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$\mathcal{H}_\infty$-Norm Computation for Large Sparse Descriptor Systems

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Continuous-Time Descriptor Systems

Given: Continuous-time LTI descriptor system

$$\Sigma : \quad \begin{cases} E\dot{x}(t) = Ax(t) + Bu(t), \\ \quad y(t) = Cx(t) \end{cases}$$

- E, $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{p \times n}$, $m, p \ll n$,
- descriptor vector $x(t) \in \mathbb{R}^n$, input vector $u(t) \in \mathbb{R}^m$, output vector $y(t) \in \mathbb{R}^p$.
- Assumptions: $\lambda E - A$ is regular, all matrices are large and sparse.

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Frequency domain representation

Transfer function $G(s) := C(sE - A)^{-1}B$

$\mathcal{H}_\infty$-Spaces and $\mathcal{H}_\infty$-Norm

Definition: the spaces $\mathcal{H}_\infty^{p\times m}(\mathrm{i}\omega)$ and $\mathcal{RH}_\infty^{p\times m}(\mathrm{i}\omega)$

- With $\mathcal{H}_\infty^{p\times m}(\mathrm{i}\omega)$ we denote the Hardy space of $p \times m$ matrix-valued functions which are analytic and bounded in the open right half-plane $\mathbb{C}^+ := \{s \in \mathbb{C} : \mathrm{Re}(s) > 0\}$.
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Definition: $\mathcal{H}_\infty$-norm

Natural norm for the space $\mathcal{RH}_\infty^{p\times m}(\mathrm{i}\omega)$:

$$\|G\|_{\mathcal{H}_\infty} := \sup_{s\in\mathbb{C}^+} \sigma_{\max}(G(s)) = \sup_{\omega\in\mathbb{R}} \sigma_{\max}(G(\mathrm{i}\omega)).$$

Applications

Model order reduction

Let $\hat{\Sigma} = (\lambda\hat{E} - \hat{A}, \hat{B}, \hat{C})$ be a reduced order model of the system Σ. The transfer function of the error system $\Sigma^{err} := \Sigma - \hat{\Sigma}$ is given by

$$G^{err}(s) = \begin{bmatrix} C & -\hat{C} \end{bmatrix} \left(s \begin{bmatrix} E & 0 \\ 0 & \hat{E} \end{bmatrix} - \begin{bmatrix} A & 0 \\ 0 & \hat{A} \end{bmatrix} \right)^{-1} \begin{bmatrix} B \\ \hat{B} \end{bmatrix}.$$

$\|G^{err}\|_{\mathcal{H}_\infty}$ is the size of the worst-case approximation error.

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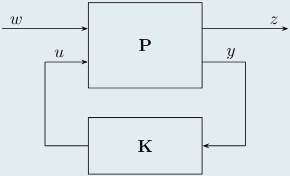
$\mathcal{H}_\infty$-control

[GREEN, LIMEBEER '95]



- Plant **P**, dynamic compensator **K**,
- noise w, estimation error z,
- $\mathcal{H}_\infty$-norm of the transfer function from w to z displays the worst-case influence of the disturbances w on the output z.

Structured Stability Radius & Pseudospectra

Definition: structured complex stability radius

$$r_{\mathbb{C}}^{\mathrm{f}}(E,A,B,C) := \inf\left\{\|\Delta\|_2 : \Pi_{\mathrm{f}}(E,A+B\Delta C,B,C)\cap \mathrm{i}\mathbb{R} \neq \emptyset\right\},$$
$$r_{\mathbb{C}}^{\infty}(E,A,B,C) := \inf\{\|\Delta\|_2 : \Pi_{\infty}(E,A+B\Delta C,B,C)\neq\emptyset \text{ or } \lambda E-(A+B\Delta C) \text{ is singular}\},$$
$$r_{\mathbb{C}}(E,A,B,C) := \min\left\{r_{\mathbb{C}}^{\mathrm{f}}(E,A,B,C),\ r_{\mathbb{C}}^{\infty}(E,A,B,C)\right\}.$$

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Definition: structured pseudospectrum of $G(s)$

$$\Pi_{\varepsilon}(E,A,B,C) = \left\{z\in\mathbb{C} : z\in \Pi_{\mathrm{f}}(E,A+B\Delta C,B,C) \text{ for some } \Delta\in\mathbb{C}^{m\times p} \text{ with } \|\Delta\|_2<\varepsilon\right\}.$$

