Linear-Quadratic Control of DAEs with an Application to Flow Control Problems

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Differential-Algebraic Systems/Descriptor Systems

Linear time-invariant differential-algebraic systems/descriptor systems

$$\frac{\mathrm{d}}{\mathrm{d}t} Ex(t) = Ax(t) + Bu(t)$$

where

- $E, A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$,
- E is typically singular, but assume that $sE - A$ is regular, i. e., $\det(sE - A) \not\equiv 0$,
- state $x \in \mathcal{L}^2_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^n)$, input $u \in \mathcal{L}^2_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^m)$.

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Typical applications

- network models: electrical circuits, gas networks, constrained multi-body systems, ...
- semi-discretization of PDEs (e. g., Navier-Stokes),
- linearization of non-linear DAEs,
- ....

The Linear-Quadratic Optimal Control Problem

Linear-quadratic optimal control problem: Minimize

$$\mathcal{J}(x,u) := \int_0^\infty \begin{pmatrix} x(\tau) \\ u(\tau) \end{pmatrix}^\mathsf{T} \begin{bmatrix} Q & S \\ S^\mathsf{T} & R \end{bmatrix} \begin{pmatrix} x(\tau) \\ u(\tau) \end{pmatrix} \mathrm{d}\tau$$

with $Q = Q^\mathsf{T} \in \mathbb{R}^{n\times n}$, $S \in \mathbb{R}^{n\times m}$, and $R = R^\mathsf{T} \in \mathbb{R}^{m\times m}$ subject to

$$\tfrac{\mathrm{d}}{\mathrm{d}t} Ex(t) = Ax(t) + Bu(t), \quad Ex(0) = Ex_0, \quad \lim_{t\to\infty} Ex(t) = 0.$$

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Problems:

- feasibility, i. e., is

  $$V^+(Ex_0) := \inf \left\{ \mathcal{J}(x,u) : \tfrac{\mathrm{d}}{\mathrm{d}t} Ex = Ax + Bu,\ Ex(0) = Ex_0,\ \lim_{t \to \infty} Ex(t) = 0 \right\}$$

  finite for all consistent x_0?
- optimal value $V^+(Ex_0)$?
- existence and uniqueness of optimal controls ($\rightsquigarrow$ regularity)?

Geometric Concepts

a) behavior:

$$\mathfrak{B}_{[E,A,B]} := \left\{(x,u) \in \mathcal{L}^2_{\mathrm{loc}}(\mathbb{R},\mathbb{R}^n) \times \mathcal{L}^2_{\mathrm{loc}}(\mathbb{R},\mathbb{R}^m) : \tfrac{\mathrm{d}}{\mathrm{d}t} Ex = Ax + Bu\right\}.$$

b) system space: smallest subspace in $\mathbb{R}^{n+m}$ such that

$$\begin{pmatrix} x(t) \\ u(t) \end{pmatrix} \in \mathcal{V}_{\mathrm{sys}} \quad \forall (x,u) \in \mathfrak{B}_{[E,A,B]} \text{ and almost all } t \in \mathbb{R}.$$

c) space of consistent initial differential variables:

$$\mathcal{V}_{\mathrm{diff}} := \left\{x_0 \in \mathbb{R}^n : \exists (x,u) \in \mathfrak{B}_{[E,A,B]} \text{ with } Ex(0) = Ex_0\right\}.$$

A system is called impulse controllable, if $\mathcal{V}_{\mathrm{diff}} = \mathbb{R}^n$.

Examples

Example 1

$$\text{Minimize} \quad \mathcal{J}(x,u) = \int_0^\infty \begin{pmatrix} x(\tau) \\ u(\tau) \end{pmatrix}^{\mathsf{T}} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{pmatrix} x(\tau) \\ u(\tau) \end{pmatrix} \mathrm{d}\tau$$

$$\text{subject to} \quad \tfrac{\mathrm{d}}{\mathrm{d}t} x(t) = -x(t), \quad x(0) = x_0, \quad \lim_{t\to\infty} x(t) = 0.$$

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Observation:

$$\mathcal{J}(x,u) = \int_0^\infty 2x(\tau)u(\tau)\mathrm{d}\tau$$

can be made arbitrarily large and negative, since $u(\cdot)$ has no influence on the dynamics.

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$\implies V(Ex_0) = -\infty$ for all $x_0 \in \mathbb{R} \implies$ infeasible problem.

Examples

Example 2

Minimize $$\mathcal{J}(x,u) = \int_0^\infty \begin{pmatrix} x(\tau) \\ u(\tau) \end{pmatrix}^\mathsf{T} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} x(\tau) \\ u(\tau) \end{pmatrix} \mathrm{d}\tau$$

subject to $$\tfrac{\mathrm{d}}{\mathrm{d}t} x(t) = -u(t), \quad x(0) = x_0, \quad \lim_{t\to\infty} x(t) = 0.$$

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Observation:

- For $x_0 \neq 0$ we have $\mathcal{J}(x,u) > 0$ for all solutions (x,u) with $x(0) = x_0$.
- For $u_k = k \cdot \chi_{[0,k^{-1}]} \cdot x_0$ we have

$$\mathcal{J}(x_k, u_k) = \frac{1}{3k} x_0^2 \overset{k\to\infty}{\longrightarrow} 0.$$

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$\Longrightarrow$ $V(Ex_0) = 0$ for all $x_0 \in \mathbb{R}$, but for $x_0 \neq 0$ there exists no solution attaining this value $\Longrightarrow$ singular problem.

