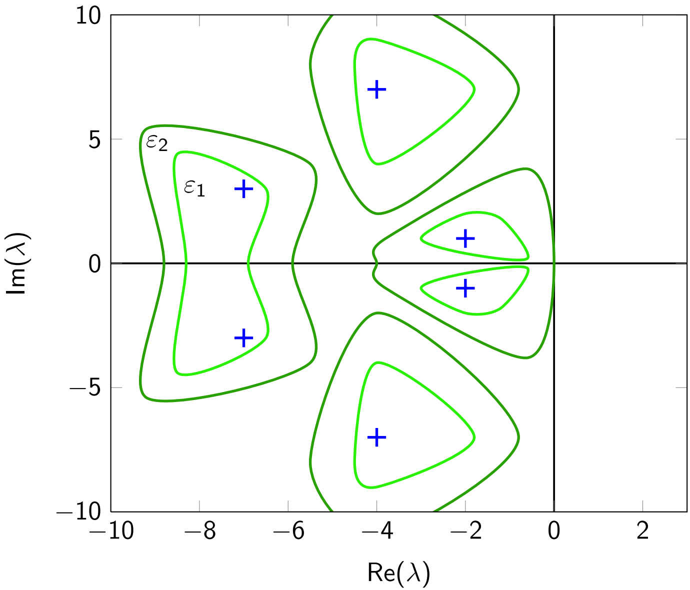

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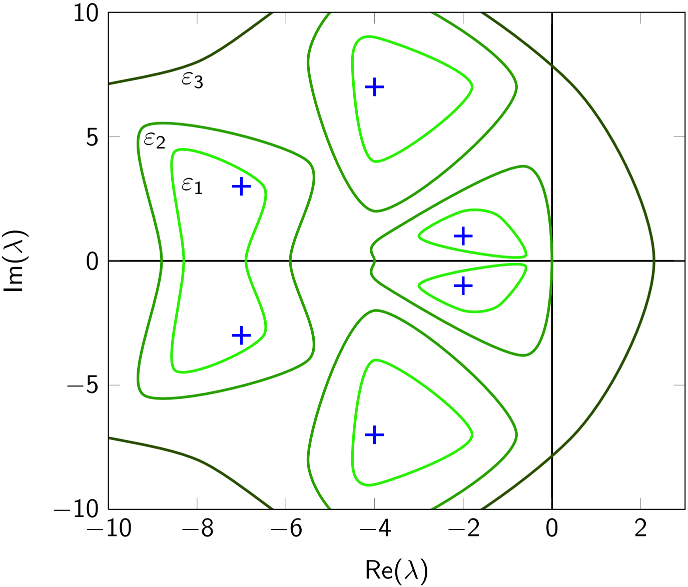

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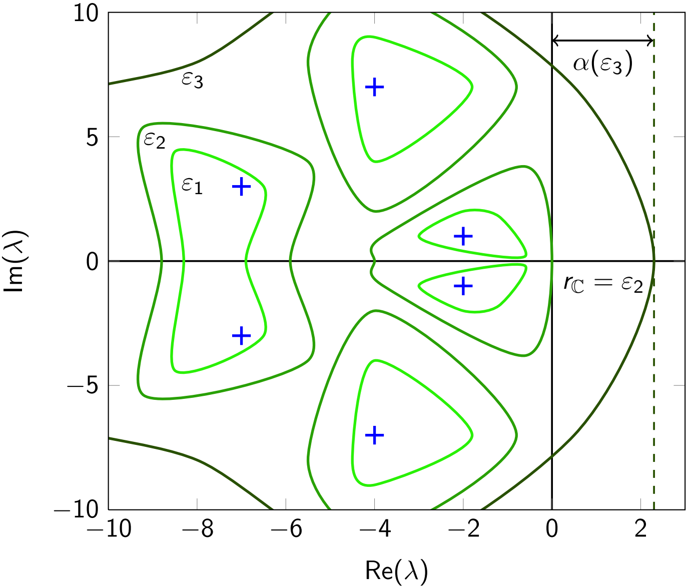

Algorithm Outline

Finding $r_{\mathbb{C}}(E, A, B, C)$ is equivalent to finding the (unique) root of $\alpha(\cdot)$.
Derivative of $\alpha(\cdot)$ can be computed! $\rightsquigarrow$ Newton's method applicable.