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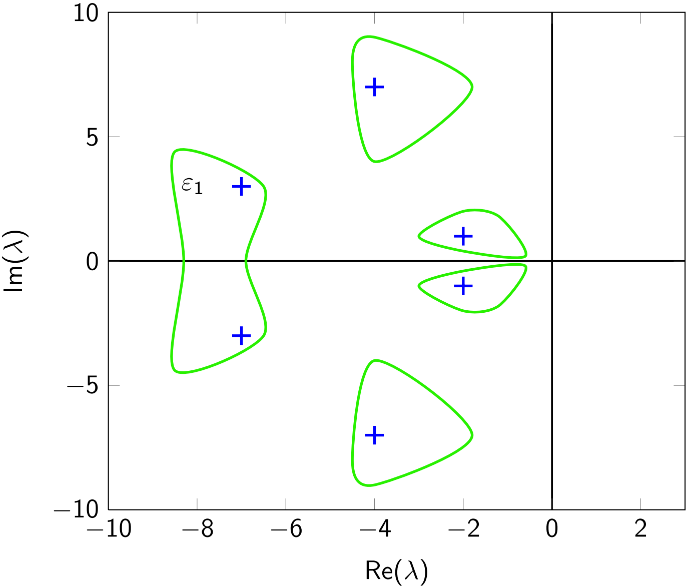
Definition: structured pseudospectral abscissa

$$\alpha_\varepsilon(E,A,B,C) := \max\left\{\operatorname{Re} z : z \in \Pi_\varepsilon(E,A,B,C)\right\}.$$

Graphical Interpretation



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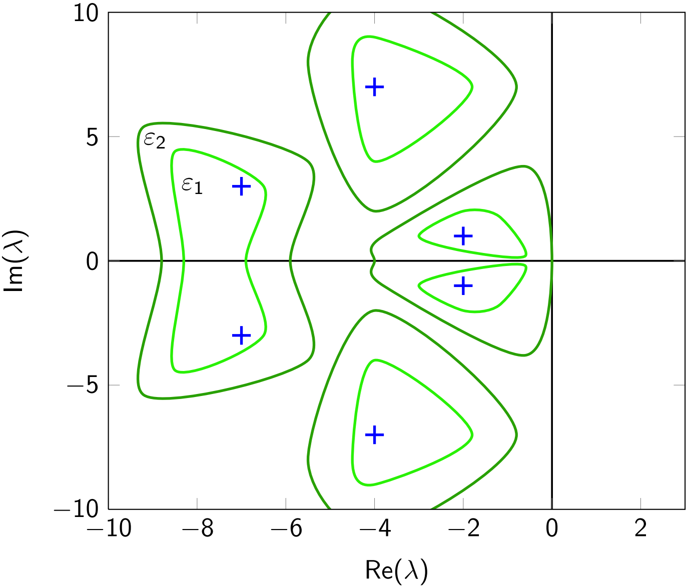

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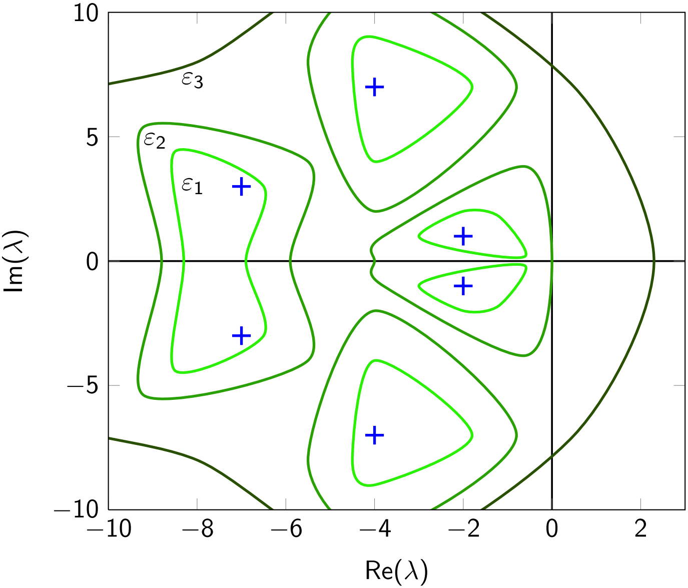

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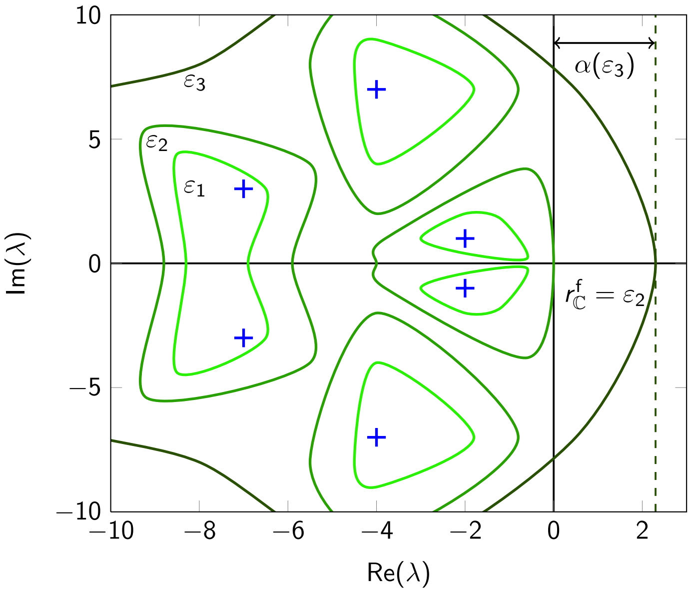

