Linear-Quadratic Optimal Control of Differential-Algebraic Equations

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Differential-Algebraic Systems/Descriptor Systems

Linear time-invariant differential-algebraic systems/descriptor systems

$$\frac{\mathrm{d}}{\mathrm{d}t}Ex(t) = Ax(t) + Bu(t)$$

where

- $E, A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$,
- assume $sE - A$ is regular, i. e., $\det(sE - A) \not\equiv 0$,
- state $x \in \mathcal{L}^2_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^n)$, input $u \in \mathcal{L}^2_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^m)$.

Typical applications

- network models: electrical circuits, gas networks, constrained multi-body systems, . . .
- semi-discretization of PDEs (e. g., Navier-Stokes),
- linearization of non-linear DAEs,
- . . . .

The Linear-Quadratic Optimal Control Problem

Linear-quadratic optimal control problem: Minimize

$$\mathcal{J}(x,u) := \int_0^\infty \begin{pmatrix} x(\tau) \\ u(\tau) \end{pmatrix}^\mathsf{T} \begin{bmatrix} Q & S \\ S^\mathsf{T} & R \end{bmatrix} \begin{pmatrix} x(\tau) \\ u(\tau) \end{pmatrix} \mathrm{d}\tau$$

with $Q = Q^\mathsf{T} \in \mathbb{R}^{n \times n}$, $S \in \mathbb{R}^{n \times m}$, and $R = R^\mathsf{T} \in \mathbb{R}^{m \times m}$ subject to

$$\tfrac{\mathrm{d}}{\mathrm{d}t} Ex(t) = Ax(t) + Bu(t), \quad Ex(0) = Ex_0, \quad \lim_{t \to \infty} Ex(t) = 0.$$

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Problems:

- feasibility, i. e., is

$$V^+(Ex_0) := \inf \left\{ \mathcal{J}(x,u) : \tfrac{\mathrm{d}}{\mathrm{d}t} Ex = Ax + Bu,\ Ex(0) = Ex_0,\ \lim_{t\to\infty} Ex(t) = 0 \right\}$$

  finite for all consistent x_0?
- optimal value $V^+(Ex_0)$?
- existence and uniqueness of optimal controls ($\rightsquigarrow$ regularity)?

Geometric Concepts

a) behavior:

$$\mathfrak{B}_{[E,A,B]} := \left\{ (x,u) \in \mathcal{L}^2_{\mathrm{loc}}(\mathbb{R},\mathbb{R}^n) \times \mathcal{L}^2_{\mathrm{loc}}(\mathbb{R},\mathbb{R}^m) : \tfrac{\mathrm{d}}{\mathrm{d}t} Ex = Ax + Bu \right\}.$$

b) system space: smallest subspace in $\mathbb{R}^{n+m}$ such that

$$\begin{pmatrix} x(t) \\ u(t) \end{pmatrix} \in \mathcal{V}_{\mathrm{sys}} \quad \forall (x,u) \in \mathfrak{B}_{[E,A,B]} \text{ and almost all } t \in \mathbb{R}.$$

c) space of consistent initial differential variables:

$$\mathcal{V}_{\mathrm{diff}} := \left\{ x_0 \in \mathbb{R}^n : \exists (x,u) \in \mathfrak{B}_{[E,A,B]} \text{ with } Ex(0) = Ex_0 \right\}.$$

A system is called impulse controllable, if $\mathcal{V}_{\mathrm{diff}} = \mathbb{R}^n$.

