

# Spectral Conditions for Symmetric Positive Real and Negative Imaginary Systems

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**Abstract**—Non-Hamiltonian spectral conditions for the class of symmetric multivariable strictly positive real and strictly negative imaginary systems are derived. They represent generalizations of known ones for strict positive realness to the cases with singular feedthrough matrix, and are novel in the context of strict negative imaginarity. Moreover, we propose a concept of strong negative imaginarity and establish its links to strict positive realness of symmetric systems. The proposed spectral conditions are useful in the corresponding assessment and enforcement procedures, as well as in quadratic stability analysis of uncertain and switched systems.

## I. INTRODUCTION

In this note we consider non-Hamiltonian spectral conditions for the positive realness [1] and negative imaginarity (e.g., [2]) assessment for the class of linear systems with symmetric transfer matrix functions [3]. This and related previous works (e.g. [4], [5], [6]) are motivated by the strong need in improving the computational efficiency which in the Hamiltonian approach may deteriorate in the case of large models. For instance, in a passivity enforcement procedure, computing the *critical purely imaginary eigenvalues* of the Hamiltonian matrix or pencil can be time dominant [5]. For small-scale systems, these can still be efficiently and robustly computed by structure-preserving algorithms [7], [8]. However, the application of these methods becomes prohibitively expensive in case of large-scale models. There also exist structure-exploiting methods for the computation of the eigenvalues of related large-scale even pencils [9], but they suffer from the disadvantage that they only return eigenvalues close to a prespecified shift. In this way it is not clear how to ensure that *all* imaginary eigenvalues can be found. However, in our approach it is only necessary to check whether a product of two matrices, typically of the form  $A(A - BD^{-1}C)$ , has no *negative real eigenvalues* which is more reasonable for sparse systems.

Our contribution related to the positive realness tests has primarily a generalization character of some spectral results that have appeared in the past several years in the literature. The closest works are [6] and [5]. In [6], the spectral conditions for *strict positive realness* have been devised for SISO systems, while in [5] these conditions are generalized to the class of symmetric multivariable systems with a nonsingular feedthrough matrix. In this respect, the novelty in the first part of our work refers to generalizing such non-Hamiltonian spectral conditions to the case where the feedthrough matrix

might be singular. To our best knowledge, our spectral conditions for *strict negative imaginarity* assessment are new. Finally, we provide interpretations of spectral conditions in the context of reciprocal systems.

## II. PRELIMINARIES

### A. Notation and Facts

We use the standard notation. The fields of real and complex numbers are denoted by  $\mathbb{R}$  and  $\mathbb{C}$ , respectively. The set of all  $m \times n$  matrices over  $\mathbb{R}$  (or  $\mathbb{C}$ ) are denoted by  $\mathbb{R}^{m \times n}$  (or  $\mathbb{C}^{m \times n}$ ), and the conjugate transpose of a matrix  $M$  by  $M^H = \overline{M}^T$ . The spectrum of  $M \in \mathbb{C}^{n \times n}$  is written as  $\Lambda(M)$ . A matrix  $M$  is symmetric if  $M = M^T$  and Hermitian if  $M = M^H$ . A Hermitian matrix is positive semidefinite (positive definite) if all its eigenvalues are nonnegative (positive), which we denote by  $M \geq 0$  ( $M > 0$ ). For convenience, we abbreviate the Hermitian and skew-Hermitian part of a matrix  $M$  by

$$\text{He}(M) := \frac{1}{2}(M + M^H), \quad \text{SHe}(M) := \frac{1}{2}(M - M^H).$$