Connection to the $\mathcal{H}_\infty$-Norm

Theorem

Let $G \in \mathcal{RH}_\infty^{p \times m}(\mathrm{i}\omega)$. Then

$$r_{\mathbb{C}}(E, A, B, C) = \begin{cases} \|G\|_{\mathcal{H}_\infty}^{-1} & \text{if } G \not\equiv 0, \\ \infty & \text{if } G \equiv 0. \end{cases}$$

standard: [Hinrichsen, Pritchard '86], descriptor: [V. '11]

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$\alpha_{r_{\mathbb{C}}^{\mathrm{f}}}(E, A, B, C) = 0.$

$\Longrightarrow$ We have to find the value ε for which the corresponding structured pseudospectrum $\Pi_\varepsilon(E, A, B, C)$ touches the imaginary axis!

Algorithm Outline

Finding $r_{\mathbb{C}}^{\mathrm{f}}(E, A, B, C)$ is equivalent to finding the (unique) root of $\alpha(\varepsilon) := \alpha_\varepsilon(E, A, B, C)$. Thus we apply a root-finding algorithm. We

- do not have derivative information $\Longrightarrow$ Newton-like method not possible,
- can evaluate $\alpha(\varepsilon)$ relatively cheaply $\Longrightarrow$ secant method.

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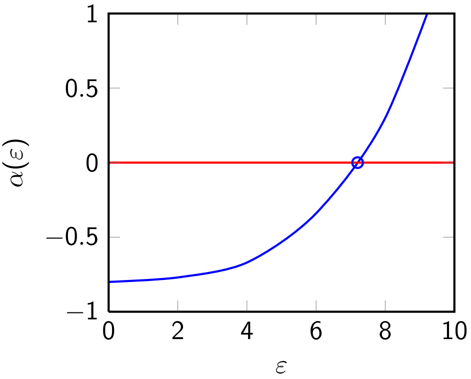

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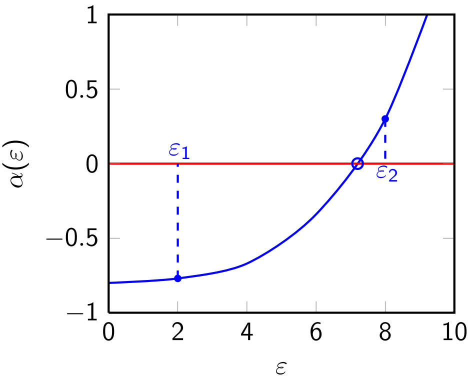

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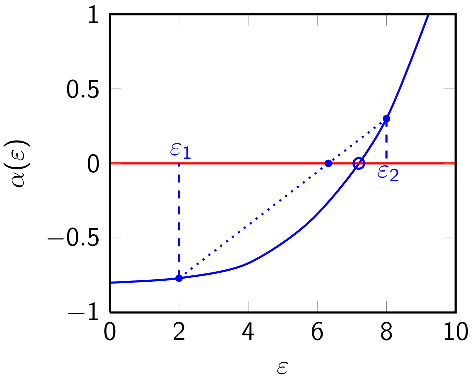

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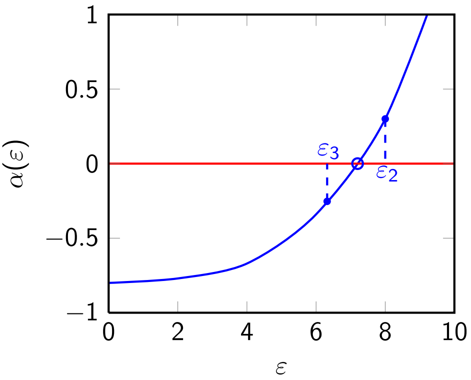

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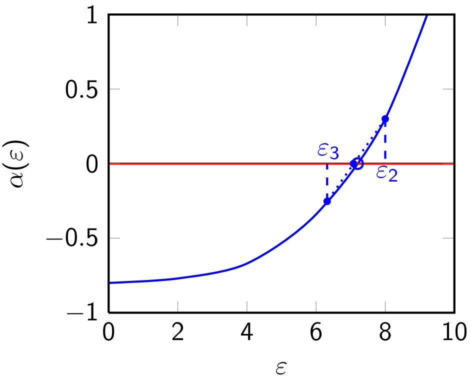