Conditions for Feasibility

Reminder: Optimal value

$$V^+(Ex_0) := \inf \left\{ \mathcal{J}(x,u) : \tfrac{\mathrm{d}}{\mathrm{d}t} Ex = Ax + Bu,\ Ex(0) = Ex_0,\ \lim_{t\to\infty} Ex(t) = 0 \right\}$$

Necessary condition for feasibility

$$V^+(Ex_0) < \infty \quad \forall\, x_0 \in \mathcal{V}_{\mathrm{diff}}$$

holds if and only if the system is behaviorally stabilizable, i. e., for all $x_0 \in \mathcal{V}_{\mathrm{diff}}$ there exists a solution (x,u) with $Ex(0) = Ex_0$ and $\lim_{t\to\infty} Ex(t) = 0$.
Algebraically:

$$\operatorname{rank} \begin{bmatrix} -\lambda E + A & B \end{bmatrix} = n \quad \forall\, \lambda \in \overline{\mathbb{C}^+}.$$

Conditions for Feasibility

(Virtual) storage functions

A function $V : E\mathcal{V}_{\mathrm{diff}} \to \mathbb{R}$ is called (virtual) storage function for the optimal control problem if V is continuous, $V(0) = 0$ and for all solutions (x, u) we have

$$V(Ex(t_1)) - V(Ex(t_2)) \leq \int_{t_1}^{t_2} \begin{pmatrix} x(\tau) \\ u(\tau) \end{pmatrix}^{\mathsf{T}} \begin{bmatrix} Q & S \\ S^{\mathsf{T}} & R \end{bmatrix} \begin{pmatrix} x(\tau) \\ u(\tau) \end{pmatrix} \mathrm{d}\tau.$$

For $t_1 = 0$, $t_2 \to \infty$, and $Ex(0) = Ex_0$, $\lim_{t\to\infty} Ex(t) = 0$ we obtain

$$V(Ex_0) \leq \mathcal{J}(x, u).$$

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For $t_1 = 0$, $t_2 \to \infty$, and $Ex(0) = Ex_0$, $\lim_{t\to\infty} Ex(t) = 0$ we obtain

$$V(Ex_0) \leq \mathcal{J}(x, u).$$

Sufficient condition for feasibility

If the system is behaviorally stabilizable and there exists a storage function, then

$$-\infty < V(Ex_0) \leq V^+(Ex_0) < \infty \quad \forall\, x_0 \in \mathcal{V}_{\mathrm{diff}}.$$

Quadratic Storage Functions

Notation

For symmetric $M, N \in \mathbb{R}^{n \times n}$ and a subspace $\mathcal{V} \subseteq \mathbb{R}^n$ we write

- $M \geq_{\mathcal{V}} N \quad :\Longleftrightarrow \quad v^{\mathsf{T}} M v \geq v^{\mathsf{T}} N v \quad \forall\, v \in \mathcal{V}$,
- $M =_{\mathcal{V}} N \quad :\Longleftrightarrow \quad v^{\mathsf{T}} M v = v^{\mathsf{T}} N v \quad \forall\, v \in \mathcal{V}$.

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Quadratic storage functions and KYP inequality [Reis, Rendel, V. '15]

The function $V : E\mathcal{V}_{\mathrm{diff}} \to \mathbb{R}$ is a quadratic storage function, i. e.,

$$V(Ex_0) = x_0^{\mathsf{T}} E^{\mathsf{T}} P E x_0 \quad \text{for} \quad P = P^{\mathsf{T}} \in \mathbb{R}^{n\times n}.$$

if and only if the KYP inequality

$$\begin{bmatrix} E^{\mathsf{T}} PA + A^{\mathsf{T}} PE + Q & E^{\mathsf{T}} PB + S \\ B^{\mathsf{T}} PE + S^{\mathsf{T}} & R \end{bmatrix} \geq_{\mathcal{V}_{\mathrm{sys}}} 0, \quad P = P^{\mathsf{T}}$$

is fulfilled.

Quadratic Storage Functions

Quadratic storage functions and KYP inequality

If $V(Ex_0) = x_0^\mathsf{T} E^\mathsf{T} P E x_0$ is a quadratic storage function, then there exist $K \in \mathbb{R}^{q \times n}$ and $L \in \mathbb{R}^{q \times m}$ such that

$$\begin{bmatrix} E^\mathsf{T} PA + A^\mathsf{T} PE + Q & E^\mathsf{T} PB + S \\ B^\mathsf{T} PE + S^\mathsf{T} & R \end{bmatrix} =_{\mathcal{V}_{\mathrm{sys}}} \begin{bmatrix} K^\mathsf{T} \\ L^\mathsf{T} \end{bmatrix} \begin{bmatrix} K & L \end{bmatrix}, \quad P = P^\mathsf{T}$$