Conditions for Feasibility

Reminder: Optimal value

$$V^+(Ex_0) := \inf \left\{ \mathcal{J}(x,u) : \tfrac{\mathrm{d}}{\mathrm{d}t} Ex = Ax + Bu,\ Ex(0) = Ex_0,\ \lim_{t\to\infty} Ex(t) = 0 \right\}$$

Necessary condition for feasibility

$$V^+(Ex_0) < \infty \quad \forall\, x_0 \in \mathcal{V}_{\mathrm{diff}}$$

holds if and only if the system is behaviorally stabilizable, i. e., for all $x_0 \in \mathcal{V}_{\mathrm{diff}}$ there exists a solution (x,u) with $Ex(0) = Ex_0$ and $\lim_{t\to\infty} Ex(t) = 0$.
Algebraically:

$$\operatorname{rank} \begin{bmatrix} -\lambda E + A & B \end{bmatrix} = n \quad \forall\, \lambda \in \overline{\mathbb{C}^+}.$$

Conditions for Feasibility

(Virtual) storage functions

A function $V : E\mathcal{V}_{\mathrm{diff}} \to \mathbb{R}$ is called (virtual) storage function for the optimal control problem if V is continuous, $V(0) = 0$ and for all solutions (x, u) we have

$$V(Ex(t_1)) - V(Ex(t_2)) \leq \int_{t_1}^{t_2} \begin{pmatrix} x(\tau) \\ u(\tau) \end{pmatrix}^{\mathsf{T}} \begin{bmatrix} Q & S \\ S^{\mathsf{T}} & R \end{bmatrix} \begin{pmatrix} x(\tau) \\ u(\tau) \end{pmatrix} \mathrm{d}\tau.$$

For $t_1 = 0$, $t_2 \to \infty$, and $Ex(0) = Ex_0$, $\lim_{t\to\infty} Ex(t) = 0$ we obtain

$$V(Ex_0) \leq \mathcal{J}(x, u).$$

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$$V(Ex_0) \leq \mathcal{J}(x, u).$$

Sufficient condition for feasibility

If the system is behaviorally stabilizable and there exists a storage function, then

$$-\infty < V(Ex_0) \leq V^{+}(Ex_0) < \infty \quad \forall\, x_0 \in \mathcal{V}_{\mathrm{diff}}.$$

Quadratic Storage Functions

Notation

For symmetric M, $N \in \mathbb{R}^{n \times n}$ and a subspace $\mathcal{V} \subseteq \mathbb{R}^n$ we write

- $M \geq_{\mathcal{V}} N \quad :\Longleftrightarrow \quad v^{\mathsf{T}} M v \geq v^{\mathsf{T}} N v \quad \forall\, v \in \mathcal{V}$,
- $M =_{\mathcal{V}} N \quad :\Longleftrightarrow \quad v^{\mathsf{T}} M v = v^{\mathsf{T}} N v \quad \forall\, v \in \mathcal{V}$.

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Quadratic storage functions and KYP inequality [Reis, Rendel, V. '15]

The function $V : E\mathcal{V}_{\text{diff}} \to \mathbb{R}$ is a quadratic storage function, i. e.,

$$V(Ex_0) = x_0^\mathsf{T} E^\mathsf{T} P E x_0 \quad \text{for} \quad P = P^\mathsf{T} \in \mathbb{R}^{n\times n}.$$

if and only if the KYP inequality

$$\begin{bmatrix} E^\mathsf{T} PA + A^\mathsf{T} PE + Q & E^\mathsf{T} PB + S \\ B^\mathsf{T} PE + S^\mathsf{T} & R \end{bmatrix} \geq_{\mathcal{V}_{\text{sys}}} 0, \quad P = P^\mathsf{T}$$

is fulfilled.

Quadratic Storage Functions

Quadratic storage functions and KYP inequality

If $V(Ex_0) = x_0^\mathsf{T} E^\mathsf{T} P E x_0$ is a quadratic storage function, then there exist $K \in \mathbb{R}^{q \times n}$ and $L \in \mathbb{R}^{q \times m}$ such that

$$\begin{bmatrix} E^\mathsf{T} P A + A^\mathsf{T} P E + Q & E^\mathsf{T} P B + S \\ B^\mathsf{T} P E + S^\mathsf{T} & R \end{bmatrix} =_{\mathcal{V}_{\mathrm{sys}}} \begin{bmatrix} K^\mathsf{T} \\ L^\mathsf{T} \end{bmatrix} \begin{bmatrix} K & L \end{bmatrix}, \quad P = P^\mathsf{T}$$

Quadratic Storage Functions

For (x, u) with $Ex(0) = Ex_0$ and $\lim_{t\to\infty} Ex(t) = 0$ it holds

$$x_0^\mathsf{T} E^\mathsf{T} P E x_0 = -\int_0^\infty \frac{\mathrm{d}}{\mathrm{d}\tau}(x(\tau)^\mathsf{T} E^\mathsf{T} P E x(\tau))\mathrm{d}\tau$$