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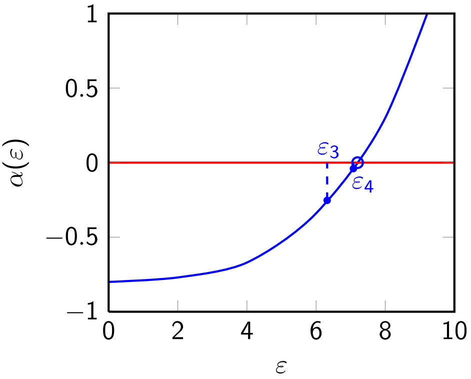

Algorithm Outline

Input: $\Sigma = (\lambda E - A, B, C)$, two initial values ε_1, ε_2, preferably close to the root.

Output: $\|G\|_{\mathcal{H}_\infty}$.

1: **for** $j = 1, 2, \ldots, k$ **do**

2: Compute $\varepsilon_{j+2} = \varepsilon_{j+1} - \frac{\varepsilon_{j+1} - \varepsilon_j}{\alpha(\varepsilon_{j+1}) - \alpha(\varepsilon_j)} \alpha(\varepsilon_{j+1})$.

3: **end for**

4: Compute $r_{\mathbb{C}}^{\infty}(E, A, B, C) = 1 / \lim\limits_{\omega \to \infty} \sigma_{\max}(G(\mathrm{i}\omega))$.

5: $\|G\|_{\mathcal{H}_\infty} = \max\left\{ \frac{1}{\varepsilon_k}, \frac{1}{r_{\mathbb{C}}^{\infty}(E,A,B,C)} \right\}$.

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Properties

- superlinear convergence,
- order of convergence is $\frac{1+\sqrt{5}}{2} \approx 1.618$,
- needs only a few evaluations of $\alpha(\varepsilon)$.

Preliminaries

Theorem

Let $z \in \mathbb{C} \backslash \Pi_{\mathsf{f}}(E, A, B, C)$ be given and $\varepsilon > 0$. Then the following statements are equivalent:

(a) $z \in \Pi_\varepsilon(E, A, B, C)$.

(b) $\sigma_{\max}(G(z)) > \varepsilon^{-1}$.

(c) There exist vectors $u \in \mathbb{C}^m$ and $v \in \mathbb{C}^p$ with $\|u\|_2 < 1$ and $\|v\|_2 < 1$ such that $z \in \Pi_{\mathsf{f}}\left(E, A + \varepsilon B u v^H C, B, C\right)$.

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Corollary

$$\Pi_\varepsilon(E, A, B, C) = \Pi_f(E, A, B, C) \cup \left\{z \in \mathbb{C} : \sigma_{\max}(G(z)) > \varepsilon^{-1}\right\}$$

with boundary

$$\partial\Pi_\varepsilon(E, A, B, C) = \left\{z \in \mathbb{C} : \sigma_{\max}(G(z)) = \varepsilon^{-1}\right\}.$$

First-Order Perturbation Theory

Strategy: Compute a sequence of suitable structured rank-1 perturbed pencils $\lambda E - (A + \varepsilon B u v^H C)$ such that one of the perturbed eigenvalues converges to the rightmost pseudopole of $(\lambda E - A, B, C)$!

[Guglielmi, Overton '11]

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Lemma [Stewart, Sun '90]

Let $x, y \in \mathbb{C}^n$ be right and left eigenvectors corresponding to a simple finite eigenvalue $\lambda = \frac{y^H Ax}{y^H Ex}$ of the pencil $\lambda E - A$. Let $\lambda E - (A + tBuv^H C)$ be a perturbed matrix pencil with eigenvalue $\tilde{\lambda}$. Then it holds

$$\tilde{\lambda} = \lambda + t\frac{y^H Buv^H Cx}{y^H Ex} + \mathcal{O}\left(t^2\right).$$

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$$\tilde{\lambda} = \lambda + t \frac{y^H B u v^H C x}{y^H E x} + \mathcal{O}\left(t^2\right).$$

Corollary

$$\left.\frac{\mathrm{d}\tilde{\lambda}(t)}{\mathrm{d}t}\right|_{t=0} = \frac{y^H B u v^H C x}{y^H E x}.$$

Construction of Structured Rank-1 Perturbations

Given: Matrix pencil $\lambda E - A$ with simple eigenvalue λ and right and left eigenvectors x, y, normalized such that $y^H Ex > 0$; vectors $u \in \mathbb{C}^m$, $v \in \mathbb{C}^p$ with $\|u\|_2 = \|v\|_2 = 1$.

