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# $\mathcal{L}_\infty$-Norm Computation for Discrete-Time Descriptor Systems

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NETWORK THEORY

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# Discrete-Time Descriptor Systems

**Given: Discrete-time LTI descriptor system**

$$\Sigma : \quad \begin{cases} Ex_{k+1} = Ax_k + Bu_k, \\ y_k = Cx_k + Du_k, \end{cases}$$

- $E$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, $D \in \mathbb{R}^{p\times m}$,
- descriptor vector $x_k \in \mathbb{R}^n$, input vector $u_k \in \mathbb{R}^m$, output vector $y_k \in \mathbb{R}^p$.
- Assumptions: $\lambda E - A$ is regular, i.e. $\det(\lambda E - A) \not\equiv 0$.

# Z-Transform and Transfer Function

Apply the Z-transform

$$\mathcal{Z}\{f\}(z) = \sum_{k=0}^{\infty} f_k z^{-k}$$

to the signal sequences $x = \{x_k\}_{k=0}^{\infty}$, $u = \{u_k\}_{k=0}^{\infty}$, $y = \{y_k\}_{k=0}^{\infty}$.

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This leads to

$$\mathcal{Z}(\Sigma): \quad \begin{cases} zE\mathcal{Z}\{x\}(z) - Ex_0 = A\mathcal{Z}\{x\}(z) + B\mathcal{Z}\{u\}(z), \\ \mathcal{Z}\{y\}(z) = C\mathcal{Z}\{x\}(z) + D\mathcal{Z}\{u\}(z). \end{cases}$$

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With $Ex_0 = 0$ we obtain

$$\mathcal{Z}\{y\}(z) = \left(C\left(zE - A\right)^{-1} B + D\right) \mathcal{Z}\{u\}(z).$$

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$$\mathcal{Z}\{y\}(z) = \underbrace{\left(C\left(zE - A\right)^{-1}B + D\right)}_{=: G(z) \text{ (transfer function)}} \mathcal{Z}\{u\}(z).$$

# $\mathcal{L}_\infty$-Spaces and $\mathcal{L}_\infty$-Norm

**Definition: the spaces $\mathcal{L}_\infty^{p\times m}\left(e^{\mathrm{i}\omega}\right)$ and $\mathcal{RL}_\infty^{p\times m}\left(e^{\mathrm{i}\omega}\right)$**

- With $\mathcal{L}_\infty^{p\times m}\left(e^{\mathrm{i}\omega}\right)$ we denote the space of $p \times m$ matrix-valued transfer functions which are bounded on the unit circle.
- With $\mathcal{RL}_\infty^{p\times m}\left(e^{\mathrm{i}\omega}\right)$ we denote the rational subspace of $\mathcal{L}_\infty^{p\times m}\left(e^{\mathrm{i}\omega}\right)$. All its elements have a descriptor system realization of the form $\Sigma$.

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## Definition: $\mathcal{L}_\infty$-norm

Natural norm for the space $\mathcal{RL}_\infty^{p\times m}\left(e^{i\omega}\right)$:

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### Remark

For stable systems (all poles of $G(z)$ are inside the unit circle), the $\mathcal{L}_\infty$-norm is equivalent to the $\mathcal{H}_\infty$-norm.

# Applications

## Model order reduction

Let $\hat{\Sigma} = (\lambda\hat{E} - \hat{A}, \hat{B}, \hat{C}, \hat{D})$ be a reduced order model of the system $\Sigma$.
The transfer function of the error system $\Sigma^{err} := \Sigma - \hat{\Sigma}$ is given by

$$G^{err}(z) = \begin{bmatrix} C & -\hat{C} \end{bmatrix} \left( z \begin{bmatrix} E & 0 \\ 0 & \hat{E} \end{bmatrix} - \begin{bmatrix} A & 0 \\ 0 & \hat{A} \end{bmatrix} \right)^{-1} \begin{bmatrix} B \\ \hat{B} \end{bmatrix} + D - \hat{D}.$$

$\|G^{err}\|_{\mathcal{H}_\infty}$ is the size of the worst-case approximation error.