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\begin{aligned}
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&= -\int_0^\infty \tfrac{\mathrm{d}}{\mathrm{d}\tau}\big(x(\tau)^{\mathsf{T}} E^{\mathsf{T}}\big) P E x(\tau) + x(\tau)^{\mathsf{T}} E^{\mathsf{T}} P \tfrac{\mathrm{d}}{\mathrm{d}\tau}\big(E x(\tau)\big)\mathrm{d}\tau \\
&= -\int_0^\infty \big(Ax(\tau) + Bu(\tau)\big)^{\mathsf{T}} P E x(\tau) + x(\tau)^{\mathsf{T}} E^{\mathsf{T}} P \big(Ax(\tau) + Bu(\tau)\big)\mathrm{d}\tau \\
&= \int_0^\infty \begin{pmatrix} x(\tau) \\ u(\tau) \end{pmatrix}^{\mathsf{T}} \begin{bmatrix} -E^{\mathsf{T}} P A - A^{\mathsf{T}} P E & -E^{\mathsf{T}} P B \\ -B^{\mathsf{T}} P E & 0 \end{bmatrix} \begin{pmatrix} x(\tau) \\ u(\tau) \end{pmatrix} \mathrm{d}\tau
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&= \int_0^\infty \begin{pmatrix} x(\tau) \\ u(\tau) \end{pmatrix}^\mathsf{T} \left(\begin{bmatrix} Q & S \\ S^\mathsf{T} & R \end{bmatrix} - \begin{bmatrix} K^\mathsf{T} K & K^\mathsf{T} L \\ L^\mathsf{T} K & L^\mathsf{T} L \end{bmatrix} \right) \begin{pmatrix} x(\tau) \\ u(\tau) \end{pmatrix} \mathrm{d}\tau
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&= \int_0^\infty \begin{pmatrix} x(\tau) \\ u(\tau) \end{pmatrix}^\mathsf{T} \left(\begin{bmatrix} Q & S \\ S^\mathsf{T} & R \end{bmatrix} - \begin{bmatrix} K^\mathsf{T} K & K^\mathsf{T} L \\ L^\mathsf{T} K & L^\mathsf{T} L \end{bmatrix} \right) \begin{pmatrix} x(\tau) \\ u(\tau) \end{pmatrix} \mathrm{d}\tau \\
&= \mathcal{J}(x, u) - \| Kx + Lu \|^2_{\mathcal{L}^2([0,\infty),\mathbb{R}^q)} .
\end{aligned}$$

Relation to the Optimal Control Problem

If the optimal control problem is feasible then the optimal value function V^+ is quadratic, i. e.,

$$V^+(Ex_0) = x_0^\mathsf{T} E^\mathsf{T} P E x_0,$$

where

$$\begin{bmatrix} E^\mathsf{T} PA + A^\mathsf{T} PE + Q & E^\mathsf{T} PB + S \\ B^\mathsf{T} PE + S^\mathsf{T} & R \end{bmatrix} =_{\mathcal{V}_{\mathrm{sys}}} \begin{bmatrix} K^\mathsf{T} \\ L^\mathsf{T} \end{bmatrix} \begin{bmatrix} K & L \end{bmatrix}, \quad P = P^\mathsf{T}$$

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Implications:

- There exists a sequence $((x_k, u_k))_{k\in\mathbb{N}}$ of $\mathcal{L}_2$-solutions with $Ex_k(0) = Ex_0$ and $\lim_{t\to\infty} Ex_k(t) = 0$ with

$$\lim_{k\to\infty} \|Kx_k + Lu_k\|^2_{\mathcal{L}^2([0,\infty),\mathbb{R}^q)} = 0.$$

- An optimal control fulfills the optimality DAE

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}\begin{pmatrix} x(t) \\ u(t) \end{pmatrix} = \begin{bmatrix} A & B \\ K & L \end{bmatrix}\begin{pmatrix} x(t) \\ u(t) \end{pmatrix}, \quad Ex(0) = Ex_0, \quad \lim_{t\to\infty} Ex(t) = 0.$$

Lur'e Equations

We have

$$V^+(Ex_0) = x_0^\mathsf{T} E^\mathsf{T} P E x_0,$$

where

$$\begin{bmatrix} E^\mathsf{T} PA + A^\mathsf{T} PE + Q & E^\mathsf{T} PB + S \\ B^\mathsf{T} PE + S^\mathsf{T} & R \end{bmatrix} =_{\mathcal{V}_{\mathrm{sys}}} \begin{bmatrix} K^\mathsf{T} \\ L^\mathsf{T} \end{bmatrix} \begin{bmatrix} K & L \end{bmatrix}, \quad P = P^\mathsf{T}. \qquad \text{(LE)}$$

We can choose q such that $K \in \mathbb{R}^{q \times n}$ and $L \in \mathbb{R}^{q \times m}$ fulfill the extra condition

$$\operatorname{rank}_{\mathbb{R}(s)} \begin{bmatrix} -sE + A & B \\ K & L \end{bmatrix} = n + q. \qquad \text{(rk)}$$

Definition

We call a triple $(P, K, L) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{q \times n} \times \mathbb{R}^{q \times m}$ solution of the Lur'e equation (LE), if (LE) and (rk) are satisfied.