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$$\operatorname{Re}\left(\left. \frac{\mathrm{d}\tilde{\lambda}(t)}{\mathrm{d}t} \right|_{t=0} \right) = \frac{\operatorname{Re}\left(y^H B u v^H C x\right)}{y^H E x} \leq \frac{\left\|y^H B\right\|_2 \|Cx\|_2}{y^H E x}.$$

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Equality holds for

$$u = \frac{B^T y}{\|B^T y\|_2}, \quad v = \frac{Cx}{\|Cx\|_2}.$$

$\Longrightarrow$ This choice of u, v yields locally maximal growth in $\operatorname{Re}(\tilde{\lambda}(t))$ as t increases from 0.

Subsequent Steps

Given: Perturbed matrix pencil $\lambda E - \hat{A} = \lambda E - (A + \varepsilon B \hat{u} \hat{v}^H C)$ with simple eigenvalue $\hat{\lambda}$, and right and left eigenvectors $\hat{x}$, $\hat{y}$, normalized such that $\hat{y}^H E \hat{x} > 0$; vectors $u \in \mathbb{C}^m$, $v \in \mathbb{C}^p$ with $\|u\|_2 = \|v\|_2 = 1$.

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$$\lambda E - \left(\hat{A} + tB\left(uv^H - \hat{u}\hat{v}^H\right)C\right),$$

which is an ε-norm rank-1 perturbation of $\lambda E - A$ for $t = 0$, $t = \varepsilon$.

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$$\begin{aligned}\operatorname{Re}\left(\left.\frac{\mathrm{d}\tilde{\lambda}(t)}{\mathrm{d}t}\right|_{t=0}\right) &= \frac{\operatorname{Re}\left(\hat{y}^H B\left(uv^H - \hat{u}\hat{v}^H\right)C\hat{x}\right)}{\hat{y}^H E\hat{x}} \\ &\leq \frac{\left\|\hat{y}^H B\right\|_2 \left\|C\hat{x}\right\|_2 - \operatorname{Re}\left(\hat{y}^H B\hat{u}\hat{v}^H C\hat{x}\right)}{\hat{y}^H E\hat{x}}.\end{aligned}$$

Subsequent Steps

Given: Perturbed matrix pencil $\lambda E - \hat{A} = \lambda E - (A + \varepsilon B \hat{u} \hat{v}^H C)$ with simple eigenvalue $\hat{\lambda}$, and right and left eigenvectors $\hat{x}$, $\hat{y}$, normalized such that $\hat{y}^H E \hat{x} > 0$; vectors $u \in \mathbb{C}^m$, $v \in \mathbb{C}^p$ with $\|u\|_2 = \|v\|_2 = 1$. Consider

$$\lambda E - \left(\hat{A} + tB \left(uv^H - \hat{u}\hat{v}^H \right) C \right),$$

which is an ε-norm rank-1 perturbation of $\lambda E - A$ for $t = 0$, $t = \varepsilon$.

$$\operatorname{Re}\left(\left. \frac{\mathrm{d}\tilde{\lambda}(t)}{\mathrm{d}t} \right|_{t=0} \right) = \frac{\operatorname{Re}\left(\hat{y}^H B \left(uv^H - \hat{u}\hat{v}^H \right) C\hat{x} \right)}{\hat{y}^H E \hat{x}}$$

$$\leq \frac{\left\| \hat{y}^H B \right\|_2 \|C\hat{x}\|_2 - \operatorname{Re}\left(\hat{y}^H B \hat{u} \hat{v}^H C \hat{x} \right)}{\hat{y}^H E \hat{x}}.$$

Again, equality holds for

$$u = \frac{B^T \hat{y}}{\|B^T \hat{y}\|_2}, \quad v = \frac{C\hat{x}}{\|C\hat{x}\|_2},$$

which is our next perturbation!

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$\Longrightarrow$ Subspace Accelerated MIMO Dominant Pole Algorithm (SAMDP)
[Rommes, Martins '06]

Dominant Poles

Assume that $\lambda E - A$ has only simple eigenvalues λ_k with left and right eigenvectors y_k and x_k such that $y_k^H E x_k = 1$. If $G(s)$ is proper then

$$G(s) = C(sE - A)^{-1}B = \sum_{k=1}^{n} \frac{R_k}{s - \lambda_k} + R_\infty$$

with residues

$$R_k = C x_k y_k^H B, \quad R_\infty = \lim_{\omega \to \infty} G(\mathrm{i}\omega).$$

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$$R_k = Cx_k y_k^H B, \quad R_\infty = \lim_{\omega \to \infty} G(\mathrm{i}\omega).$$

Observation: If λ_j is close to the imaginary axis and $\|R_j\|_2$ is large, then for $\omega \approx \mathrm{Im}(\lambda_j)$

$$G(\mathrm{i}\omega) \approx \frac{R_j}{-\mathrm{Re}(\lambda_j)} + \sum_{\substack{k=1\\k\neq j}}^{n} \frac{R_k}{\mathrm{i}\omega - \lambda_k} + R_\infty$$

and therefore $\|G(\mathrm{i}\omega)\|_2$ is large, too.