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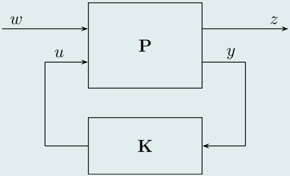
## $\mathcal{H}_\infty$-control [Green, Limebeer '95]




- Plant **P**, dynamic compensator **K**,
- noise $w$, estimation error $z$,
- $\mathcal{H}_\infty$-norm of the transfer function from $w$ to $z$ displays the worst-case influence of the disturbances $w$ on the output $z$.

# Basic Theorems

**Theorem** [Genin, Van Dooren, Vermaut '98, V '10]

Assume that the matrix pencil $\lambda E - A$ is regular and has no unitary eigenvalues, $\gamma > 0$ is not a singular value of $D$ and $\omega_0 \in [0, 2\pi)$. Then, $\gamma$ is a singular value of $G(e^{\mathrm{i}\omega_0})$ if and only if $e^{\mathrm{i}\omega_0}$ is an eigenvalue of the matrix pencil

$$\lambda N_\gamma - M_\gamma := \begin{bmatrix} \lambda E - A + BD^T S^{-1} C & -BB^T + BD^T S^{-1} DB^T \\ \lambda C^T S^{-1} C & E^T - \lambda A^T + \lambda C^T S^{-1} DB^T \end{bmatrix}$$

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## Corollary

Let $\gamma > \min_{\omega \in [0, 2\pi)} \sigma_{\max}\left(G\left(e^{\mathrm{i}\omega}\right)\right)$ be not a singular value of D. Then, $\|G\|_{\mathcal{L}_\infty} \geq \gamma$ if and only if $\lambda N_\gamma - M_\gamma$ has unitary eigenvalues.

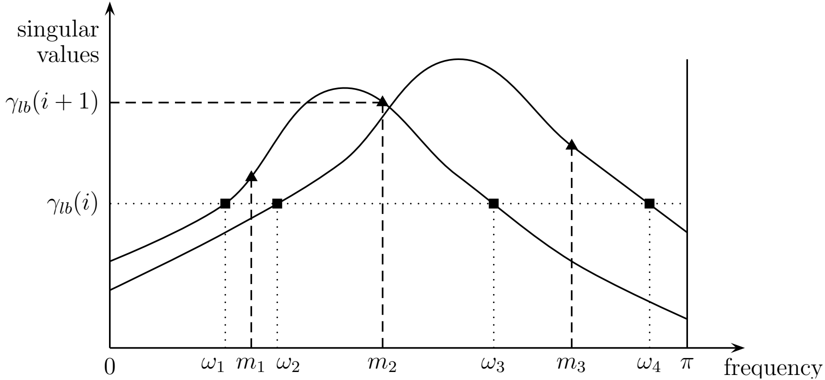
# Sketch of the Algorithm

**Input:** Discrete-time linear time-invariant descriptor system with transfer function $G \in \mathcal{RL}_\infty^{p\times m}\left(e^{i\omega}\right)$, tolerance $\varepsilon$.

**Output:** $\|G\|_{\mathcal{L}_\infty}$.

```
Compute an initial value γ_lb < ‖G‖_L∞.
repeat
   Set γ := (1 + ε)γ_lb.
   Compute the unitary eigenvalues of the matrix pencil λN_γ − M_γ.
   if no unitary eigenvalues then
      break.
   else
      Set {e^{iω_1}, ..., e^{iω_k}} = unitary eigenvalues with ω_j ∈ [0, π),
      j = 1, ..., k.
      Set m_j = 1/2 (ω_j + ω_{j+1}), j = 1, ..., k − 1.
      Compute the largest singular value of G(e^{im_j}), j = 1, ..., k − 1.
      Set γ_lb = max_j σ_max(G(e^{im_j})).
   end if
until break
Set ‖G‖_L∞ = γ_lb.
```

# Graphical Interpretation



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- relative error is at most $\varepsilon$ (assuming exact arithmetic).
- Computing time is affected by number of frequency points in each step $\Longrightarrow$ generally last step takes less time than the first, good choice of initial value can reduce CPU time drastically.
- Care must be taken of the eigenvalue computation as it is important to catch all unitary eigenvalues $\Longrightarrow$ use certain matrix pencil transformations and a structure-preserving algorithm to compute the eigenvalues.

# Choice of the Initial Value $\gamma_{lb}$

1. Evaluate $G(z)$ at the boundary frequencies $\omega = 0$ and $\omega = \pi$, i.e. compute $\sigma_{\max}\left(G\left(e^{\mathrm{i}\cdot 0}\right)\right)$ and $\sigma_{\max}\left(G\left(e^{\mathrm{i}\pi}\right)\right)$.