Stabilizing Solutions

Solutions of the Lur'e equation corresponding to V^+ have further structure: A solution (P, K, L) of the Lur'e equation is called stabilizing, if

$$\operatorname{rank} \begin{bmatrix} -\lambda E + A & B \\ K & L \end{bmatrix} = n + q \quad \forall \lambda \in \mathbb{C}^+.$$

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Existence and uniqueness [Reis, Rendel, V. '15]

Assume that the KYP inequality is solvable. Then we have:

- **Existence:** system is behaviorally stabilizable $\Rightarrow$ $\exists$ stabilizing solution.
- **Uniqueness:** (P_1, K_1, L_1), (P_2, K_2, L_2) stabilizing solutions $\Rightarrow$ $E^\mathsf{T} P_1 E =_{\mathcal{V}_\mathrm{diff}} E^\mathsf{T} P_2 E$.
- **Extremality:** (P, K, L) is stabilizing solution of the Lur'e equation $\Rightarrow$ $E^\mathsf{T} P E \geq_{\mathcal{V}_\mathrm{diff}} E^\mathsf{T} \tilde{P} E$ for all solutions $\tilde{P}$ of the KYP inequality.

Existence and Uniqueness of Optimal Controls

Reminder: optimality DAE

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}\begin{pmatrix} x(t) \\ u(t) \end{pmatrix} = \begin{bmatrix} A & B \\ K & L \end{bmatrix}\begin{pmatrix} x(t) \\ u(t) \end{pmatrix}, \quad Ex(0) = Ex_0, \quad \lim_{t\to\infty} Ex(t) = 0.$$

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Corollary

- If for all $x_0 \in \mathcal{V}_{\mathrm{diff}}$ there exists an optimal control, then

$$\operatorname{rank}\begin{bmatrix} -\lambda E + A & B \\ K & L \end{bmatrix} = n + q \quad \forall\, \lambda \in \overline{\mathbb{C}^+}.$$

- If for all $x_0 \in \mathcal{V}_{\mathrm{diff}}$ there exists a unique optimal control, then

$$\operatorname{rank}\begin{bmatrix} -\lambda E + A & B \\ K & L \end{bmatrix} = n + m \quad \forall\, \lambda \in \overline{\mathbb{C}^+}.$$

Remark: An equivalence of such statements can be proven for impulse controllable systems.

Approaches Based on Matrix Inequalities

Our KYP Inequality

$$\begin{bmatrix} E^\mathsf{T} P A + A^\mathsf{T} P E + Q & E^\mathsf{T} P B + S \\ B^\mathsf{T} P E + S^\mathsf{T} & R \end{bmatrix} \geq_{\mathcal{V}_{\mathrm{sys}}} 0, \quad P = P^\mathsf{T}$$

Approaches Based on Matrix Inequalities

Our KYP Inequality

$$\begin{bmatrix} E^{\mathsf{T}}PA + A^{\mathsf{T}}PE + Q & E^{\mathsf{T}}PB + S \\ B^{\mathsf{T}}PE + S^{\mathsf{T}} & R \end{bmatrix} \geq_{\mathcal{V}_{\mathrm{sys}}} 0, \quad P = P^{\mathsf{T}}$$

- **Geerts '89–'94:**

$$\begin{bmatrix} E^{\mathsf{T}}PA + A^{\mathsf{T}}PE + Q & E^{\mathsf{T}}PB + S \\ B^{\mathsf{T}}PE + S^{\mathsf{T}} & R \end{bmatrix} \geq 0, \quad P = P^{\mathsf{T}}$$

Remarks:

- assumptions: $\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} = \begin{bmatrix} C^T \\ D^T \end{bmatrix} \begin{bmatrix} C & D \end{bmatrix} \geq 0$, however $sE - A \in \mathbb{R}[s]^{k \times n}$ may be singular,
- (E, A, B) is impulse controllable $\Rightarrow \exists$ maximal solution of the LMI solving the optimal control problem,
- no theory for non-impulse controllable systems!

Approaches Based on Matrix Inequalities

Our KYP Inequality

$$\begin{bmatrix} E^{\mathsf{T}}PA + A^{\mathsf{T}}PE + Q & E^{\mathsf{T}}PB + S \\ B^{\mathsf{T}}PE + S^{\mathsf{T}} & R \end{bmatrix} \geq_{\mathcal{V}_{\mathrm{sys}}} 0, \quad P = P^{\mathsf{T}}$$

- **Brüll '11:**

$$\begin{bmatrix} A^{\mathsf{T}}P_1 + P_1^{\mathsf{T}}A + Q & A^{\mathsf{T}}P_2 + P_1^{\mathsf{T}}B + S \\ B^{\mathsf{T}}P_1 + P_2^{\mathsf{T}}A + S^{\mathsf{T}} & B^{\mathsf{T}}P_2 + P_2^{\mathsf{T}}B + R \end{bmatrix} \geq 0,$$

$$E^{\mathsf{T}}P_1 = P_1^{\mathsf{T}}E, \quad E^{\mathsf{T}}P_2 = 0$$

Remarks:

- assumptions: (E, A, B) is completely controllable,
- also generalization of the KYP inequality to higher-order and behavioral systems.

