The Kalman-Yakubovich-Popov Inequality for Differential-Algebraic Equations

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Introduction

Consider a control system

$$\dot{x}(t) = Ax(t) + Bu(t)$$

with $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, state $x : \mathbb{R} \to \mathbb{R}^n$, input $u : \mathbb{R} \to \mathbb{R}^m$.

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Kalman-Yakubovich-Popov Inequality

Let $Q = Q^T \in \mathbb{R}^{n \times n}$, $S \in \mathbb{R}^{n \times m}$, $R = R^T \in \mathbb{R}^{m \times m}$.
Want to know: $\exists\ P \in \mathbb{R}^{n \times n}$ such that

$$\begin{bmatrix} PA + A^T P + Q & PB + S \\ B^T P + S^T & R \end{bmatrix} \geq 0, \quad P = P^T?$$

Linear-Quadratic Optimal Control

Problem Statement

$$\mathcal{J}(x,u) = \int_0^\infty \begin{pmatrix} x(\tau) \\ u(\tau) \end{pmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{pmatrix} x(\tau) \\ u(\tau) \end{pmatrix} \mathrm{d}\tau$$

Problem: Determine

$$V^+(x_0) := \inf \left\{ \mathcal{J}(x,u) : \dot{x} = Ax + Bu,\ x(0) = x_0,\ \lim_{t\to\infty} x(t) = 0 \right\}.$$

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Assume that P solves the KYP inequality. Then for $t_2 \geq t_1$ we have

$$x(t_2)^T P x(t_2) - x(t_1)^T P x(t_1)$$
$$= \int_{t_1}^{t_2} \frac{\mathrm{d}}{\mathrm{d}\tau} x(\tau)^T P x(\tau) \mathrm{d}\tau$$

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$$x(t_2)^T P x(t_2) - x(t_1)^T P x(t_1) \geq - \int_{t_1}^{t_2} \begin{pmatrix} x(\tau) \\ u(\tau) \end{pmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{pmatrix} x(\tau) \\ u(\tau) \end{pmatrix} \mathrm{d}\tau$$

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For $t_1 = 0,\ t_2 \to \infty$, we get $x_0^T P x_0 \leq \mathcal{J}(x,u)$.

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For $t_1 = 0$, $t_2 \to \infty$, we get $x_0^T P x_0 \leq \mathcal{J}(x,u)$. If (A, B) is stabilizable, then there exists a maximal solution P_+ such that $V^+(x_0) = x_0^T P_+ x_0$.

Further Characterizations [Willems '71]

Let (A, B) be controllable. Then the following are equivalent:

a) The optimal control problem is feasible, i.e., $V^+(x_0) \in \mathbb{R}$ for all $x_0 \in \mathbb{R}$

b) The KYP inequality

$$\begin{bmatrix} PA + A^T P + Q & PB + S \\ B^T P + S^T & R \end{bmatrix} \geq 0$$

has a symmetric solution P.

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has a symmetric solution P.

c) The Popov function

$$\Phi(s) = \begin{bmatrix} (-\bar{s}I_n - A)^{-1}B \\ I_m \end{bmatrix}^H \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} (sI_n - A)^{-1}B \\ I_m \end{bmatrix} \in \mathbb{R}(s)^{m \times m}$$

fulfills $\Phi(\mathrm{i}\omega) \geq 0$ for all $\mathrm{i}\omega \notin \Lambda(A)$.

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d) It holds

$$\int_0^t \begin{pmatrix} x(\tau) \\ u(\tau) \end{pmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{pmatrix} x(\tau) \\ u(\tau) \end{pmatrix} \mathrm{d}\tau \geq 0$$

for all smooth solutions (x, u) with $x(0) = x(t) = 0$.

Further Applications [V. '15]

a) energy analysis of dynamical systems $\rightsquigarrow$ dissipativity, cyclo-dissipativity, passivity, etc.

b) spectral factorization:

$$\Phi(s) = W^H(-\bar{s})W(s)$$

c) normalized coprime factorization:

$$G(s) = N(s)M^{-1}(s) \in \mathbb{R}(s)^{p \times m}$$

where $N(s) \in \mathcal{RH}_\infty^{p \times m}$, $M(s) \in \mathcal{RH}_\infty^{m \times m}$, and

$$N^H(-\bar{s})N(s) + M^H(-\bar{s})M(s) = I_m$$

d) inner/outer factorization:

$$G(s) = G_{\mathrm{i}}(s)G_{\mathrm{o}}(s) \in \mathbb{R}(s)^{p \times m}$$

where $G_{\mathrm{i}}(s) \in \mathbb{R}(s)^{p \times q}$ is an inner function and $G_{\mathrm{o}}(s) \in \mathbb{R}(s)^{q \times m}$ is an outer function

Controllability and Stabilizability Concepts

Consider a descriptor system

$$E\dot{x}(t) = Ax(t) + Bu(t)$$

with $sE - A \in \mathbb{R}[s]^{n \times n}$ regular, $B \in \mathbb{R}^{n \times m}$, state $x : \mathbb{R} \to \mathbb{R}^n$, input $u : \mathbb{R} \to \mathbb{R}^m$.