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Sketch of the algorithm

1. Choose initial ε.
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Computation of $\alpha(\varepsilon)$ – Perturbation Strategy

“Theorem”

The whole structured pseudospectrum can be obtained by using only rank-1 perturbations, i.e., $\Delta = uv^H$ with vectors u, v.

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Strategy [Guglielmi, Overton '11]

Compute a sequence of suitable structured rank-1 perturbed pencils $\lambda E - (A + \varepsilon B u v^H C)$ such that one of the perturbed eigenvalues converges to the rightmost pseudoeigenvalue of $\lambda E - A$!

First-Order Perturbation Theory

Lemma [Stewart, Sun '90]

Let x, y be right and left eigenvectors corresponding to a simple finite eigenvalue λ of the pencil $\lambda E - A$. Let $\lambda E - (A + tBuv^H C)$ be a perturbed matrix pencil with eigenvalue $\tilde{\lambda}$. Then it holds

$$\tilde{\lambda} = \lambda + t\frac{y^H Buv^H Cx}{y^H Ex} + \mathcal{O}\left(t^2\right).$$

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Corollary

$$\left.\frac{\mathrm{d}\tilde{\lambda}(t)}{\mathrm{d}t}\right|_{t=0} = \frac{y^H Buv^H Cx}{y^H Ex}.$$

Construction of Structured Rank-1 Perturbations

Given:

- Pencil $\lambda E - A$ with simple eigenvalue λ, right/left eigenvectors x, y, $y^H E x > 0$;
- vectors $u \in \mathbb{C}^m$, $v \in \mathbb{C}^p$ with $\|u\|_2 = \|v\|_2 = 1$.

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Equality holds for

$$u = \frac{B^T y}{\|B^T y\|_2}, \quad v = \frac{Cx}{\|Cx\|_2}.$$

Subsequent Steps

Given:

- Perturbed pencil $\lambda E - \hat{A} = \lambda E - (A + \varepsilon B \hat{u} \hat{v}^H C)$ with simple eigenvalue $\hat{\lambda}$, left/right eigenvectors $\hat{x}$, $\hat{y}$, $\hat{y}^H E \hat{x} > 0$;
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Consider

$$\lambda E - \left(\hat{A} + tB \left(uv^H - \hat{u}\hat{v}^H \right) C \right),$$

which is an ε-norm rank-1 perturbation of $\lambda E - A$ for $t = 0$, $t = \varepsilon$.

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$$\operatorname{Re}\left(\left. \frac{\mathrm{d}\tilde{\lambda}(t)}{\mathrm{d}t} \right|_{t=0} \right) = \frac{\operatorname{Re}\left(\hat{y}^H B \left(uv^H - \hat{u}\hat{v}^H \right) C \hat{x} \right)}{\hat{y}^H E \hat{x}} \leq \frac{\left\| \hat{y}^H B \right\|_2 \left\| C \hat{x} \right\|_2 - \operatorname{Re}\left(\hat{y}^H B \hat{u} \hat{v}^H C \hat{x} \right)}{\hat{y}^H E \hat{x}}.$$

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Again, equality holds for

$$u = \frac{B^T \hat{y}}{\|B^T \hat{y}\|_2}, \quad v = \frac{C\hat{x}}{\|C\hat{x}\|_2}.$$

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$\Longrightarrow$ Subspace Accelerated MIMO Dominant Pole Algorithm (SAMDP)

[Rommes, Martins '06]

Dominant Poles

Assume that $\lambda E - A$ has only simple eigenvalues λ_k with left and right eigenvectors y_k and x_k such that $y_k^H E x_k = 1$. If $G(\cdot)$ is proper then

$$G(s) = C(sE - A)^{-1}B = \sum_{k=1}^{n} \frac{R_k}{s - \lambda_k} + R_\infty$$

with residues

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Observation: If λ_j is close to the imaginary axis and $\|R_j\|_2$ is large, then

$$G(\mathrm{i}\omega) \approx \frac{R_j}{-\operatorname{Re}(\lambda_j)} + \sum_{\substack{k=1 \\ k \neq j}}^{n} \frac{R_k}{\mathrm{i}\omega - \lambda_k} + R_\infty$$

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for $\omega \approx \operatorname{Im}(\lambda_j)$ and therefore $\|G(\mathrm{i}\omega)\|_2$ is large, too.