Approaches Based on Matrix Equations

Our Lur'e Equation

$$\begin{bmatrix} E^\mathsf{T}XA + A^\mathsf{T}XE + Q & E^\mathsf{T}XB + S \\ B^\mathsf{T}XE + S^\mathsf{T} & R \end{bmatrix} =_{\mathcal{V}_{\mathrm{sys}}} \begin{bmatrix} K^\mathsf{T} \\ L^\mathsf{T} \end{bmatrix} \begin{bmatrix} K & L \end{bmatrix}, \quad X = X^\mathsf{T}$$

Approaches Based on Matrix Equations

Our Lur'e Equation

$$\begin{bmatrix} E^{\mathsf{T}}XA + A^{\mathsf{T}}XE + Q & E^{\mathsf{T}}XB + S \\ B^{\mathsf{T}}XE + S^{\mathsf{T}} & R \end{bmatrix} =_{\mathcal{V}_{\mathrm{sys}}} \begin{bmatrix} K^{\mathsf{T}} \\ L^{\mathsf{T}} \end{bmatrix} \begin{bmatrix} K & L \end{bmatrix}, \quad X = X^{\mathsf{T}}$$

- If $E = I_n$, then our Lur'e equation reduces to the standard Lur'e equation [Reis '11]

$$\begin{bmatrix} A^{\mathsf{T}}X + XA + Q & XB + S \\ B^{\mathsf{T}}X + S^{\mathsf{T}} & R \end{bmatrix} = \begin{bmatrix} K^{\mathsf{T}} \\ L^{\mathsf{T}} \end{bmatrix} \begin{bmatrix} K & L \end{bmatrix}, \quad X = X^{\mathsf{T}}.$$

Approaches Based on Matrix Equations

Our Lur'e Equation

$$\begin{bmatrix} E^\mathsf{T}XA + A^\mathsf{T}XE + Q & E^\mathsf{T}XB + S \\ B^\mathsf{T}XE + S^\mathsf{T} & R \end{bmatrix} =_{\mathcal{V}_{\text{sys}}} \begin{bmatrix} K^\mathsf{T} \\ L^\mathsf{T} \end{bmatrix} \begin{bmatrix} K & L \end{bmatrix}, \quad X = X^\mathsf{T}$$

- If $E = I_n$, then our Lur'e equation reduces to the standard Lur'e equation [Reis '11]

$$\begin{bmatrix} A^\mathsf{T}X + XA + Q & XB + S \\ B^\mathsf{T}X + S^\mathsf{T} & R \end{bmatrix} = \begin{bmatrix} K^\mathsf{T} \\ L^\mathsf{T} \end{bmatrix} \begin{bmatrix} K & L \end{bmatrix}, \quad X = X^\mathsf{T}.$$

- If further R is invertible, then $\operatorname{rank} \begin{bmatrix} K & L \end{bmatrix} = m$ and K and L can be eliminated to obtain an algebraic Riccati equation

$$A^\mathsf{T}X + XA + Q - (XB + S)R^{-1}(B^\mathsf{T}X + S^\mathsf{T}) = 0, \quad X = X^\mathsf{T}.$$

Approaches Based on Matrix Equations

Our Lur'e Equation

$$\begin{bmatrix} E^\mathsf{T} XA + A^\mathsf{T} XE + Q & E^\mathsf{T} XB + S \\ B^\mathsf{T} XE + S^\mathsf{T} & R \end{bmatrix} =_{\mathcal{V}_{\mathrm{sys}}} \begin{bmatrix} K^\mathsf{T} \\ L^\mathsf{T} \end{bmatrix} \begin{bmatrix} K & L \end{bmatrix}, \quad X = X^\mathsf{T}$$

- **Kurina '93:** generalized algebraic Riccati equation:

$$A^\mathsf{T} X + XA + Q - \left(XB + S\right) R^{-1} \left(B^\mathsf{T} X + S^T\right) = 0, \quad E^\mathsf{T} X = X^\mathsf{T} E$$

Remarks:

- assumptions: $\begin{bmatrix} Q & S \\ S^\mathsf{T} & R \end{bmatrix} \geq 0$, $R > 0$, (E, A, B) impulse controllable,
- solution analysis: Katayama/Minamino '92 and Katayama/Kawamoto/Takaba '99, in particular solvability of an "algebraic quadratic matrix equation" necessary for existence of stabilizing solutions.

Approaches Based on Matrix Equations

Our Lur'e Equation

$$\begin{bmatrix} E^\mathsf{T}XA + A^\mathsf{T}XE + Q & E^\mathsf{T}XB + S \\ B^\mathsf{T}XE + S^\mathsf{T} & R \end{bmatrix} =_{\mathcal{V}_{\mathrm{sys}}} \begin{bmatrix} K^\mathsf{T} \\ L^\mathsf{T} \end{bmatrix} \begin{bmatrix} K & L \end{bmatrix}, \quad X = X^\mathsf{T}$$