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a) impulse controllable $\Leftrightarrow$ rank $\begin{bmatrix} E & AS_\infty & B \end{bmatrix} = n$, where $\operatorname{im} S_\infty = \ker E$,

b) behavioral controllable $\Leftrightarrow$ rank $\begin{bmatrix} \lambda E - A & B \end{bmatrix} = n$ for all $\lambda \in \mathbb{C}$,

c) behavioral stabilizable $\Leftrightarrow$ rank $\begin{bmatrix} \lambda E - A & B \end{bmatrix} = n$ for all $\lambda \in \overline{\mathbb{C}^+}$,

d) behavioral anti-stabilizable $\Leftrightarrow$ rank $\begin{bmatrix} \lambda E - A & B \end{bmatrix} = n$ for all $\lambda \in \overline{\mathbb{C}^-}$,

Kalman-Yakubovich-Popov Lemma

Define Popov function:

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Kalman-Yakubovich-Popov Lemma [Reis, Rendel, V. '14]

a) If there exists a symmetric matrix P such that

$$\begin{bmatrix} E^T PA + A^T PE + Q & E^T PB + S \\ B^T PE + S^T & R \end{bmatrix} \geq_{\mathcal{V}_{\mathrm{sys}}} 0, \qquad (1)$$

then $\Phi(\mathrm{i}\omega) \geq 0$ for all $\mathrm{i}\omega \notin \Lambda(E, A)$.

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then $\Phi(\mathrm{i}\omega) \geq 0$ for all $\mathrm{i}\omega \notin \Lambda(E, A)$.

b) If (E, A, B) is behavioral controllable and $\Phi(\mathrm{i}\omega) \geq 0$ for all $\mathrm{i}\omega \notin \Lambda(E, A)$, then there exists a symmetric matrix P such that (1) holds.

KYP Inequality and System Space

$$\begin{bmatrix} E^T PA + A^T PE + Q & E^T PB + S \\ B^T PE + S^T & R \end{bmatrix} \geq_{\mathcal{V}_{\rm sys}} 0,$$

a) **Notation:**

$$M \geq_{\mathcal{V}} 0 \quad \Leftrightarrow \quad v^T M v \geq 0 \quad \forall v \in \mathcal{V}$$

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a) **Notation:**

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b) **System space $\mathcal{V}_{\mathrm{sys}}$:** the smallest subspace of $\mathbb{R}^{n+m}$ in which solution trajectories evolve, i.e., if (x, u) solves $E\dot{x} = Ax + Bu$, then

$$\begin{pmatrix} x(t) \\ u(t) \end{pmatrix} \in \mathcal{V}_{\mathrm{sys}} \quad \forall t \in \mathbb{R}.$$

Tools for the Proof

Feedback Equivalence Form

There exist nonsingular W, $T \in \mathbb{R}^{n\times n}$ and $F \in \mathbb{R}^{m\times n}$ such that

$$W \begin{bmatrix} sE - A & B \end{bmatrix} \begin{bmatrix} T & 0 \\ -FT & I_m \end{bmatrix} = \begin{bmatrix} sI_{n_1} - A_{11} & 0 & 0 & B_1 \\ 0 & -I_{n_2} & sE_{23} & B_2 \\ 0 & 0 & sE_{33} - I_{n_3} & 0 \end{bmatrix},$$

where E_{33} is nilpotent.

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where E_{33} is nilpotent.

Structure of the system space in FEF:

$$\mathcal{V}_{\mathrm{sys},F} = \left\{ \begin{pmatrix} x_1 \\ -B_2 u \\ 0_{n_3 \times 1} \\ u \end{pmatrix} \in \mathbb{R}^{n+m} : x_1 \in \mathbb{R}^{n_1},\, u \in \mathbb{R}^m \right\}$$

Proof idea: Construct basis matrix of the system space and use the KYP lemma for ODE systems.