Dominant Poles

Definition

An eigenvalue $\lambda_j \in \Lambda(E, A)$ is called dominant pole of $G(s)$, if

$$\frac{\|R_k\|_2}{|\mathrm{Re}(\lambda_k)|} < \frac{\|R_j\|_2}{|\mathrm{Re}(\lambda_j)|}, \quad k = 1, \ldots, n, \quad k \neq j.$$

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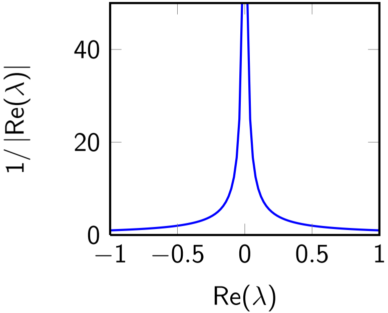
Problem with this Approach:

Poles loose dominance when they have crossed the imaginary axis

$\Longrightarrow$ need an alternative dominance measure!



Dominant Poles – New Definition

Definition

An eigenvalue $\lambda_j \in \Lambda(E, A)$ is called (exponentially) dominant pole of $G(s)$, if

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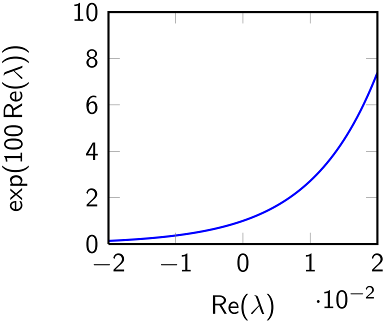
Remarks:

- β is a weighting factor which can be choosen to weigh the relevance of real part with the residual.
- For our purpose: $\beta = 100$ (high weight on real part).
- SAMDP can also be used to generate very good initial estimates of the $\mathcal{H}_\infty$-norm.



The Complete Algorithm

Input: $\Sigma = (\lambda E - A, B, C)$, perturbation level ε, tolerance on relative change τ.

Output: $\alpha_\varepsilon(E, A, B, C)$.

Compute the dominant pole λ_0 of $(\lambda E - A, B, C)$ with left and right eigenvectors y_0 and x_0.

Compute the perturbation $\hat{A} = A + \varepsilon \frac{BB^T y_0 x_0^H C^T C}{\|B^T y_0\|_2 \|C x_0\|_2}$.

for $j = 1, 2, \ldots$ **do**

  Compute the dominant pole λ_j of $(\lambda E - \hat{A}, B, C)$ with left and right eigenvectors y_j and x_j.

  if $|\mathrm{Re}\,(\lambda_j) - \mathrm{Re}\,(\lambda_{j-1})| < \tau\, |\mathrm{Re}\,(\lambda_j)|$ **then**

    Set $k = j$.

    Break.

  end if

  Compute the perturbation $\hat{A} = A + \varepsilon \frac{BB^T y_j x_j^H C^T C}{\|B^T y_j\|_2 \|C x_j\|_2}$.

end for

$\alpha_\varepsilon(E, A, B, C) = \mathrm{Re}\,(\lambda_k)$.

Example 1 – M20PI_n

Model with $n = 1182,\ m = p = 3$.

Results:

$\|G\|_{\mathcal{H}_\infty} = 3.87260, \quad t = 6.03s, \quad \alpha_{r_{\mathbb{C}}^{\mathrm{f}}}(E, A, B, C) = \text{-}3.9700\text{e-}13.$

Table: Convergence History

	k			
	1	2	3	4
$\mathrm{Re}(\lambda_{\mathrm{dom}})$	-6.7945e-02	-6.0215e+00	-3.7397e-04	3.6222e-11
	2.3140e-03	-6.0212e+00	-3.4533e-05	3.9094e-11
	3.0285e-03	—	-3.2591e-05	3.8420e-11
	3.0355e-03	—	-3.2572e-05	—
	3.0356e-03	—	—	—
ε_k	2.58250e-01	2.06600e-01	2.58224e-01	2.58224e-01