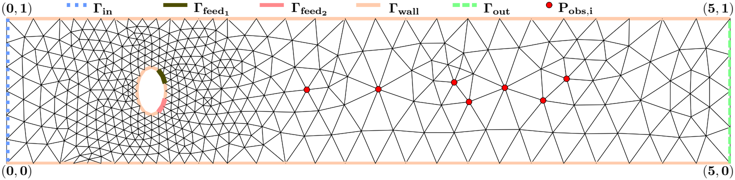
- **Mehrmann '91:** generalized algebraic Riccati equation:

$$A^\mathsf{T}XE + E^\mathsf{T}XA + Q - \left(E^\mathsf{T}XB + S\right)R^{-1}\left(B^\mathsf{T}XE + S^\mathsf{T}\right) = 0, \quad X = X^\mathsf{T}$$

Remarks:

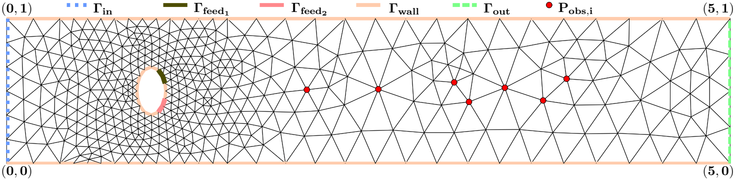
- assumptions: $\begin{bmatrix} Q & S \\ S^\mathsf{T} & R \end{bmatrix} \geq 0$, $R > 0$, (E, A, B) impulse controllable,
- solution of optimal control problems using even boundary value problems.

Example: von Kármán vortex street



Goal: stabilization of the fields towards a desired stationary solution $(\mathbf{w}(x), \chi_s(x))$.

Example: von Kármán vortex street



Goal: stabilization of the fields towards a desired stationary solution $(\mathbf{w}(x), \chi_s(x))$.

Model: linearized Navier-Stokes equation [Bänsch, Benner, Saak, Weichelt '15]

$$\frac{\partial}{\partial t}\mathbf{z}(t,x) - \frac{1}{\mathrm{Re}}\Delta\mathbf{z}(t,x) + (\mathbf{w}(x)\cdot\nabla)\mathbf{z}(t,x) + (\mathbf{z}(t,x)\cdot\nabla)\mathbf{w}(x) + \nabla\mathbf{p}(t,x) = 0,$$

$$\operatorname{div}\mathbf{z}(t,x) = 0,$$

+initial conditions/+boundary conditions,

Here:

- $\mathbf{z}(t,x)$ - difference between actual and desired velocities,
- $\mathbf{p}(t,x)$ - difference between actual and desired pressure.

Discretization and Optimal Control Problem

Spatial discretization

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix}\begin{pmatrix} z(t) \\ p(t) \end{pmatrix} = \begin{bmatrix} A & G^{\mathsf{T}} \\ G & 0 \end{bmatrix}\begin{pmatrix} z(t) \\ p(t) \end{pmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u(t), \qquad \text{(DAE)}$$

where $M > 0$ and G has full row rank.

Discretization and Optimal Control Problem

Spatial discretization

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix}\begin{pmatrix} z(t) \\ p(t) \end{pmatrix} = \begin{bmatrix} A & G^{\mathsf{T}} \\ G & 0 \end{bmatrix}\begin{pmatrix} z(t) \\ p(t) \end{pmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u(t), \qquad \text{(DAE)}$$

where $M > 0$ and G has full row rank.

Optimal control problem

Minimize

$$\mathcal{J}(z,p,u) := \int_0^\infty \begin{pmatrix} z(\tau) \\ p(\tau) \\ u(\tau) \end{pmatrix}^{\mathsf{T}} \begin{bmatrix} Q & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & R \end{bmatrix} \begin{pmatrix} z(\tau) \\ p(\tau) \\ u(\tau) \end{pmatrix} \mathrm{d}\tau$$

with $Q \geq 0$ and $R > 0$ subject to (DAE) and

$$z(0) = z_0, \quad \lim_{t\to\infty} z(t) = 0.$$

Lur'e Equation and System Space

Lur'e equation

$$\begin{bmatrix} MX_{11}A + A^{\mathsf{T}}X_{11}M + MX_{12}G + G^{\mathsf{T}}X_{12}^{\mathsf{T}}M + Q & MX_{11}G^{\mathsf{T}} & MX_{11}B_1 \\ GX_{11}M & 0 & 0 \\ B_1^{\mathsf{T}}X_{11}M & 0 & R \end{bmatrix}$$
$$=_{\mathcal{V}_{\mathrm{sys}}} \begin{bmatrix} K_1^{\mathsf{T}} \\ K_2^{\mathsf{T}} \\ L^{\mathsf{T}} \end{bmatrix} \begin{bmatrix} K_1 & K_2 & L \end{bmatrix}, \quad X = X^{\mathsf{T}} = \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^{\mathsf{T}} & X_{22} \end{bmatrix}$$

Lur'e Equation and System Space

Lur'e equation

$$\begin{bmatrix} MX_{11}A + A^{\mathsf{T}}X_{11}M + MX_{12}G + G^{\mathsf{T}}X_{12}^{\mathsf{T}}M + Q & MX_{11}G^{\mathsf{T}} & MX_{11}B_1 \\ GX_{11}M & 0 & 0 \\ B_1^{\mathsf{T}}X_{11}M & 0 & R \end{bmatrix} =_{\mathcal{V}_{\mathrm{sys}}} \begin{bmatrix} K_1^{\mathsf{T}} \\ K_2^{\mathsf{T}} \\ L^{\mathsf{T}} \end{bmatrix} \begin{bmatrix} K_1 & K_2 & L \end{bmatrix}, \quad X = X^{\mathsf{T}} = \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^{\mathsf{T}} & X_{22} \end{bmatrix}$$