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   2. Compute $\sigma_{\max}\left(G\left(e^{\mathrm{i}\mu_j}\right)\right)$, where

$$\mu_j := \max\left\{\frac{1}{4}\left|\psi_j\right|^2, \omega_j^2 - \nu_j^2\right\}^{\frac{1}{2}} \quad \text{for } \omega_j \in (0, \pi),$$

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# The Eigenvalue Problem

**Reminder: original matrix pencil**

$$\lambda N_\gamma - M_\gamma := \begin{bmatrix} \lambda E - A + BD^T S^{-1} C & -BB^T + BD^T S^{-1} DB^T \\ \lambda C^T S^{-1} C & E^T - \lambda A^T + \lambda C^T S^{-1} DB^T \end{bmatrix},$$

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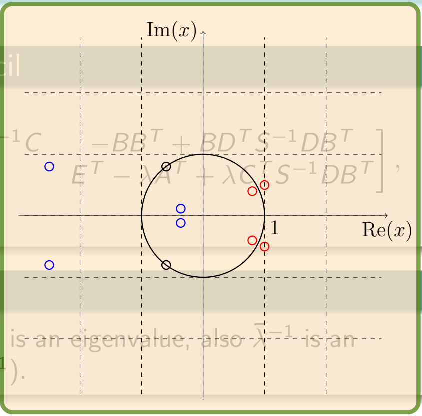

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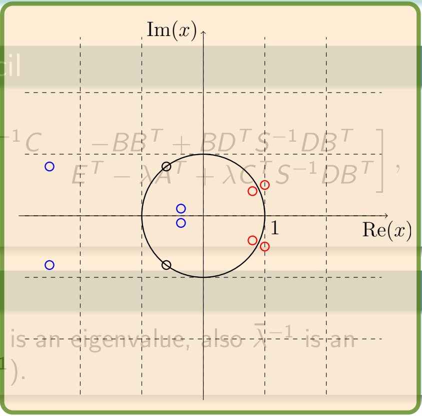


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- If $\gamma$ is close to a singular value of $D$, $S$ is ill-conditioned,
- forming "matrix-times-its-transpose" products numerically unstable.
  $\Longrightarrow$ Explicitly forming $\lambda N_\gamma - M_\gamma$ must be avoided!

# An Extended Eigenvalue Problem

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Multiply $\lambda N_\gamma - M_\gamma$ by $P = \begin{bmatrix} I_n & 0 \\ 0 & -\gamma I_n \end{bmatrix}$ from the left and by $Q = \begin{bmatrix} I_n & 0 \\ 0 & \frac{1}{\gamma} I_n \end{bmatrix}$ from the right in order to get the matrix pencil

$$\lambda \tilde{N}_\gamma - \tilde{M}_\gamma = \begin{bmatrix} \lambda E - A & 0 \\ 0 & \lambda A^T - E^T \end{bmatrix} - \begin{bmatrix} B & 0 \\ 0 & -\lambda C^T \end{bmatrix} \begin{bmatrix} -D & \gamma I_p \\ \gamma I_m & -D^T \end{bmatrix}^{-1} \begin{bmatrix} C & 0 \\ 0 & B^T \end{bmatrix}.$$

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Exploit Schur complement structure and get the extended matrix pencil

$$\lambda \mathcal{N} - \mathcal{M}_\gamma = \lambda \left[\begin{array}{cc|cc} E & 0 & 0 & 0 \\ 0 & A^T & 0 & C^T \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] - \left[\begin{array}{cc|cc} A & 0 & B & 0 \\ 0 & E^T & 0 & 0 \\ \hline C & 0 & D & -\gamma I_p \\ 0 & B^T & -\gamma I_m & D^T \end{array}\right]$$

which has the same eigenvalues as $\lambda N_\gamma - M_\gamma$ with additional infinite eigenvalues.

