Tegernsee Workshop
Tegernsee
June 25-28, 2014

Singular Linear-Quadratic Optimal Control of DAEs and Descriptor Lur'e Equations

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Problem Formulation

Minimize

$$\mathcal{J}(x_0, u) = \int_0^\infty \begin{pmatrix} x(t) \\ u(t) \end{pmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{pmatrix} x(t) \\ u(t) \end{pmatrix} \mathrm{d}t$$

subject to

$$E\dot{x}(t) = Ax(t) + Bu(t), \quad Ex(0) = Ex_0, \quad \lim_{t\to\infty} Ex(t) = 0$$

where

- $E,\ A \in \mathbb{R}^{n\times n},\ B \in \mathbb{R}^{n\times m}$,
- $Q = Q^T \in \mathbb{R}^{n\times n},\ S \in \mathbb{R}^{n\times m},\ R = R^T \in \mathbb{R}^{m\times m}$.

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Definitions

Minimizer: A function $\hat{u} \in \mathcal{L}^2_{\mathrm{loc}}(\mathbb{R}^+, \mathbb{R}^m)$ such that

$$\mathcal{J}(x_0, \hat{u}) = \inf \left\{ \mathcal{J}(x_0, u) : u \in \mathcal{L}^2_{\mathrm{loc}}(\mathbb{R}^+, \mathbb{R}^m) \right\}.$$

Optimal value:

$$V(Ex_0) = \mathcal{J}(x_0, \hat{u}).$$

State of the Art

Questions

1. Feasibility: Does a minimizer exist for all initial values x_0?
2. Construction: If yes, how can we construct such a minimizer?

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Literature

- for ODEs ($E = I_n$) and $R > 0$: classical theory with algebraic Riccati equations

$$A^T X + XA + Q - \big(XB + S\big) R^{-1} \big(B^T X + S^T\big) = 0, \quad X = X^T$$

[Lancaster, Rodman '95]

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- generalization to DAEs: generalized algebraic Riccati equations

$$A^T XE + E^T XA + Q - (E^T XB + S)R^{-1}(B^T XE + S^T) = 0, \quad X = X^T,$$

[Mehrmann '91]

$$A^T X + X^T A + Q - (X^T B + S)R^{-1}(B^T X + S^T) = 0, \quad E^T X = X^T E,$$

however need artificial side conditions to be solvable [Kawamoto, Takaba, Katayama '99]

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- generalization to DAEs: generalized algebraic Riccati equations [Mehrmann '91], [Kawamoto, Takaba, Katayama '99]
- generalization to singular control, i.e., $R \geq 0$: Lur'e equations

$$\begin{aligned} A^T X + XA + Q &= K^T K, \\ XB + S &= K^T L, \quad X = X^T, \\ R &= L^T L \end{aligned}$$

[Reis '11]

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- generalization to singular control, i.e., $R \geq 0$: Lur'e equations [Reis '11]
- Here: singular control for DAEs: solution theory of descriptor Lur'e equations [Reis, V. '14]

Descriptor Lur'e Equations

$$\begin{aligned} A^T X + X^T A + Q &= K^T K + V_\infty^T \Sigma V_\infty, \\ X^T B + S &= K^T L + V_\infty^T \Sigma W_\infty, \quad E^T X = X^T E \\ R &= L^T L + W_\infty^T \Sigma W_\infty, \end{aligned}$$

has a solution

$$(X, K, L, V_\infty, W_\infty, \Sigma) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{p \times n} \times \mathbb{R}^{p \times m} \times \mathbb{R}^{n-r \times n} \times \mathbb{R}^{n-r \times m} \times \mathbb{R}^{n-r \times n-r}$$

where

- $r = \operatorname{rank} E$,
- p is minimal,
- Σ is a signature matrix,
- $\ker \begin{bmatrix} V_\infty & W_\infty \end{bmatrix} = \mathcal{V} := \left\{ \begin{pmatrix} x \\ u \end{pmatrix} \in \mathbb{R}^{n+m} : Ax + Bu \in \operatorname{im} E \right\}.$

Even Matrix Pencils

Descriptor Lur'e equations have a close relationship to matrix pencils

$$s\mathcal{E} - \mathcal{A} = \begin{bmatrix} 0 & -sE + A & B \\ sE^T + A^T & Q & S \\ B^T & S^T & R \end{bmatrix}.$$

The pencil $s\mathcal{E} - \mathcal{A}$ is called even, since $\mathcal{E} = -\mathcal{E}^T$ and $\mathcal{A} = \mathcal{A}^T$.