Let  $M$  be partitioned as

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}.$$

Assume that  $A$  is nonsingular. Then the Schur complement of  $M$  with respect to  $A$  is defined by  $M/A := D - CA^{-1}B$ . Furthermore, according to the Schur's formula it holds

$$\det(M) = \det(M/A) \det(A). \quad (1)$$

If  $M$  and  $A$  are both nonsingular, so is  $M/A$ , and

$$M^{-1} = \begin{bmatrix} A^{-1} + A^{-1}B(M/A)^{-1}CA^{-1} & -A^{-1}B(M/A)^{-1} \\ -(M/A)^{-1}CA^{-1} & (M/A)^{-1} \end{bmatrix} \quad (2)$$

The inertia of a Hermitian matrix is defined by the triplet

$$\text{In}(M) = \{\pi(M), \nu(M), \delta(M)\}$$

where  $\pi(M), \nu(M), \delta(M)$  are the numbers of the positive, negative, and zero eigenvalues of  $M$  counted with multiplicities, respectively [10]. Obviously, the inertia of a nonsingular Hermitian matrix and its inverse are the same. Similar Hermitian matrices have the same inertia, too. Moreover, according to the classical Sylvester's Law of Inertia, for given  $n \times n$  Hermitian matrices  $M$  and  $N$ , there exists a nonsingular  $\Pi$  such that  $M = \Pi^*N\Pi$  if and only if  $\text{In}(M) = \text{In}(N)$ . If  $M$  is Hermitian and  $A$  is invertible, according to the Haynsworth Inertia Additivity Formula (abbr. HIAF, [10]), we have

$$\text{In}(M) = \text{In}(A) + \text{In}(M/A). \quad (3)$$

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(Of course a similar equation holds also w.r.t. the block  $D$  of  $M$ .) In this work, we consider often times the inertia of the matrix interval  $M + \alpha N$ , where  $M, N$  are Hermitian,  $\alpha \in T \subseteq \mathbb{R}$  and  $T$  is connected. If further  $M$  is non-singular, then for a sufficiently small  $\alpha$  it holds

$$\text{In}(M + \alpha N) = \text{In}(M), \quad (4)$$

and the following three statements are equivalent [11]:

(A1)  $-1/\alpha$  is an eigenvalue of  $M^{-1}N$ ;

(A2)  $I + \alpha M^{-1}N$  is singular;

(A3)  $M + \alpha N$  is singular.

In these observations, we necessarily have  $\alpha \neq 0$ . Note that if any of the latter conditions are satisfied for  $\alpha \rightarrow \infty$ , this indicates a zero eigenvalue of  $N$ . As a consequence of the observations (A1)–(A3), the following are equivalent:

(B1)  $\text{In}(M + \alpha N) = \text{const}$ ;

(B2)  $-1/\alpha$  is not an eigenvalue of the matrix  $M^{-1}N$ ;

where  $\alpha \in T$ . Hence, the inertia of such a matrix interval is a piecewise constant function of  $\alpha$ . Using (4), we then have:

$$\lim_{\alpha \rightarrow \infty} \text{In}(M + \alpha N) = \text{In}(N). \quad (5)$$

Finally, note that if  $N$  has a zero eigenvalue of algebraic multiplicity  $r$ , then the product  $MN$  may have a zero eigenvalue of an algebraic multiplicity larger than  $r$ , even if  $M$  is nonsingular.

### B. Symmetric systems

In this paper,  $\mathcal{R}^{m \times m}$  and  $\mathcal{RH}_{\infty}^{m \times m}$  denote the spaces of all proper real-rational and proper real-rational stable  $m \times m$  transfer function matrices, respectively. The realization of a transfer matrix  $G(\cdot) \in \mathcal{R}^{m \times m}$  is symbolically denoted by

$$\left[ \begin{array}{c|c} A & B \\ \hline C & D \end{array} \right] \sim G(s) := C(sI_n - A)^{-1}B + D. \quad (6)$$

where  $A \in \mathbb{R}^{n \times n}$ ,  $B \in \mathbb{R}^{n \times m}$ ,  $C \in \mathbb{R}^{m \times n}$ ,  $D \in \mathbb{R}^{m \times m}$ . If the underlying realization is minimal we will use  $\overset{\text{min}}{\sim}$  instead of  $\sim$  in (6).