System space

Define projector $\Pi = I_{n_z} - G^{\mathsf{T}}(GM^{-1}G^{\mathsf{T}})^{-1}GM^{-1}$. We have

- $\frac{\mathrm{d}}{\mathrm{d}t}Mz(t) = \Pi Az(t) + \Pi B_1 u(t)$,
- $p(t) = -(GM^{-1}G^{\mathsf{T}})^{-1}GM^{-1}Az(t) - (GM^{-1}G^{\mathsf{T}})^{-1}GM^{-1}B_1u(t)$,

Lur'e Equation and System Space

Lur'e equation

$$\begin{bmatrix} MX_{11}A + A^\mathsf{T} X_{11} M + MX_{12}G + G^\mathsf{T} X_{12}^\mathsf{T} M + Q & MX_{11}G^\mathsf{T} & MX_{11}B_1 \\ GX_{11}M & 0 & 0 \\ B_1^\mathsf{T} X_{11} M & 0 & R \end{bmatrix}$$

$$=_{\mathcal{V}_{\mathrm{sys}}} \begin{bmatrix} K_1^\mathsf{T} \\ K_2^\mathsf{T} \\ L^\mathsf{T} \end{bmatrix} \begin{bmatrix} K_1 & K_2 & L \end{bmatrix}, \quad X = X^\mathsf{T} = \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^\mathsf{T} & X_{22} \end{bmatrix}$$

System space

Define projector $\Pi = I_{n_z} - G^\mathsf{T}(GM^{-1}G^\mathsf{T})^{-1}GM^{-1}$. We have

- $\frac{\mathrm{d}}{\mathrm{d}t} Mz(t) = \Pi A z(t) + \Pi B_1 u(t)$,
- $p(t) = -(GM^{-1}G^\mathsf{T})^{-1}GM^{-1}Az(t) - (GM^{-1}G^\mathsf{T})^{-1}GM^{-1}B_1 u(t)$,

$$\Longrightarrow \mathcal{V}_{\mathrm{sys}} = \operatorname{im} \begin{bmatrix} I_{n_z} & 0 \\ -(GM^{-1}G^\mathsf{T})^{-1}GM^{-1}A & -(GM^{-1}G^\mathsf{T})^{-1}GM^{-1}B_1 \\ 0 & I_m \end{bmatrix}$$

Reduced Lur'e Equation

Reduced Lur'e Equation

Projection on $\mathcal{V}_{\mathrm{sys}}$ leads to

$$\begin{bmatrix} MX_{11}\Pi A + A^{\mathsf{T}}\Pi^{\mathsf{T}}X_{11}M + MX_{12}G + G^{\mathsf{T}}X_{12}^{\mathsf{T}}M + Q & MX_{11}\Pi B_1 \\ B_1^{\mathsf{T}}\Pi^{\mathsf{T}}X_{11}M & R \end{bmatrix} = \begin{bmatrix} K^{\mathsf{T}} \\ L^{\mathsf{T}} \end{bmatrix} \begin{bmatrix} K & L \end{bmatrix}, \quad GX_{11}M = 0.$$

Projected Riccati Equation

Use Schur complement:

$$MX_{11}\Pi A + A^{\mathsf{T}}\Pi^{\mathsf{T}}X_{11}M + MX_{12}G + G^{\mathsf{T}}X_{12}^{\mathsf{T}}M + Q \\ - MX_{11}\Pi B_1 R^{-1}B_1^{\mathsf{T}}\Pi^{\mathsf{T}}X_{11}M = 0, \quad X_{11} = X_{11}^{\mathsf{T}}.$$

Projected Riccati Equation

Use Schur complement:

$$MX_{11}\Pi A + A^{\mathsf T}\Pi^{\mathsf T}X_{11}M + MX_{12}G + G^{\mathsf T}X_{12}^{\mathsf T}M + Q \\ - MX_{11}\Pi B_1R^{-1}B_1^{\mathsf T}\Pi^{\mathsf T}X_{11}M = 0, \quad X_{11} = X_{11}^{\mathsf T}.$$

Left-multiply by Π and right-multiply by $\Pi^{\mathsf T}$ and use $\Pi M = M\Pi^T$ and $G\Pi^{\mathsf T} = 0$:

$$M\Pi^{\mathsf T}X_{11}\Pi A\Pi^{\mathsf T} + \Pi A^{\mathsf T}\Pi^{\mathsf T}X_{11}\Pi M + \Pi Q\Pi^{\mathsf T} \\ - M\Pi^{\mathsf T}X_{11}\Pi B_1R^{-1}B_1^{\mathsf T}\Pi^{\mathsf T}X_{11}\Pi^{\mathsf T}M = 0, \quad X_{11} = X_{11}^{\mathsf T}.$$

Projected Riccati Equation

Use Schur complement:

$$M X_{11} \Pi A + A^{\mathsf{T}} \Pi^{\mathsf{T}} X_{11} M + M X_{12} G + G^{\mathsf{T}} X_{12}^{\mathsf{T}} M + Q - M X_{11} \Pi B_1 R^{-1} B_1^{\mathsf{T}} \Pi^{\mathsf{T}} X_{11} M = 0, \quad X_{11} = X_{11}^{\mathsf{T}}.$$

