A Greedy Subspace Method for Computing the $\mathcal{L}_\infty$-Norm

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We consider the computation of the $\mathcal{L}_\infty$-norm for a general class of $\mathcal{L}_\infty$-functions and focus on the case where the function is represented in terms of large-scale matrix-valued factors. We propose a subspace projection method to obtain reduced approximations of this function by interpolation techniques. The $\mathcal{L}_\infty$-norms are computed for the resulting reduced functions, then the subspaces are refined by means of the optimizer of the $\mathcal{L}_\infty$-norm of the reduced function. In this way we obtain much better performance compared to existing methods.

1 Problem Statement

In this paper we consider functions of the form

$$H : \Omega \to \mathbb{C}^{p\times m}, \quad H(s) := C(s)D(s)^{-1}B(s), \tag{1}$$

where $\Omega \subseteq \mathbb{C}$ is assumed to be an open domain enclosing the imaginary axis $\mathrm{i}\mathbb{R}$. Moreover, we assume that the matrix-valued functions $B : \Omega \to \mathbb{C}^{n\times m}$, $C : \Omega \to \mathbb{C}^{p\times n}$, and $D : \Omega \to \mathbb{C}^{n\times n}$ are defined by

$$\begin{aligned} B(s) &:= f_1(s)B_1 + \cdots + f_{\kappa_B}(s)B_{\kappa_B}, \\ C(s) &:= g_1(s)C_1 + \cdots + g_{\kappa_C}(s)C_{\kappa_C}, \\ D(s) &:= h_1(s)D_1 + \cdots + h_{\kappa_D}(s)D_{\kappa_D}, \end{aligned} \tag{2}$$

for given matrices $B_1, \ldots, B_{\kappa_B} \in \mathbb{C}^{n\times m}$, $C_1, \ldots, C_{\kappa_C} \in \mathbb{C}^{p\times n}$, $D_1, \ldots, D_{\kappa_D} \in \mathbb{C}^{n\times n}$ and given functions $f_1, \ldots, f_{\kappa_B}$, $g_1, \ldots, g_{\kappa_C}, h_1, \ldots, h_{\kappa_D} : \Omega \to \mathbb{C}$ that are meromorphic in Ω.

In this paper we suppose the function H (more precisely, its restriction to the imaginary axis) to be an element of the normed space

$$\mathcal{L}_\infty^{p\times m} := \left\{ H|_{\mathrm{i}\mathbb{R}} \,\middle|\, H : \Omega \to \mathbb{C}^{p\times m} \text{ is analytic for an open domain } \Omega \subseteq \mathbb{C} \text{ with } \mathrm{i}\mathbb{R} \subset \Omega \text{ and } \sup_{\omega\in\mathbb{R}} \|H(\mathrm{i}\omega)\|_2 < \infty \right\}.$$

For ease of notation, we write $H \in \mathcal{L}_\infty^{p\times m}$ instead of $H|_{\mathrm{i}\mathbb{R}} \in \mathcal{L}_\infty^{p\times m}$. The $\mathcal{L}_\infty$-norm of a function $H \in \mathcal{L}_\infty^{p\times m}$ is defined by

$$\|H\|_{\mathcal{L}_\infty} := \sup_{\omega\in\mathbb{R}} \|H(\mathrm{i}\omega)\|_2 = \sup_{\omega\in\mathbb{R}} \sigma_{\max}(H(\mathrm{i}\omega)),$$

where $\sigma_{\max}(\cdot)$ denotes the maximum singular value of its matrix argument. In this paper we address the computation of the $\mathcal{L}_\infty$-norm with an emphasis on the case where n is large with $n \gg m$, p and thus the evaluation of $H(s)$ is expensive.