A function  $G(\cdot) \in \mathcal{R}^{m \times m}$  (6) is said to be internally symmetric if  $A = A^T$ ,  $C = B^T$  and  $D = D^T$ , and externally symmetric (or, simply symmetric) if  $G^T(s) = G(s)$  for all  $s \in \mathbb{C} \setminus \Lambda(A)$ . Obviously, internal symmetry implies symmetry, but the converse is not necessarily true. The two concepts are related by the following standard fact (see [3], [12], [4]). Let  $G(\cdot) \in \mathcal{R}^{m \times m}$  be symmetric with

$$G(\cdot) \overset{\text{min}}{\sim} \left[ \begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]. \quad (7)$$

Then, there always exists a unique equivalent minimal symmetric descriptor realization

$$G(s) = \hat{B}^T (s\hat{E} - \hat{A})^{-1} \hat{B} + \hat{D} \quad (8)$$

with nonsingular  $\hat{E} = \hat{E}^T \in \mathbb{R}^{n \times n}$ ,  $\hat{A} = \hat{A}^T$ ,  $\hat{D} = \hat{D}^T$ , and

$$A = \hat{A}\hat{E}^{-1}, \quad B = \hat{B}, \quad C = \hat{B}^T\hat{E}^{-1}, \quad D = \hat{D}. \quad (9)$$

Herein  $\hat{A}$  is nonsingular, but not necessarily Hurwitz.

## III. SPECTRAL CONDITIONS FOR STRICT POSITIVE REALNESS

**Definition 1** ([13]). A matrix function  $G(\cdot) \in \mathcal{R}^{m \times m}$  (6) is positive real (abbr. PR) if

(D1)  $G(\cdot)$  is analytic in the open right half-plane  $\mathbb{C}^+ := \{s \in \mathbb{C} : \text{Re}(s) > 0\}$ ; and

(D2)  $\text{He}(G(s)) \geq 0$  for all  $s \in \mathbb{C}^+$ .

A matrix function  $G(\cdot) \in \mathcal{R}^{m \times m}$  is said to be strictly positive real (abbr. SPR) if there exists an  $\varepsilon > 0$  such that  $G(s - \varepsilon)$  is positive real. Finally,  $G(\cdot)$  is strongly positive real (abbr. SSPR) if it is strictly positive real and  $D + D^T > 0$ .

The following theorem provides an SPRness test.

**Theorem 1** (e.g., [14], [13]). A matrix function  $G(\cdot) \in \mathcal{RH}_{\infty}^{m \times m}$  from (6) is SPR if and only if the following conditions hold:

(E1)  $\text{He}(G(j\omega)) > 0$  for all  $\omega \in \mathbb{R}$ ; and

(E2)  $\lim_{\omega \rightarrow \infty} \omega^{2(m-r)} \det(\text{He}(G(j\omega))) > 0$ , where  $r = \text{rank}(D + D^T)$ .

A standard algebraic SPR test is given by the SPR lemma:

**Theorem 2** (e.g., [13]). Consider a minimal realization of a matrix function  $G(\cdot) \in \mathcal{R}^{m \times m}$  in (6). Then,  $G(\cdot)$  is SPR if and only if there exist matrices  $P \in \mathbb{R}^{n \times n}$ ,  $L \in \mathbb{R}^{p \times n}$ , and  $W \in \mathbb{R}^{p \times m}$ , such that  $P = P^T > 0$  and

$$\begin{aligned} A^T P + PA &= -\varepsilon P - L^T L \\ PB - C^T &= -L^T W \\ D + D^T &= W^T W. \end{aligned} \quad (10)$$

The PR lemma, is obtained by setting  $\varepsilon = 0$  and allowing  $P = P^T \geq 0$ . For strictly proper matrix functions (i.e.,  $D = 0$ ), (10) reduces to the non-strict LMIs:

$$A^T P + PA < 0, \quad PB = C^T. \quad (11)$$

A Hamiltonian spectral test for (E1) is the following, based on, e.g., [15].