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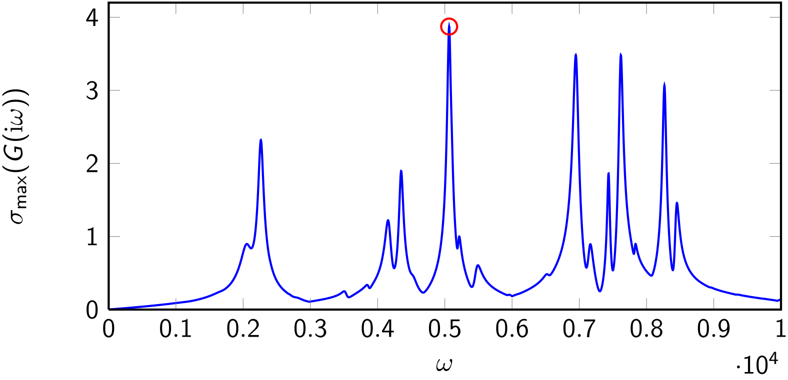

Figure: Transfer function plot with computed $\mathcal{H}_\infty$-norm

Example 2 – Mimo8x8_System

Model with $n = 13309,\ m = p = 8$.

Results:

$\|G\|_{\mathcal{H}_\infty} = 0.0534292, \quad t = 106.62s, \quad \alpha_{r^{\mathrm{f}}_{\mathbb{C}}}(E, A, B, C) = 2.6335\text{e-}13.$

Table: Convergence History

	k = 1	2	3	4
$\mathrm{Re}(\lambda_{\mathrm{dom}})$	-6.2051e-03	-4.8351e-02	-9.0793e-05	-1.4183e-09
	-6.3276e-04	-4.8266e-02	-9.4865e-06	-1.3415e-09
	3.8109e-06	-4.8253e-02	3.0458e-06	-1.3273e-09
	1.1425e-04	—	5.4062e-06	-1.3245e-09
	1.3425e-04	—	5.8487e-06	-1.3241e-09
	1.3794e-04	—	5.9315e-06	—
	1.3862e-04	—	5.9470e-06	—
	1.3875e-04	—	5.9498e-06	—
ε_k	1.87276e+01	1.49821e+01	1.87168e+01	1.87164e+01

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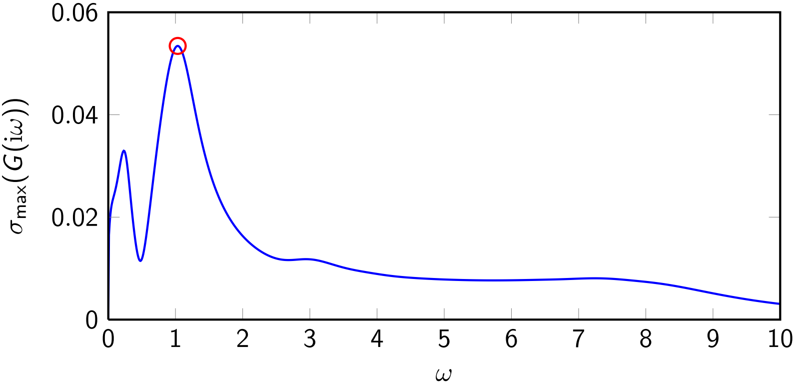

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Example 3 – Mimo46x46_System

Model with $n = 13250$, $m = p = 46$.

Results:

$\|G\|_{\mathcal{H}_\infty} = 205.631$, $t = 167.43s$, $\alpha_{r^{\mathrm{f}}_{\mathbb{C}}}(E, A, B, C) = 4.2864\text{e-}14$.

Table: Convergence History

	k = 1	2	3
$\mathrm{Re}(\lambda_{\mathrm{dom}})$	-9.0777e-05	-6.6047e-03	-3.6061e-06
	2.0799e-06	-6.6018e-03	2.7127e-09
	2.1973e-06	—	7.3774e-09
	2.1976e-06	—	7.3907e-09
	—	—	7.3907e-09
ε_k	4.86342e-03	3.89073e-03	4.86309e-03

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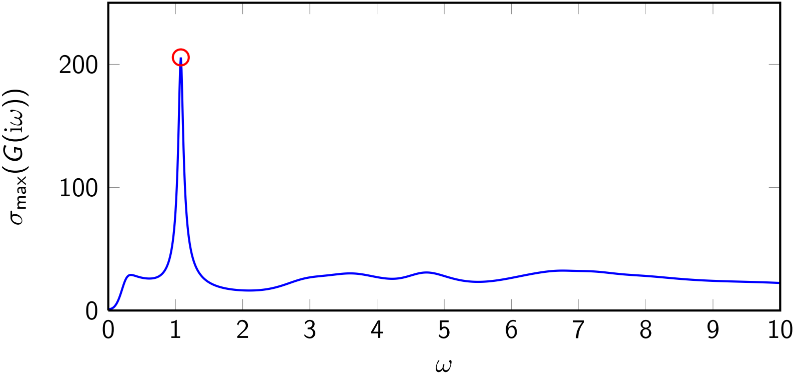