# Recovery of Structure

By performing signed block row and column permutations we get the D-type matrix pencil

$$\lambda\hat{\mathcal{N}} - \hat{\mathcal{M}}_\gamma = \lambda \left[\begin{array}{c|ccc} 0 & -E^T & 0 & 0 \\ \hline A & 0 & 0 & 0 \\ C & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] - \left[\begin{array}{c|ccc} 0 & -A^T & -C^T & 0 \\ \hline E & 0 & 0 & B \\ 0 & 0 & -\gamma I_p & D \\ 0 & B^T & D^T & -\gamma I_m \end{array}\right].$$

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## D-type matrix pencils

[Xu '06]

- have the general form $\lambda \mathcal{E}_D - \mathcal{A}_D = \lambda \begin{bmatrix} 0 & F \\ -G^T & 0 \end{bmatrix} - \begin{bmatrix} 0 & G \\ -F^T & H \end{bmatrix}$ with symmetric $H$,
- have symplectic eigenstructure + additional infinite eigenvalues.

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**Definition: generalized Cayley transform**

$$\mathbf{c}(\mathcal{A}, \mathcal{E}) := \lambda\,(\mathcal{A} + \mathcal{E}) - (\mathcal{A} - \mathcal{E})\,.$$

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**Application to D-type matrix pencils** [Xu '06]

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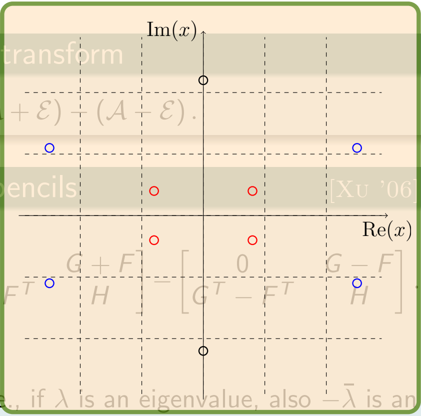


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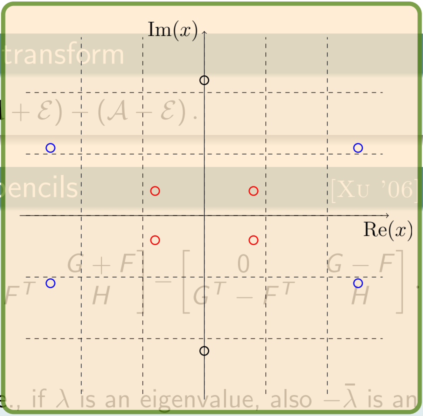


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- Hamiltonian eigensymmetry, i.e., if $\lambda$ is an eigenvalue, also $-\bar{\lambda}$ is an eigenvalue (and therefore $\bar{\lambda}$ and $-\lambda$), and additional eigenvalues 1,
- unitary eigenvalues of $\lambda\mathcal{E}_D - \mathcal{A}_D$ are mapped to purely imaginary eigenvalues of $\lambda\tilde{\mathcal{E}} - \tilde{\mathcal{A}}$.

# Transformation to an Even Matrix Pencil

**Problem**

$$\lambda\tilde{\mathcal{E}} - \tilde{\mathcal{A}} = \lambda \begin{bmatrix} 0 & G+F \\ -G^T - F^T & H \end{bmatrix} - \begin{bmatrix} 0 & G-F \\ G^T - F^T & H \end{bmatrix}$$

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## Solution: Additional drop/add transformation [Xu '06]

$$\lambda\mathcal{E}_C - \mathcal{A}_C := \mathbf{d}\left(\tilde{\mathcal{A}}, \tilde{\mathcal{E}}\right) := \begin{bmatrix} (1-\lambda)I & 0 \\ 0 & I \end{bmatrix} \left(\lambda\tilde{\mathcal{E}} - \tilde{\mathcal{A}}\right) \begin{bmatrix} I & 0 \\ 0 & \frac{1}{1-\lambda} I \end{bmatrix}.$$

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- **d**-transformation is $\lambda$-dependent with poles for $\lambda = 1, \infty$
  $\Longrightarrow$ multiplicities of these eigenvalues may have changed.

# Application to Our Problem

## Application of $\mathbf{d}(\mathbf{c}(\cdot))$ to our problem

$$\lambda\bar{\mathcal{N}} - \bar{\mathcal{M}}_\gamma = \lambda \left[\begin{array}{c|ccc} 0 & -A^T - E^T & -C^T & 0 \\ \hline A+E & 0 & 0 & 0 \\ C & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] - \left[\begin{array}{c|ccc} 0 & -A^T + E^T & -C^T & 0 \\ \hline -A+E & 0 & 0 & B \\ -C & 0 & -\gamma I_p & D \\ 0 & B^T & D^T & -\gamma I_m \end{array}\right].$$

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We transform the eigenvalue problem into skew-Hamiltonian/Hamiltonian structure to use a structure-preserving algorithm.