Lur'e Equations

Consider the Lur'e equation

$$\begin{bmatrix} E^T XA + A^T XE + Q & E^T XB + S \\ B^T XE + S^T & R \end{bmatrix} =_{\mathcal{V}_{\rm sys}} \begin{bmatrix} K^T \\ L^T \end{bmatrix} \begin{bmatrix} K & L \end{bmatrix}, \quad X = X^T. \qquad (2)$$

A triple $(X, K, L) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{q\times n} \times \mathbb{R}^{q\times m}$ is called solution of the Lur'e equation (2), if it fulfills (2) and

$$\operatorname{rank}_{\mathbb{R}(s)} \begin{bmatrix} -sE + A & B \\ K & L \end{bmatrix} = n + q.$$

Remarks:

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Remarks:

- $P = X$ solves the KYP inequality,
- $q = \operatorname{rank}_{\mathbb{R}(s)} \Phi(s)$,
- with a basis matrix $M_{\mathcal{V}_{\mathrm{sys}}}$ of $\mathcal{V}_{\mathrm{sys}}$, q minimizes the rank of

$$M_{\mathcal{V}_{\mathrm{sys}}}^T \begin{bmatrix} E^TPA + A^TPE + Q & E^TPB + S \\ B^TPE + S^T & R \end{bmatrix} M_{\mathcal{V}_{\mathrm{sys}}}$$

among all solutions of the KYP inequality ($\rightsquigarrow P = X$ is called rank-minimizing solution).

Existence of Stabilizing and Anti-Stabilizing Solutions

A solution (X, K, L) of the Lur'e equation is called

a) stabilizing, if

$$\operatorname{rank} \begin{bmatrix} -\lambda E + A & B \\ K & L \end{bmatrix} = n + q \quad \forall \lambda \in \mathbb{C}^{+},$$

b) anti-stabilizing, if

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Existence [Reis, Rendel, V. '14]

Assume that the KYP inequality is solvable. Then it holds:

a) (E, A, B) is behavioral stabilizable $\Rightarrow \exists$ stabilizing solution,

b) (E, A, B) is behavioral anti-stabilizable $\Rightarrow \exists$ anti-stabilizing solution.

Uniqueness and Extremality

Extremality [Reis, Rendel, V. '14]

a) (X, K, L) is stabilizing solution of the Lur'e equation $\Rightarrow E^T X E \geq_{\mathcal{V}_{\mathrm{diff}}} E^T P E$ for all solutions P of the KYP inequality,

b) (X, K, L) is anti-stabilizing solution of the Lur'e equation $\Rightarrow E^T X E \leq_{\mathcal{V}_{\mathrm{diff}}} E^T P E$ for all solutions P of the KYP inequality,

where

$$\mathcal{V}_{\mathrm{diff}} := \{x_0 \in \mathbb{R}^n : \exists (x, u) \text{ solving } E\dot{x} = Ax + Bu \text{ with } Ex(0) = Ex_0\}$$

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$$\mathcal{V}_{\rm diff} := \{x_0 \in \mathbb{R}^n : \exists (x, u) \text{ solving } E\dot{x} = Ax + Bu \text{ with } Ex(0) = Ex_0\}$$

Uniqueness [Reis, Rendel, V. '14]

(X_1, K_1, L_1), (X_2, K_2, L_2) (anti-)stabilizing solutions of the Lur'e equation $\Rightarrow$ $E^T X_1 E =_{\mathcal{V}_{\rm diff}} E^T X_2 E$

Approaches Based on Matrix Inequalities

Our KYP Inequality

$$\begin{bmatrix} E^T PA + A^T PE + Q & E^T PB + S \\ B^T PE + S^T & R \end{bmatrix} \geq_{\mathcal{V}_{\mathrm{sys}}} 0, \quad P = P^T$$

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- **Geerts '89–'94:**

$$\begin{bmatrix} E^T PA + A^T PE + Q & E^T PB + S \\ B^T PE + S^T & R \end{bmatrix} \geq 0,$$

Remarks:

- assumptions: $\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} = \begin{bmatrix} C^T \\ D^T \end{bmatrix} \begin{bmatrix} C & D \end{bmatrix} \geq 0$, however $sE - A \in \mathbb{R}[s]^{k \times n}$ may be singular
- (E, A, B) is impulse controllable $\Rightarrow \exists$ maximal solution of the LMI solving the optimal control problem
- no rank-minimization property!

Approaches Based on Matrix Inequalities

Our KYP Inequality

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- **Brüll ’11:**

$$\begin{bmatrix} P_1^T A + A^T P_1 + Q & P_1^T B + A^T P_2 + S \\ P_2^T A + B^T P_1 + S^T & P_2^T B + B^T P_2 + R \end{bmatrix} \geq 0,$$
$$E^T P_1 = P_1^T E, \quad E^T P_2 = 0$$

Remarks:

- assumptions: (E, A, B) is completely controllable, i.e, behavioral controllable and rank $\begin{bmatrix} E & B \end{bmatrix} = n$
- also generalization of the KYP inequality to higher-order and behavioral systems

Approaches Based on Matrix Inequalities

Our KYP Inequality

$$\begin{bmatrix} E^T PA + A^T PE + Q & E^T PB + S \\ B^T PE + S^T & R \end{bmatrix} \geq_{\mathcal{V}_{\mathrm{sys}}} 0, \quad P = P^T$$