$\Longrightarrow$ Compute the dominant poles $= \lambda_j$ with largest $\dfrac{\|R_j\|_2}{|\operatorname{Re}(\lambda_j)|}$!

Sketch of the Algorithm

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1. Choose dominant eigenvalue λ with right/left eigenvectors x, y.
2. Construct the perturbed pencil $\lambda E - \left(A + \varepsilon \frac{BB^T y x^H C^T C}{\|B^T y\|_2 \|Cx\|_2}\right)$.
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Remark

The dominant pole algorithm is also used to determine the initial ε!

Numerical Example – M20PI_n, $n = 1182$, $p = m = 3$

Phase 1: Searching initial pole:

1. inner iteration: psa = -0.0679449.
2. inner iteration: psa = 0.00231404.
3. inner iteration: psa = 0.00302846.
4. inner iteration: psa = 0.00303551.
5. inner iteration: psa = 0.00303558.

Iteration stagnates, pseudospectral abscissa found.

1. outer iteration: epsilon = 0.25825, psa = 0.00303558.

Phase 2: Iteration over epsilon:

1. inner iteration: psa = 1.61449e-08.
2. inner iteration: psa = 1.62557e-08.
3. inner iteration: psa = 1.62568e-08.

Iteration stagnates, pseudospectral abscissa found.

1. outer iteration: epsilon = 0.258224, psa = 1.62568e-08.

The H-infinity-norm is attained for fopt = 5064.12. The norm value is 3.8726.
The algorithm needed 2 outer and 8 inner iterations.
The runtime is 4.03271 seconds.

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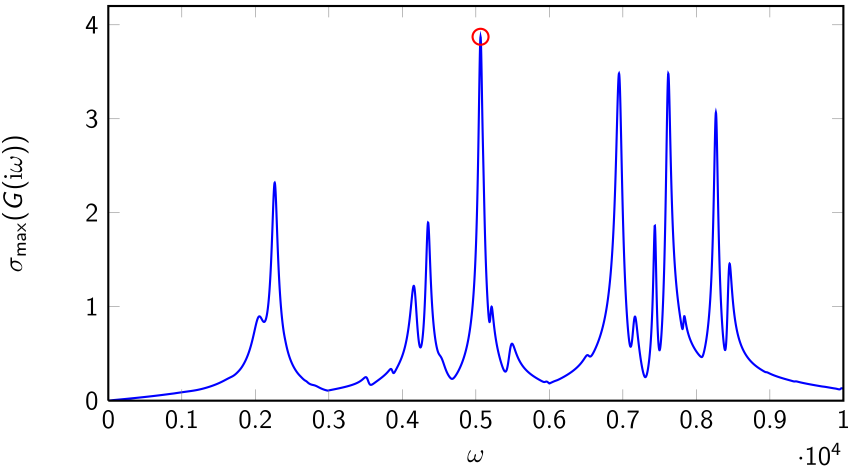

Figure: Transfer function plot with computed norm value

$\mathcal{H}_\infty$-Norm and Even Matrix Pencils

Consider the even matrix pencil

$$\mathcal{H}_\gamma(\lambda) := \left[\begin{array}{cc|cc} 0 & -\lambda E^T - A^T & -C^T & 0 \\ \lambda E - A & 0 & 0 & -B \\ \hline -C & 0 & \gamma I_p & 0 \\ 0 & -B^T & 0 & \gamma I_m \end{array}\right].$$

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Under some technical conditions, γ is a singular of $G(\mathrm{i}\omega_0)$ if and only if $\mathcal{H}_\gamma(\lambda)$ has the eigenvalue $\mathrm{i}\omega_0$.