Quadratic Storage Functions

For (x, u) with $Ex(0) = Ex_0$ and $\lim_{t\to\infty} Ex(t) = 0$ it holds

$$x_0^\mathsf{T} E^\mathsf{T} P E x_0 = -\int_0^\infty \frac{\mathrm{d}}{\mathrm{d}\tau}(x(\tau)^\mathsf{T} E^\mathsf{T} P E x(\tau))\mathrm{d}\tau$$

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&= -\int_0^\infty \tfrac{\mathrm{d}}{\mathrm{d}\tau}(x(\tau)^\mathsf{T} E^\mathsf{T}) P E x(\tau) + x(\tau)^\mathsf{T} E^\mathsf{T} P \tfrac{\mathrm{d}}{\mathrm{d}\tau}(E x(\tau))\mathrm{d}\tau
\end{aligned}
$$

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&= -\int_0^\infty (Ax(\tau) + Bu(\tau))^\mathsf{T} P E x(\tau) + x(\tau)^\mathsf{T} E^\mathsf{T} P (Ax(\tau) + Bu(\tau))\mathrm{d}\tau \\
&= \int_0^\infty \begin{pmatrix} x(\tau) \\ u(\tau) \end{pmatrix}^\mathsf{T} \begin{bmatrix} -E^\mathsf{T} P A - A^\mathsf{T} P E & -E^\mathsf{T} P B \\ -B^\mathsf{T} P E & 0 \end{bmatrix} \begin{pmatrix} x(\tau) \\ u(\tau) \end{pmatrix} \mathrm{d}\tau
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&= \int_0^\infty \begin{pmatrix} x(\tau) \\ u(\tau) \end{pmatrix}^\mathsf{T} \begin{bmatrix} -E^\mathsf{T} P A - A^\mathsf{T} P E & -E^\mathsf{T} P B \\ -B^\mathsf{T} P E & 0 \end{bmatrix} \begin{pmatrix} x(\tau) \\ u(\tau) \end{pmatrix} \mathrm{d}\tau \\
&= \int_0^\infty \begin{pmatrix} x(\tau) \\ u(\tau) \end{pmatrix}^\mathsf{T} \left(\begin{bmatrix} Q & S \\ S^\mathsf{T} & R \end{bmatrix} - \begin{bmatrix} K^\mathsf{T} K & K^\mathsf{T} L \\ L^\mathsf{T} K & L^\mathsf{T} L \end{bmatrix} \right) \begin{pmatrix} x(\tau) \\ u(\tau) \end{pmatrix} \mathrm{d}\tau
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&= \int_0^\infty \begin{pmatrix} x(\tau) \\ u(\tau) \end{pmatrix}^\mathsf{T} \left(\begin{bmatrix} Q & S \\ S^\mathsf{T} & R \end{bmatrix} - \begin{bmatrix} K^\mathsf{T} K & K^\mathsf{T} L \\ L^\mathsf{T} K & L^\mathsf{T} L \end{bmatrix} \right) \begin{pmatrix} x(\tau) \\ u(\tau) \end{pmatrix} \mathrm{d}\tau \\
&= \mathcal{J}(x, u) - \|Kx + Lu\|^2_{\mathcal{L}^2([0,\infty),\mathbb{R}^q)} .
\end{aligned}$$

Relation to the Optimal Control Problem

If the optimal control problem is feasible then the optimal value function V^+ is quadratic, i. e.,

$$V^+(Ex_0) = x_0^\mathsf{T} E^\mathsf{T} P E x_0,$$

where

$$\begin{bmatrix} E^\mathsf{T}PA + A^\mathsf{T}PE + Q & E^\mathsf{T}PB + S \\ B^\mathsf{T}PE + S^\mathsf{T} & R \end{bmatrix} =_{\mathcal{V}_{\mathrm{sys}}} \begin{bmatrix} K^\mathsf{T} \\ L^\mathsf{T} \end{bmatrix} \begin{bmatrix} K & L \end{bmatrix}, \quad P = P^\mathsf{T}$$

and

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and

$$V^+(Ex_0) = \mathcal{J}(x,u) - \|Kx + Lu\|^2_{\mathcal{L}^2([0,\infty),\mathbb{R}^q)}.$$