- **Camlibel/Frasca ’07:**

$$\begin{bmatrix} E^T PA + A^T PE & E^T PB + C^T \\ B^T PE + C & D + D^T \end{bmatrix} \geq_{\mathcal{V}_{\mathrm{sys}}} 0, \quad P = P^T \leq 0$$

(in our notation)

Remarks:

- assumptions: (E, A, B, C, D) is in minimal realization, implying complete controllability
- the above is a modification of the KYP inequality for assessing passivity

Approaches Based on Matrix Equations

Our Lur'e Equation

$$\begin{bmatrix} E^T XA + A^T XE + Q & E^T XB + S \\ B^T XE + S^T & R \end{bmatrix} =_{\mathcal{V}_{\mathrm{sys}}} \begin{bmatrix} K^T \\ L^T \end{bmatrix} \begin{bmatrix} K & L \end{bmatrix}, \quad X = X^T$$

Approaches Based on Matrix Equations

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- If $E = I_n$, then our Lur'e equation reduces to the standard Lur'e equation [Reis '11]

$$\begin{bmatrix} XA + A^T X + Q & XB + S \\ B^T X + S^T & R \end{bmatrix} = \begin{bmatrix} K^T \\ L^T \end{bmatrix} \begin{bmatrix} K & L \end{bmatrix}, \quad X = X^T.$$

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$$\begin{bmatrix} XA + A^T X + Q & XB + S \\ B^T X + S^T & R \end{bmatrix} = \begin{bmatrix} K^T \\ L^T \end{bmatrix} \begin{bmatrix} K & L \end{bmatrix}, \quad X = X^T.$$

- If further R is invertible, then rank $\begin{bmatrix} K & L \end{bmatrix} = m$ and K and L can be eliminated to obtain an algebraic Riccati equation

$$XA + A^T X + Q - (XB + S) R^{-1} (B^T X + S^T) = 0, \quad X = X^T.$$

Approaches Based on Matrix Equations

Our Lur'e Equation

$$\begin{bmatrix} E^T XA + A^T XE + Q & E^T XB + S \\ B^T XE + S^T & R \end{bmatrix} =_{\mathcal{V}_{\text{sys}}} \begin{bmatrix} K^T \\ L^T \end{bmatrix} \begin{bmatrix} K & L \end{bmatrix}, \quad X = X^T$$

- **Kurina '93:** generalized algebraic Riccati equation:

$$XA + A^T X + Q - \big(XB + S\big) R^{-1} \big(B^T X + S^T\big) = 0, \quad E^T X = X^T E$$

Remarks:

- assumptions: $\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \geq 0$, $R > 0$, (E, A, B) impulse controllable
- solution analysis: Katayama/Minamino '92 and Katayama/Kawamoto/Takaba '99, in particular solvability of an "algebraic quadratic matrix equation" necessary for existence of stabilizing solutions

Approaches Based on Matrix Equations

Our Lur'e Equation

$$\begin{bmatrix} E^T XA + A^T XE + Q & E^T XB + S \\ B^T XE + S^T & R \end{bmatrix} =_{\mathcal{V}_{\text{sys}}} \begin{bmatrix} K^T \\ L^T \end{bmatrix} \begin{bmatrix} K & L \end{bmatrix}, \quad X = X^T$$

- **Mehrmann '91:** generalized algebraic Riccati equation:

$$E^T XA + A^T XE + Q - \left(E^T XB + S\right) R^{-1} \left(B^T XE + S^T\right) = 0, \quad X = X^T$$

Remarks:

- assumptions: $\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \geq 0$, $R > 0$, (E, A, B) impulse controllable
- solution of optimal control problems using even boundary value problems

Conclusions

Our KYP inequality and Lur'e equation extend currently known formulations of the KYP inequality, the Lur'e equation, and the (generalized) algebraic Riccati equation. In particular, we do not require

- $\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \geq 0$,
- invertibility of R,
- impulse controllability.

Conclusions

Our KYP inequality and Lur'e equation extend currently known formulations of the KYP inequality, the Lur'e equation, and the (generalized) algebraic Riccati equation. In particular, we do not require

- $\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \geq 0$,
- invertibility of R,
- impulse controllability.

This has direct consequences for

- linear-quadratic optimal control problems,
- the analysis of dissipative and cyclo-dissipative systems,
- the factorization of rational functions.

Outlook

Open Problems:

- singular $sE - A \in \mathbb{R}[s]^{k \times n}$?
- time-varying problems?
- discrete-time problems?
- robust control?
- numerical solution? $\rightsquigarrow$ PhD thesis Olaf Rendel

Outlook

Open Problems:

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Thanks for Listening!

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