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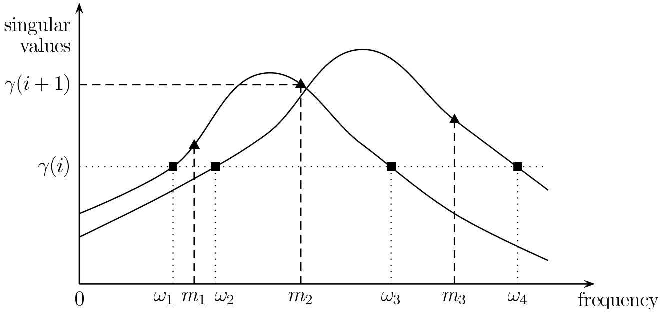
"Theorem" [Benner, Sima, V. '12]

let $\gamma > \min_{\omega\in\mathbb{R}} \sigma_{\max}(G(\mathrm{i}\omega))$. Then, under some technical conditions $\|G\|_{\mathcal{H}_\infty} \geq \gamma$ if and only if $\mathcal{H}_\gamma(\lambda)$ has purely imaginary eigenvalues.

Algorithm Outline



General Algorithm

1. Choose an initial value for γ.
2. Determine purely imaginary eigenvalues of $\mathcal{H}_\gamma(\lambda)$.
3. If imaginary eigenvalues exist, increase γ and repeat Step 2.
4. If no imaginary eigenvalues exist, γ is an upper bound for $\|G\|_{\mathcal{H}_\infty}$.

Eigenvalue Computation

- Eigenvalues computed using the structure-preserving even IRA method.

  [Mehrmann, Schröder, Simoncini '12]

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  This is an open problem!
- **But:** Heuristically, the $\mathcal{H}_\infty$-norm is attained close to a dominant pole! $\Longrightarrow$ Use the imaginary parts of the dominant poles as shifts!

  [LOWE, V. '13]

Numerical Example – M20PI_n, $n = 1182$, $p = m = 3$

The 6 most dominant poles with associated dominance values are:
-28.763 + 5064.528i with dominance 3.714211
-41.310 + 6951.137i with dominance 3.413362
-32.264 + 7614.848i with dominance 3.376605
-28.867 + 8266.724i with dominance 3.046229
-36.085 + 11972.542i with dominance 2.313766
-39.964 + 2263.478i with dominance 2.141803

The 6 most dominant poles are used to calculate the initial gamma value.

The initial lower bound of gamma is 3.872214 at fopt = 5064.528178.

For cycle 1:
The positive imaginary eigenvalues for shift = 5067.060442039i are:
lambda = 5063.711555996i
lambda = 5064.524040687i
There are 6 shifts, and 1 shifts that produce imaginary eigenvalues.
There are 24 eigenvalues, and 4 imaginary eigenvalues.
The lower bound of gamma is 3.872601 at fopt = 5064.117782.

For cycle 2:
There are 1 shifts, and 0 shifts that produce imaginary eigenvalues.
There are 4 eigenvalues, and 0 imaginary eigenvalues.

The L-infinity-norm is 3.8726 at fopt = 5064.12.
The runtime is 1.72567 seconds.

Some More Timings

Table: Timings for some larger test examples

example	n	m	p	time in s: pseudospectra	time in s: even pencil
bips07_1693	13275	4	4	167.10	31.98
bips07_1998	15066	4	4	102.11	29.99
bips07_2476	16861	4	4	146.18	31.62
bips07_3078	21128	4	4	91.05	34.73
xingo_afonso_itaipu	13250	1	1	39.24	16.80
mimo8x8_system	13309	8	8	78.47	23.25
mimo28x28_system	13251	28	28	85.36	35.45
mimo46x46_system	13250	46	46	115.91	49.13

Comparison of the Methods

"Pseudospectra method"

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– slower than "even pencil method"
– perturbed pencils are complex $\rightsquigarrow$ complex arithmetics
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"Even pencil method"

+ faster than "pseudospectra method"
+ works under less restrictive conditions than "pseudospectra method"
– larger number of user-defined parameters
– no guaranteed global optimization

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Thanks for Listening!

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