Implications:

- There exists a sequence $((x_k, u_k))_{k\in\mathbb{N}}$ of $\mathcal{L}_2$-solutions with $Ex_k(0) = Ex_0$ and $\lim_{t\to\infty} Ex_k(t) = 0$ with

$$\lim_{k\to\infty} \|Kx_k + Lu_k\|^2_{\mathcal{L}^2([0,\infty),\mathbb{R}^q)} = 0.$$

- An optimal control fulfills the optimality DAE

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}\begin{pmatrix} x(t) \\ u(t) \end{pmatrix} = \begin{bmatrix} A & B \\ K & L \end{bmatrix}\begin{pmatrix} x(t) \\ u(t) \end{pmatrix}, \quad Ex(0) = Ex_0, \quad \lim_{t\to\infty} Ex(t) = 0.$$

Lur'e Equations

We have

$$V^{+}(Ex_0) = x_0^{\mathsf{T}} E^{\mathsf{T}} P E x_0,$$

where

$$\begin{bmatrix} E^{\mathsf{T}}PA + A^{\mathsf{T}}PE + Q & E^{\mathsf{T}}PB + S \\ B^{\mathsf{T}}PE + S^{\mathsf{T}} & R \end{bmatrix} =_{\mathcal{V}_{\mathrm{sys}}} \begin{bmatrix} K^{\mathsf{T}} \\ L^{\mathsf{T}} \end{bmatrix} \begin{bmatrix} K & L \end{bmatrix}, \quad P = P^{\mathsf{T}}. \qquad \text{(LE)}$$

We can choose q such that $K \in \mathbb{R}^{q \times n}$ and $L \in \mathbb{R}^{q \times m}$ fulfill the extra condition

$$\operatorname{rank}_{\mathbb{R}(s)} \begin{bmatrix} -sE + A & B \\ K & L \end{bmatrix} = n + q. \qquad \text{(rk)}$$

Definition

We call a triple $(P, K, L) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{q \times n} \times \mathbb{R}^{q \times m}$ solution of the Lur'e equation (LE), if (LE) and (rk) are satisfied.

Stabilizing Solutions

Solutions of the Lur'e equation corresponding to V^+ have further structure: A solution (P, K, L) of the Lur'e equation is called stabilizing, if

$$\operatorname{rank} \begin{bmatrix} -\lambda E + A & B \\ K & L \end{bmatrix} = n + q \quad \forall \lambda \in \mathbb{C}^+.$$

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Existence and uniqueness [Reis, Rendel, V. '15]

Assume that the KYP inequality is solvable. Then we have:

- **Existence:** system is behaviorally stabilizable $\Rightarrow \exists$ stabilizing solution.
- **Uniqueness:** (P_1, K_1, L_1), (P_2, K_2, L_2) stabilizing solutions $\Rightarrow$ $E^\mathsf{T} P_1 E =_{\mathcal{V}_{\text{diff}}} E^\mathsf{T} P_2 E$.
- **Extremality:** (P, K, L) is stabilizing solution of the Lur'e equation $\Rightarrow$ $E^\mathsf{T} P E \geq_{\mathcal{V}_{\text{diff}}} E^\mathsf{T} \tilde{P} E$ for all solutions $\tilde{P}$ of the KYP inequality.