Figure: Transfer function plot with computed $\mathcal{H}_\infty$-norm

Conclusions and Outlook

Conclusions

- Introduction of relations between the structured pseudospectra and the $\mathcal{H}_\infty$-norm of descriptor systems,
- development of an iterative algorithm for the computation of the $\mathcal{H}_\infty$-norm by iterating over the structured pseudospectral abscissa,

Open Problems

- Discrete-time systems $\Longrightarrow$ computation of the structured pseudospectral radius,
- how to graph structured pseudospectra? $\Longrightarrow$ EigTool for structured pseudospectra,
- real stability radii, passivity radius?

Thank you for your Attention!

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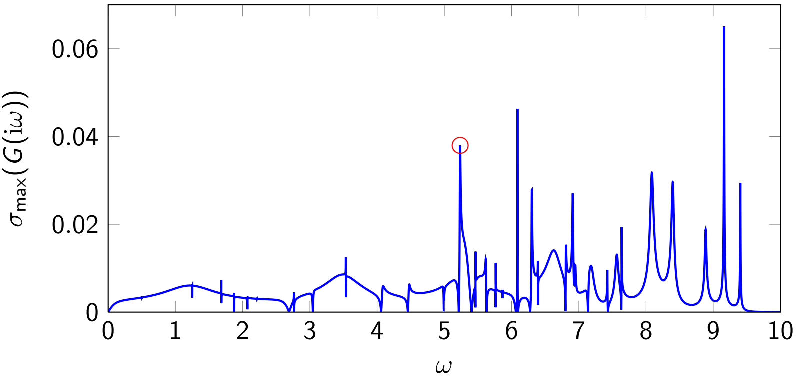
Example 4 – Peec (Does Not Work)

Model with $n = 480$, $m = p = 1$, lots of peaks close to $\mathrm{i}\mathbb{R}$!

Results:

$\|G\|_{\mathcal{H}_\infty} = 0.0379802$, $t = 23.20s$, $\alpha_{r_\mathbb{C}}(E, A, B, C) = 6.1976\text{e-}11$.



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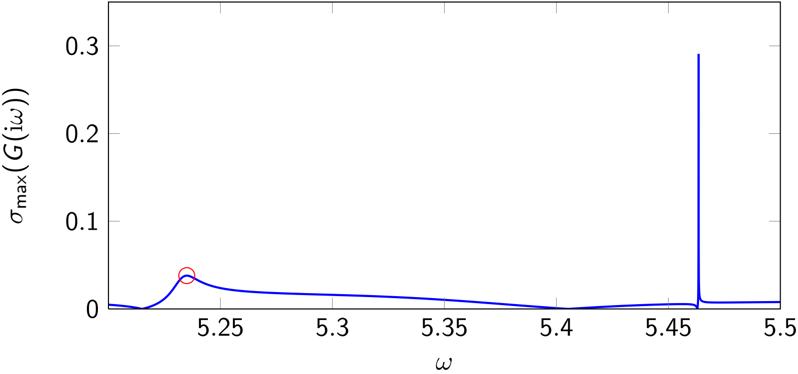

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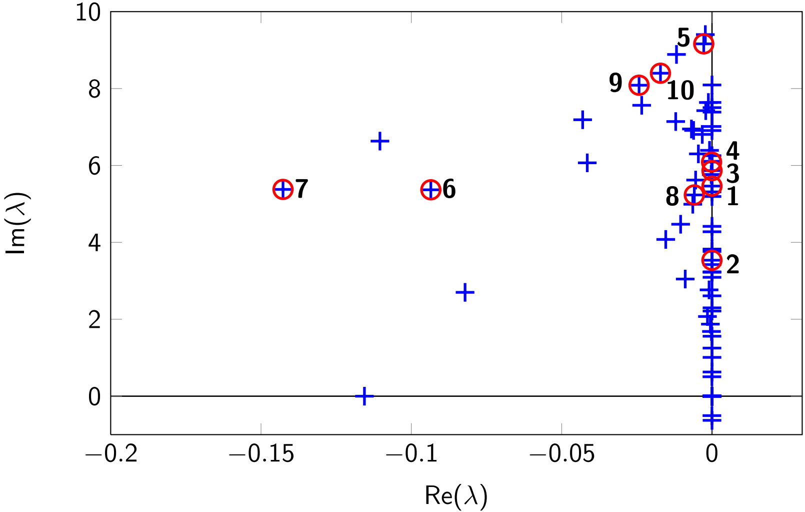

Figure: Eigenvalues (blue pluses) and the 10 most dominant poles (red circles)