2 The Method

Our approach makes use of ideas from the field of model order reduction, i. e., instead of computing the $\mathcal{L}_\infty$-norm of H directly, we construct reduced functions of the form

$$\widetilde{H} : \Omega \to \mathbb{C}^{p\times m}, \quad \widetilde{H}(s) := \widetilde{C}(s)\widetilde{D}(s)^{-1}\widetilde{B}(s), \tag{3}$$

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where

$$\widetilde{B}(s) := f_1(s)\widetilde{B}_1 + \cdots + f_{\kappa_B}(s)\widetilde{B}_{\kappa_B}, \quad \widetilde{B}_j := W^*B_j, \quad j = 1, \ldots, \kappa_B,$$
$$\widetilde{C}(s) := g_1(s)\widetilde{C}_1 + \cdots + g_{\kappa_C}(s)\widetilde{C}_{\kappa_C}, \quad \widetilde{C}_j := C_jV, \quad j = 1, \ldots, \kappa_C,$$
$$\widetilde{D}(s) := h_1(s)\widetilde{D}_1 + \cdots + h_{\kappa_D}(s)\widetilde{D}_{\kappa_D}, \quad \widetilde{D}_j := W^*D_jV, \quad j = 1, \ldots, \kappa_D,$$

and the matrices $V, W \in \mathbb{C}^{n\times k}$ with $k \ll n$ are appropriately chosen. Since $\widetilde{D}$ is of much smaller dimension than D, the $\mathcal{L}_\infty$-norm of $\widetilde{H}$ can be computed much more efficiently.

In our method we compute a sequence of such reduced functions $\widetilde{H}_1, \widetilde{H}_2, \ldots$, whose $\mathcal{L}_\infty$-norms converge to $\|H\|_{\mathcal{L}_\infty}$. For the construction of the reduced functions we use the interpolation technique of [1]. Assume first that $m = p$ and that r interpolation points $\mathrm{i}\omega_1, \ldots, \mathrm{i}\omega_r$ are given. Then $\widetilde{H}_r$ is defined by $\widetilde{H}$ as in (3) with V, W set equal to the particular choices

$$V_r = \begin{bmatrix} D(\mathrm{i}\omega_1)^{-1}B(\mathrm{i}\omega_1) & \ldots & D(\mathrm{i}\omega_r)^{-1}B(\mathrm{i}\omega_r) \end{bmatrix}, \quad W_r = \begin{bmatrix} D(\mathrm{i}\omega_1)^{-*}C(\mathrm{i}\omega_1)^* & \ldots & D(\mathrm{i}\omega_r)^{-*}C(\mathrm{i}\omega_r)^* \end{bmatrix}. \quad (5)$$

These choices for V and W give rise to the Hermite interpolation properties [1, 2]

$$H(\mathrm{i}\omega_k) = \widetilde{H}_r(\mathrm{i}\omega_k), \quad H'(\mathrm{i}\omega_k) = \widetilde{H}_r'(\mathrm{i}\omega_k), \quad k = 1, \ldots, r. \quad (6)$$

Note that the number of columns of V_r and W_r have to be equal so that $\widetilde{D}(s)^{-1}$ exists. This is violated by the choices in (5) unless $m = p$. The following choices overcome this difficulty when $m \neq p$ while preserving the Hermite interpolation properties (6):

$$V_r = \begin{bmatrix} D(\mathrm{i}\omega_1)^{-1}B(\mathrm{i}\omega_1)H(\mathrm{i}\omega_1)^* & \ldots & D(\mathrm{i}\omega_r)^{-1}B(\mathrm{i}\omega_r)H(\mathrm{i}\omega_r)^* \end{bmatrix}, \quad W_r \text{ as in (5)}, \quad \text{if } m > p,$$
$$W_r = \begin{bmatrix} D(\mathrm{i}\omega_1)^{-*}C(\mathrm{i}\omega_1)^*H(\mathrm{i}\omega_1) & \ldots & D(\mathrm{i}\omega_r)^{-*}C(\mathrm{i}\omega_r)^*H(\mathrm{i}\omega_r) \end{bmatrix}, \quad V_r \text{ as in (5)}, \quad \text{if } m < p.$$