Existence and Uniqueness of Optimal Controls

Reminder: optimality DAE

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}\begin{pmatrix} x(t) \\ u(t) \end{pmatrix} = \begin{bmatrix} A & B \\ K & L \end{bmatrix}\begin{pmatrix} x(t) \\ u(t) \end{pmatrix}, \quad Ex(0) = Ex_0, \quad \lim_{t\to\infty} Ex(t) = 0.$$

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Corollary

- If for all $x_0 \in \mathcal{V}_{\mathrm{diff}}$ there exists an optimal control, then

$$\operatorname{rank}\begin{bmatrix} -\lambda E + A & B \\ K & L \end{bmatrix} = n + q \quad \forall\, \lambda \in \overline{\mathbb{C}^+}.$$

- If for all $x_0 \in \mathcal{V}_{\mathrm{diff}}$ there exists a unique optimal control, then

$$\operatorname{rank}\begin{bmatrix} -\lambda E + A & B \\ K & L \end{bmatrix} = n + m \quad \forall\, \lambda \in \overline{\mathbb{C}^+}.$$

Remark: An equivalence of such statements can be proven for impulse controllable systems.

Even Matrix Pencils

Assume that (E, A, B) is impulse controllable. Lur'e equations have a close relationship to matrix pencils

$$s\mathcal{E} - \mathcal{A} := \begin{bmatrix} 0 & -sE + A & B \\ sE^{\mathsf{T}} + A^{\mathsf{T}} & Q & S \\ B^{\mathsf{T}} & S^{\mathsf{T}} & R \end{bmatrix} \in \mathbb{R}[s]^{(2n+m)\times(2n+m)}.$$

The pencil $s\mathcal{E} - \mathcal{A}$ is even, since $\mathcal{E} = -\mathcal{E}^{\mathsf{T}}$ and $\mathcal{A} = \mathcal{A}^{\mathsf{T}}$.

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The pencil $s\mathcal{E} - \mathcal{A}$ is even, since $\mathcal{E} = -\mathcal{E}^\mathsf{T}$ and $\mathcal{A} = \mathcal{A}^\mathsf{T}$.

Problems:

- construction of its solution via deflating subspaces,
- characterization of existence and uniqueness of optimal controls.

Deflating Subspaces [Van Dooren '83]

- A subspace $\mathcal{Y} \subseteq \mathbb{R}^N$ with basis matrix $Y \in \mathbb{R}^{N\times k}$ is called deflating subspace of $s\mathcal{E} - \mathcal{A} \in \mathbb{R}[s]^{N\times N}$ if there exist $Z \in \mathbb{R}^{N\times \ell}$ and $s\tilde{\mathcal{E}} - \tilde{\mathcal{A}} \in \mathbb{R}[s]^{\ell\times k}$ of full row rank such that
$$(s\mathcal{E} - \mathcal{A})Y = Z(s\tilde{\mathcal{E}} - \tilde{\mathcal{A}}).$$
- A subspace $\mathcal{Y}$ is called $\mathcal{E}$-neutral if $y_1^\mathsf{T}\mathcal{E}y_2 = 0$ for all $y_1,\ y_2 \in \mathcal{Y}$.

Construction of Solutions

Theorem

If there exists a solution (X, K, L) to the Lur'e equation, then there exist

$$Y = \begin{bmatrix} Y_\mu \\ Y_x \\ Y_u \end{bmatrix} \in \mathbb{R}^{(2n+m)\times(n+m)}, \quad Z = \begin{bmatrix} Z_\mu \\ Z_x \\ Z_u \end{bmatrix} \in \mathbb{R}^{(2n+m)\times(n+q)} \text{ such that}$$

(a) the space $\mathcal{Y} = \operatorname{im} Y$ is $\mathcal{E}$-neutral and of dimension $n+m$;

(b) $\mathcal{V}_{\mathrm{sys}} \subseteq \operatorname{im} \begin{bmatrix} Y_x \\ Y_u \end{bmatrix}$;

(c) $\operatorname{rank} EY_x = \operatorname{rank} E$;

(d) there exist $s\tilde{\mathcal{E}} - \tilde{\mathcal{A}} \in \mathbb{R}[s]^{(n+q)\times(n+m)}$ with $\operatorname{rank}_{\mathbb{R}(s)} \left(s\tilde{\mathcal{E}} - \tilde{\mathcal{A}}\right) = n+q$ such that

$$\begin{bmatrix} 0 & -sE + A & B \\ sE^{\mathsf{T}} + A^{\mathsf{T}} & Q & S \\ B^{\mathsf{T}} & S^{\mathsf{T}} & R \end{bmatrix} \begin{bmatrix} Y_\mu \\ Y_x \\ Y_u \end{bmatrix} = \begin{bmatrix} Z_\mu \\ Z_x \\ Z_u \end{bmatrix} \left(s\tilde{\mathcal{E}} - \tilde{\mathcal{A}}\right).$$