Left-multiply by Π and right-multiply by Π^{T} and use $\Pi M = M \Pi^{T}$ and $G \Pi^{\mathsf{T}} = 0$:

$$M \Pi^{\mathsf{T}} X_{11} \Pi A \Pi^{\mathsf{T}} + \Pi A^{\mathsf{T}} \Pi^{\mathsf{T}} X_{11} \Pi M + \Pi Q \Pi^{\mathsf{T}} - M \Pi^{\mathsf{T}} X_{11} \Pi B_1 R^{-1} B_1^{\mathsf{T}} \Pi^{\mathsf{T}} X_{11} \Pi^{\mathsf{T}} M = 0, \quad X_{11} = X_{11}^{\mathsf{T}}.$$

Then $\tilde{X} = \Pi^{\mathsf{T}} X_{11} \Pi$ is a solution of the projected algebraic Riccati equation

[Benner, Stykel '14], [Heiland '15]

$$M \tilde{X} \Pi A \Pi^{\mathsf{T}} + \Pi A^{\mathsf{T}} \Pi^{\mathsf{T}} \tilde{X} M + \Pi Q \Pi^{\mathsf{T}} - M \tilde{X} \Pi B_1 R^{-1} B_1^{\mathsf{T}} \Pi^{\mathsf{T}} \tilde{X} M = 0, \quad \tilde{X} = \tilde{X}^{\mathsf{T}} = \Pi^{\mathsf{T}} \tilde{X} \Pi.$$

Projected Riccati Equation

Use Schur complement:

$$M X_{11} \Pi A + A^{\mathsf{T}} \Pi^{\mathsf{T}} X_{11} M + M X_{12} G + G^{\mathsf{T}} X_{12}^{\mathsf{T}} M + Q \\ - M X_{11} \Pi B_1 R^{-1} B_1^{\mathsf{T}} \Pi^{\mathsf{T}} X_{11} M = 0, \quad X_{11} = X_{11}^{\mathsf{T}}.$$

Left-multiply by Π and right-multiply by Π^{T} and use $\Pi M = M \Pi^{T}$ and $G \Pi^{\mathsf{T}} = 0$:

$$M \Pi^{\mathsf{T}} X_{11} \Pi A \Pi^{\mathsf{T}} + \Pi A^{\mathsf{T}} \Pi^{\mathsf{T}} X_{11} \Pi M + \Pi Q \Pi^{\mathsf{T}} \\ - M \Pi^{\mathsf{T}} X_{11} \Pi B_1 R^{-1} B_1^{\mathsf{T}} \Pi^{\mathsf{T}} X_{11} \Pi^{\mathsf{T}} M = 0, \quad X_{11} = X_{11}^{\mathsf{T}}.$$

Then $\tilde{X} = \Pi^{\mathsf{T}} X_{11} \Pi$ is a solution of the projected algebraic Riccati equation [Benner, Stykel '14], [Heiland '15]

$$M \tilde{X} \Pi A \Pi^{\mathsf{T}} + \Pi A^{\mathsf{T}} \Pi^{\mathsf{T}} \tilde{X} M + \Pi Q \Pi^{\mathsf{T}} \\ - M \tilde{X} \Pi B_1 R^{-1} B_1^{\mathsf{T}} \Pi^{\mathsf{T}} \tilde{X} M = 0, \quad \tilde{X} = \tilde{X}^{\mathsf{T}} = \Pi^{\mathsf{T}} \tilde{X} \Pi.$$

Also related to the constrained algebraic Riccati equation [Heiland '14]

$$A^{\mathsf{T}} X_{11} M + M^{\mathsf{T}} X_{11} A - M^{\mathsf{T}} X_{11} B_1 R^{-1} B_1^{\mathsf{T}} X_{11} M \\ + M X_{12} G + G^{\mathsf{T}} X_{12}^{\mathsf{T}} M + Q = 0, \quad X_{11} = X_{11}^{\mathsf{T}}, \quad G X_{11} M = 0.$$

Conclusions and Further Results

Conclusion

- feasibility conditions in terms of existence of storage functions,
- Lur'e equations and their solution properties,
- existence and uniqueness conditions for optimal controls,
- application to flow control problems.

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Conclusion

- feasibility conditions in terms of existence of storage functions,
- Lur'e equations and their solution properties,
- existence and uniqueness conditions for optimal controls,
- application to flow control problems.

Further results

- relation to even boundary value problems of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} 0 & E & 0 \\ -E^{\mathsf{T}} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{pmatrix} \mu(t) \\ x(t) \\ u(t) \end{pmatrix} = \begin{bmatrix} 0 & A & B \\ A^{\mathsf{T}} & Q & S \\ B^{\mathsf{T}} & S^{\mathsf{T}} & R \end{bmatrix}\begin{pmatrix} \mu(t) \\ x(t) \\ u(t) \end{pmatrix},$$

$$Ex(0) = Ex_0, \quad \lim_{t\to\infty} E^{\mathsf{T}}\mu(t) = 0,$$

- analysis of the associated even matrix pencil.

Thank you for your Attention!

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