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Deflating subspaces

- A subspace $\mathcal{Y} := \operatorname{im} Y$ is called deflating subspace of $s\mathcal{E} - \mathcal{A}$ if there exist Z, $\tilde{\mathcal{E}}$, and $\tilde{\mathcal{A}}$ of appropriate dimensions such that

$$(s\mathcal{E} - \mathcal{A})Y = Z(s\tilde{\mathcal{E}} - \tilde{\mathcal{A}}).$$

- A subspace $\mathcal{Y}$ is called $\mathcal{E}$-neutral if $x^T \mathcal{E} y = 0$ for all $x, y \in \mathcal{Y}$.

Even Kronecker Canonical Form

Definition: even Kronecker canonical form [THOMPSON '76]

Let $s\mathcal{E} - \mathcal{A}$ be an even pencil. Then there exists a nonsingular $U \in \mathbb{C}^{n \times n}$ such that $U^H(s\mathcal{E} - \mathcal{A})U = \operatorname{diag}(\mathcal{D}_1(s), \ldots, \mathcal{D}_k(s))$ where each $\mathcal{D}_j(s)$, $j = 1, \ldots, k$ is of one of the following structures:

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Type 1 (finite non-imaginary eigenvalues):

$$\left[\begin{array}{cccc|cccc} & & & & -s+\mu_j & -1 & & \\ & & & & & \ddots & \ddots & \\ & & & & & & \ddots & -1 \\ & & & & & & & -s+\mu_j \\ \hline s+\bar{\mu}_j & & & & & & & \\ -1 & \ddots & & & & & & \\ & \ddots & \ddots & & & & & \\ & & -1 & s+\bar{\mu}_j & & & & \end{array}\right] \in \mathbb{C}[s]^{2\ell_j \times 2\ell_j},$$

with finite non-imaginary eigenvalues $\mu_j \in \mathbb{C}^+$, $-\bar{\mu}_j \in \mathbb{C}^-$.

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Type 2 (finite imaginary eigenvalues):

$$\varepsilon_j \begin{bmatrix} & & 1 & -s\mathrm{i} - \mu_j \\ & \iddots & \iddots & \\ 1 & \iddots & & \\ -s\mathrm{i} - \mu_j & & & \end{bmatrix} \in \mathbb{C}[s]^{\ell_j \times \ell_j},$$

with finite imaginary eigenvalues $\mathrm{i}\mu_j \in \mathrm{i}\mathbb{R}$ and block signature $\varepsilon_j \in \{-1, 1\}$.

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Type 3 (infinite eigenvalues):

$$\varepsilon_j \begin{bmatrix} & & si & 1 \\ & \iddots & \iddots & \\ si & \iddots & & \\ 1 & & & \end{bmatrix} \in \mathbb{C}[s]^{\ell_j \times \ell_j},$$

with block signature $\varepsilon_j \in \{-1, 1\}$.

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Type 4 (singular structure):

$$\left[\begin{array}{ccc|cccc} & & & 1 & -s & & \\ & & & & \ddots & \ddots & \\ & & & & & 1 & -s \\ \hline 1 & & & & & & \\ s & \ddots & & & & & \\ & \ddots & 1 & & & & \\ & & s & & & & \end{array}\right] \in \mathbb{C}[s]^{(2\ell_j+1)\times(2\ell_j+1)}.$$

Feasibility Conditions

Theorem

Under some stabilizability conditions, the following statements are equivalent:

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3. There exists a p such that in the EKCF of $s\mathcal{E} - \mathcal{A}$, the blocks have the following structure:
   (a) All blocks of Type 2 (imaginary evs) have even size and negative signature.

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   (c) There exist p blocks of Type 3 with positive signature.

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   (d) The remaining $2(n - r)$ blocks of Type 3 are either of even size; or the number of odd-sized blocks with positive and negative signature is equal.

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   (a) All blocks of Type 2 (imaginary evs) have even size and negative signature.
   (b) There exist exactly $2(n - r) + p$ blocks of Type 3 (infinite evs).
   (c) There exist p blocks of Type 3 with positive signature.
   (d) The remaining $2(n - r)$ blocks of Type 3 are either of even size; or the number of odd-sized blocks with positive and negative signature is equal.
   (e) There exist exactly $m - p$ blocks of Type 4 (singular structure).