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(d) there exist $s\tilde{\mathcal{E}} - \tilde{\mathcal{A}} \in \mathbb{R}[s]^{(n+q)\times(n+m)}$ with $\operatorname{rank}_{\mathbb{R}(s)} \left(s\tilde{\mathcal{E}} - \tilde{\mathcal{A}}\right) = n+q$ such that

$$\begin{bmatrix} 0 & -sE + A & B \\ sE^{\mathsf{T}} + A^{\mathsf{T}} & Q & S \\ B^{\mathsf{T}} & S^{\mathsf{T}} & R \end{bmatrix} \begin{bmatrix} Y_\mu \\ Y_x \\ Y_u \end{bmatrix} = \begin{bmatrix} Z_\mu \\ Z_x \\ Z_u \end{bmatrix} \left(s\tilde{\mathcal{E}} - \tilde{\mathcal{A}}\right).$$

Remark: The converse holds true under some additional assumptions.

Construction of Solutions

$$\begin{bmatrix} 0 & -sE + A & B \\ sE^{\mathsf{T}} + A^{\mathsf{T}} & Q & S \\ B^{\mathsf{T}} & S^{\mathsf{T}} & R \end{bmatrix} \begin{bmatrix} Y_\mu \\ Y_x \\ Y_u \end{bmatrix} = \begin{bmatrix} Z_\mu \\ Z_x \\ Z_u \end{bmatrix} \left(s\tilde{\mathcal{E}} - \tilde{\mathcal{A}}\right)$$

Construction of Solutions

$$\begin{bmatrix} 0 & -sE+A & B \\ sE^{\mathsf{T}}+A^{\mathsf{T}} & Q & S \\ B^{\mathsf{T}} & S^{\mathsf{T}} & R \end{bmatrix} \begin{bmatrix} Y_\mu \\ Y_x \\ Y_u \end{bmatrix} = \begin{bmatrix} Z_\mu \\ Z_x \\ Z_u \end{bmatrix} \left(s\tilde{\mathcal{E}} - \tilde{\mathcal{A}}\right)$$

Remarks

- Y_x, Y_u can be chosen such that $\begin{bmatrix} Y_x \\ Y_u \end{bmatrix}$ is invertible. With $\begin{bmatrix} Y_x^- & Y_u^- \end{bmatrix} := \begin{bmatrix} Y_x \\ Y_u \end{bmatrix}^{-1}$, we obtain $P = Y_\mu Y_x^-$.
- $s\tilde{\mathcal{E}} - \tilde{\mathcal{A}}$ is equivalent to $\begin{bmatrix} -sE+A & B \\ K & L \end{bmatrix}$.
- (E, A, B) behaviorally stabilizable $\Rightarrow$ construct stabilizing solution by choosing a semi-stable deflating subspace.
- Existence and uniqueness of optimal controls can be directly read off the spectral structure of $s\mathcal{E} - \mathcal{A}$.

Comparison to Other Approaches

Our Lur'e Equation

$$\begin{bmatrix} E^\mathsf{T} PA + A^\mathsf{T} PE + Q & E^\mathsf{T} PB + S \\ B^\mathsf{T} PE + S^\mathsf{T} & R \end{bmatrix} =_{\mathcal{V}_{\mathrm{sys}}} \begin{bmatrix} K^\mathsf{T} \\ L^\mathsf{T} \end{bmatrix} \begin{bmatrix} K & L \end{bmatrix}, \quad P = P^\mathsf{T}$$