# Transformation to sH/H Structure [V.' 10]

1. Introduce artificial inputs or outputs to the descriptor system to get the same number of inputs and outputs. Set $q = \max\{m, p\}$ and append $B$, $C$, $D$, by an appropriate amount of zero rows or columns.

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$\lambda\tilde{\mathcal{N}} - \tilde{\mathcal{M}}_\gamma$ is a skew-Hamiltonian/Hamiltonian matrix pencil!

# Skew-Hamiltonian/Hamiltonian Matrix Pencils

## Definition and properties

Let $\mathcal{J} := \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$. The matrix pencil $\lambda\mathcal{S} - \mathcal{H} \in \mathbb{R}^{2n \times 2n}$ is skew-Hamiltonian/Hamiltonian if $\mathcal{S}$ is skew-Hamiltonian ($(\mathcal{S}\mathcal{J})^T = -\mathcal{S}\mathcal{J}$) and $\mathcal{H}$ is Hamiltonian ($(\mathcal{H}\mathcal{J})^T = \mathcal{H}\mathcal{J}$).

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**Block structure:** $\lambda\mathcal{S} - \mathcal{H} = \lambda \begin{bmatrix} F & G \\ H & F^T \end{bmatrix} - \begin{bmatrix} R & S \\ T & -R^T \end{bmatrix}$ with skew-symmetric $G$, $H$, and symmetric $S$, $T$.

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[Benner, Byers, Mehrmann, Xu '99]

- uses special orthogonal transformations on an embedded matrix pencil to get a skew-Hamiltonian/Hamiltonian Schur-like form,
- simple, finite, purely imaginary eigenvalues stay on the imaginary axis $\Longrightarrow$ robust detection of interesting eigenvalues!

# Example Description and Results

## Example description

Example from modeling a mass-spring-damper system with:

- $n = 11$ descriptor variables, $m = 1$ input, $p = 1$ output,
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## Computation of the initial value

- $\sigma_{\max}\left(G\left(e^{\mathrm{i}\cdot 0}\right)\right) = 4.14620127792975600 \cdot 10^{-3}$,
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Other test frequencies:

| $\omega$ | $\sigma_{\max}\left(G\left(e^{\mathrm{i}\omega}\right)\right)$ |
|---|---|
| 1.572768 | $5.23947523458074063 \cdot 10^{-3}$ |
| 1.250212 | $4.91055265529189832 \cdot 10^{-3}$ |
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# $\gamma$-Iteration

Desired accuracy: $\varepsilon = 1000\mathbf{u}$ ($\mathbf{u}$ = machine precision).

| iteration | $\gamma_{lb}$ | imag. eig. of $\lambda\tilde{\mathcal{N}} - \tilde{\mathcal{M}}_\gamma$ |
|---|---|---|
| 1 | $5.50049659572314498 \cdot 10^{-3}$ | 1.431624i<br>633862.192397i |
| 2 | $5.57395403422781491 \cdot 10^{-3}$ | 1.791283i<br>3.177910i |
| 3 | $5.59298056369935456 \cdot 10^{-3}$ | 2.202331i<br>2.319735i |
| 4 | $5.59316018833811313 \cdot 10^{-3}$ | 2.258739i<br>2.259758i |
| 5 | $5.59316020191237984 \cdot 10^{-3}$ | none |

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| 2 | $5.573954034227814 91 \cdot 10^{-3}$ | 1.791283i<br>3.177910i |
| 3 | $5.592980563699354 56 \cdot 10^{-3}$ | 2.202331i<br>2.319735i |
| 4 | $5.593160188338113 13 \cdot 10^{-3}$ | 2.258739i<br>2.259758i |
| 5 | $5.593160201912379 84 \cdot 10^{-3}$ | none |

## Results

- $\|G\|_{\mathcal{L}_\infty} = 5.59316020191237984 \cdot 10^{-3}$,
- peak frequency: $\omega_p = 2.3081845795498692$.

# Conclusions and Outlook

## What we have done

- $\mathcal{L}_\infty$-norm computation for discrete-time descriptor systems by using structured matrix pencils,
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# Thank you for your Attention!

# References

## References


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