Construction of Solutions

Theorem

Under some weak condition, the following statements are equivalent:

1. There exists a solution $(X, K, L, V_\infty, W_\infty, \Sigma)$ to the descriptor Lur'e equation.
2. There exist $Y = \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix} \in \mathbb{R}^{2n+m \times n+m}$, $\quad Z = \begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \end{bmatrix} \in \mathbb{R}^{2n+m \times n+p}$

   such that

   - the space $\mathcal{Y} = \operatorname{im} Y$ is $\mathcal{E}$-neutral and of dimension $n + m$;
   - $\mathcal{V} \subset \operatorname{im} \begin{bmatrix} Y_2 \\ Y_3 \end{bmatrix}$;
   - $\operatorname{rank} EY_2 = r$;
   - there exist $\tilde{\mathcal{E}}, \tilde{\mathcal{A}} \in \mathbb{R}^{n+p \times n+m}$

$$\begin{bmatrix} 0 & -sE + A & B \\ sE^T + A^T & Q & S \\ B^T & S^T & R \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix} = \begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \end{bmatrix} \left(s\tilde{\mathcal{E}} - \tilde{\mathcal{A}} \right).$$

Construction of the Solution

Remaining Questions

- How to ensure that rank $EY_2 = r$?
- How to find a solution that solves the optimal control problem?

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Theorem

Let

$$\begin{bmatrix} 0 & -sE + A & B \\ sE^T + A^T & Q & S \\ B^T & S^T & R \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix} = \begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \end{bmatrix} \left(s\tilde{\mathcal{E}} - \tilde{\mathcal{A}} \right),$$

where Y is $\mathcal{E}$-neutral, of dimension $n + m$, and

$$\mathcal{V} := \left\{ \begin{pmatrix} x \\ u \end{pmatrix} \in \mathbb{R}^{n+m} : Ax + Bu \in \operatorname{im} E \right\} \subset \operatorname{im} \begin{bmatrix} Y_2 \\ Y_3 \end{bmatrix}.$$

If for every eigenvalue λ of $s\tilde{\mathcal{E}} - \tilde{\mathcal{A}}$, $-\bar{\lambda}$ is not an uncontrollable mode of (E, A, B), then rank $EY_2 = r$.

Construction of the Solution

Remaining Questions

- How to ensure that rank $EY_2 = r$?
- How to find a solution that solves the optimal control problem?

Theorem

If some conditions hold and the optimal control problem is feasible then

- there exists a solution $(X^+, K^+, L^+, V_\infty^+, W_\infty^+, \Sigma^+)$ of the descriptor Lur'e equation, and
- $(s\mathcal{E} - \mathcal{A})Y = Z(s\tilde{\mathcal{E}} - \tilde{\mathcal{A}})$,

such that the following properties hold:

- **Stabilization**: $s\tilde{\mathcal{E}} - \tilde{\mathcal{A}} = \begin{bmatrix} -sE + A & B \\ K^+ & L^+ \end{bmatrix}$ has only eigenvalues in the closed left half-plane.
- **Maximalility**: For $X^+ = Y_1 Y_2^-$ and every other solution X it holds $E^T X \leq E^T X^+$.

Optimal Control Signal

Optimal control signal

The optimal control signal $\hat{u}(\cdot)$ fulfills the DAE

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + B\hat{u}(t), \quad Ex(0) = Ex_0 \\ 0 &= K^{+}x(t) + L^{+}\hat{u}(t). \end{aligned}$$

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Optimal costs

$$V(Ex_0) = x_0^T E^T X^+ x_0.$$

Conclusions

Presented in this talk

1. Singular optimal control problems for differential-algebraic equations,
2. characterization of the optimal control via the maximal solution of a generalized Lur'e equation,
3. solvability criteria and construction of solution using even matrix pencils.

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3. solvability criteria and construction of solution using even matrix pencils.

Not presented in this talk

1. Frequency domain analysis (Popov functions),
2. applications:
   (a) characterization of (lossless) (cyclo-)dissipative systems,
   (b) factorization of rational matrices (spectral factorization/normalized coprime factorizations/inner-outer factorization).

Thanks for your Attention!

References

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