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- If $E = I_n$, then our Lur'e equation reduces to the standard Lur'e equation [Reis '11]

$$\begin{bmatrix} A^{\mathsf{T}}P + PA + Q & PB + S \\ B^{\mathsf{T}}P + S^{\mathsf{T}} & R \end{bmatrix} = \begin{bmatrix} K^{\mathsf{T}} \\ L^{\mathsf{T}} \end{bmatrix} \begin{bmatrix} K & L \end{bmatrix}, \quad P = P^{\mathsf{T}}.$$

Comparison to Other Approaches

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- If further R is invertible, then rank $\begin{bmatrix} K & L \end{bmatrix} = m$ and K and L can be eliminated to obtain an algebraic Riccati equation

$$A^\mathsf{T}P + PA + Q - \left(PB + S\right)R^{-1}\left(B^\mathsf{T}P + S^\mathsf{T}\right) = 0, \quad P = P^\mathsf{T}.$$

Comparison to Other Approaches

Our Lur'e Equation

$$\begin{bmatrix} E^{\mathsf{T}}PA + A^{\mathsf{T}}PE + Q & E^{\mathsf{T}}PB + S \\ B^{\mathsf{T}}PE + S^{\mathsf{T}} & R \end{bmatrix} =_{\mathcal{V}_{\mathrm{sys}}} \begin{bmatrix} K^{\mathsf{T}} \\ L^{\mathsf{T}} \end{bmatrix} \begin{bmatrix} K & L \end{bmatrix}, \quad P = P^{\mathsf{T}}$$

- **Kurina '93:** generalized algebraic Riccati equation:

$$A^{\mathsf{T}}P + PA + Q - \left(PB + S\right)R^{-1}\left(B^{\mathsf{T}}P + S^{T}\right) = 0, \quad E^{\mathsf{T}}P = P^{\mathsf{T}}E$$

Remarks:

- assumptions: $\begin{bmatrix} Q & S \\ S^{\mathsf{T}} & R \end{bmatrix} \geq 0$, $R > 0$, (E, A, B) impulse controllable,
- solution analysis: Katayama/Minamino '92 and Katayama/Kawamoto/Takaba '99, in particular solvability of an "algebraic quadratic matrix equation" necessary for existence of stabilizing solutions.

Comparison to Other Approaches

Our Lur'e Equation

$$\begin{bmatrix} E^\mathsf{T}PA + A^\mathsf{T}PE + Q & E^\mathsf{T}PB + S \\ B^\mathsf{T}PE + S^\mathsf{T} & R \end{bmatrix} =_{\mathcal{V}_{\mathrm{sys}}} \begin{bmatrix} K^\mathsf{T} \\ L^\mathsf{T} \end{bmatrix} \begin{bmatrix} K & L \end{bmatrix}, \quad P = P^\mathsf{T}$$

- **Mehrmann '91:** generalized algebraic Riccati equation:

$$A^\mathsf{T}PE + E^\mathsf{T}PA + Q - \left(E^\mathsf{T}PB + S\right)R^{-1}\left(B^\mathsf{T}PE + S^\mathsf{T}\right) = 0, \quad P = P^\mathsf{T}$$

Remarks:

- assumptions: $\begin{bmatrix} Q & S \\ S^\mathsf{T} & R \end{bmatrix} \geq 0$, $R > 0$, (E, A, B) impulse controllable,
- solution of optimal control problems using even boundary value problems.

Conclusions and Further Results

Conclusion

- feasibility conditions in terms of existence of storage functions,
- Lur'e equations and their solution properties,
- existence and uniqueness conditions for optimal controls,
- construction via deflating subspaces of even matrix pencils.

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Conclusion

- feasibility conditions in terms of existence of storage functions,
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Thank you for your Attention!

References

- T. Reis. Lur'e equations and even matrix pencils, *Linear Algebra Appl.*, 434:152–173, 2011.
- M. Voigt. *On Linear-Quadratic Optimal Control and Robustness of Differential-Algebraic Systems*, Dissertation, Otto-von-Guericke-Universität Magdeburg, 2015.
- T. Reis, O. Rendel, and M. Voigt. The Kalman-Yakubovich-Popov inequality for differential-algebraic systems, *Linear Algebra Appl.*, 485:153–193, 2015.