Once we have constructed $\widetilde{H}_r$, we compute its $\mathcal{L}_\infty$-norm by established methods for the small-scale case such as [3–5]. Using the point $\mathrm{i}\omega_{r+1}$ where the $\mathcal{L}_\infty$-norm of $\widetilde{H}_r$ is attained, we update the projection spaces

$$V_{r+1} := \begin{bmatrix} V_r & D(\mathrm{i}\omega_{r+1})^{-1}B(\mathrm{i}\omega_{r+1}) \end{bmatrix} \quad (\text{or } V_{r+1} := \begin{bmatrix} V_r & D(\mathrm{i}\omega_{r+1})^{-1}B(\mathrm{i}\omega_{r+1})H(\mathrm{i}\omega_{r+1})^* \end{bmatrix}),$$
$$W_{r+1} := \begin{bmatrix} W_r & D(\mathrm{i}\omega_{r+1})^{-*}C(\mathrm{i}\omega_{r+1})^* \end{bmatrix} \quad (\text{or } W_{r+1} := \begin{bmatrix} W_r & D(\mathrm{i}\omega_{r+1})^{-*}C(\mathrm{i}\omega_{r+1})^*H(\mathrm{i}\omega_{r+1}) \end{bmatrix}).$$

Under the assumption of convergence (that we usually observe), we can prove a superlinear rate of convergence to a local maximizer of $\sigma_{\max}(H(\mathrm{i}\cdot))$. The proof makes use of the Hermite interpolation property (6) and is elaborated in [2].

3 A Numerical Example

A MATLAB implementation of our method has been made available on the internet[1]. In this section we briefly demonstrate this implementation on a numerical example. As a benchmark we take the `S80PI_n` example[2], which is a descriptor system with $n = 4182$ states and $m = p = 1$ input and output, respectively. We obtain a transfer function $H(s) = C(sE - A)^{-1}B$ which is of the form (1). We start initially with 10 equally spaced interpolation points in the interval $[0.1, 10000]$. Only 5 additional iterations are needed so that the points where the $\mathcal{L}_\infty$-norms of two consecutive reduced functions are attained are located at a relative distance of at most 10^{-6}. Our method correctly computes $\|H\|_{\mathcal{L}_\infty} = 3.37016$ which is attained at $\omega = 6961.49$. The runtime is only 0.20 s compared to 3.96 s required by the pseudospectral method from [6].

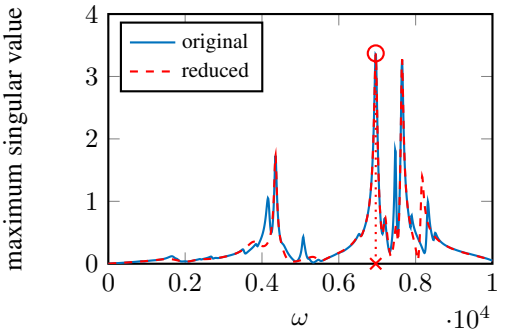

Fig. 1 Original and final reduced transfer functions obtained for the `S80PI_n` example. The red cross and circle indicate the location of the maximizer and the $\mathcal{L}_\infty$-norm of the reduced function, respectively.

[1] see `http://www.math.tu-berlin.de/index.php?id=186267&L=1`
[2] available from `http://sites.google.com/site/rommes/software`

References

[1] C. Beattie and S. Gugercin, Systems Control Lett. **58**(3), 225–232 (2009).
[2] N. Aliyev, P. Benner, E. Mengi, P. Schwerdtner, and M. Voigt, Large-scale computation of $\mathcal{L}_\infty$-norms by a greedy subspace method, 2017, arXiv:1705.10086.
[3] S. Boyd and V. Balakrishnan, Systems Control Lett. **15**(1), 1–7 (1990).
[4] P. Benner, V. Sima, and M. Voigt, IEEE Trans. Automat. Control **57**(1), 233–238 (2012).
[5] E. Mengi, E. A. Yildirim, and M. Kiliç, SIAM J. Matrix Anal. Appl. **35**(2), 699–724 (2014).
[6] P. Benner and M. Voigt, Math. Control Signals Systems **26**(2), 303–338 (2014).