**Theorem 3.** Let  $A$  be Hurwitz. Then (E1) is equivalent to the existence of an  $\omega_0 \in \mathbb{R}$  such that  $\text{He}(G(j\omega_0)) > 0$  and the non-existence of finite, purely imaginary eigenvalues of the even pencil

$$\left[ \begin{array}{ccc} 0 & -\lambda I_n + A & B \\ \lambda I_n + A^T & 0 & C^T \\ B^T & C & D + D^T \end{array} \right]. \quad (12)$$

The main result in this section follows.

**Theorem 4.** Let  $G(\cdot) \in \mathcal{RH}_{\infty}^{m \times m}$  be a symmetric matrix function with a minimal state-space realization (7) and let  $D \geq 0$  with  $r = \text{rank}(D)$ . Then,  $G(\cdot)$  is SPR if and only if the following conditions hold true:

(F1)  $D - CA^{-1}B > 0$ ;

(F2)  $A^{-1}(A^{-1} + A^{-1}B(D - CA^{-1}B)^{-1}CA^{-1})$  has no negative real eigenvalues, and

(F3) a zero eigenvalue of algebraic multiplicity  $(m - r)$ .

*Proof.* Due to the symmetry of  $G(\cdot)$ , we have  $G^H(j\omega) = G(-j\omega)$  for  $\omega \in \mathbb{R}$ , and

$$\text{He}(G(j\omega)) = \frac{1}{2} (G(j\omega) + G^H(j\omega)) = \text{Re}(G(-j\omega)).$$

Using the equivalent descriptor realization (8), we have

$$\text{He}(G(j\omega)) = \hat{D} - \text{Re} \left( \hat{B}^T (j\omega \hat{E} + \hat{A})^{-1} \hat{B} \right),$$

which we further write in the form  $\text{He}G(j\omega) = \hat{D} - \hat{B}^T (X + jY)$ , where  $\text{Re} \left( (j\omega \hat{E} + \hat{A})(X + jY) \right) = \hat{B}$  and  $\text{Im} \left( (j\omega \hat{E} + \hat{A})(X + jY) \right) = 0$ . In the matrix notation it follows that

$$\begin{bmatrix} \hat{A} & \omega \hat{E} \\ \omega \hat{E} & -\hat{A} \end{bmatrix} \begin{bmatrix} X \\ -Y \end{bmatrix} = \begin{bmatrix} \hat{B} \\ 0 \end{bmatrix},$$

yielding a matrix description of the Hermitian matrix

$$\begin{aligned} \text{He}(G(j\omega)) &= \hat{D} - \begin{bmatrix} \hat{B}^T & 0 \end{bmatrix} \begin{bmatrix} X \\ -Y \end{bmatrix} \\ &= \hat{D} - \begin{bmatrix} \hat{B}^T & 0 \end{bmatrix} \begin{bmatrix} \hat{A} & \omega \hat{E} \\ \omega \hat{E} & -\hat{A} \end{bmatrix}^{-1} \begin{bmatrix} \hat{B} \\ 0 \end{bmatrix}, \end{aligned}$$

which is the Schur complement of

$$U = \left[ \begin{array}{cc|c} \hat{A} & \omega \hat{E} & \hat{B} \\ \omega \hat{E} & -\hat{A} & 0 \\ \hline \hat{B}^T & 0 & \hat{D} \end{array} \right] \quad (13)$$

with respect to the leading  $2 \times 2$  block. Next, we define  $V$  by introducing a permutation matrix  $\Pi$  such that  $U = \Pi^* V \Pi$ :

$$V = \left[ \begin{array}{cc|c} \hat{A} & \hat{B} & \omega \hat{E} \\ \hat{B}^T & \hat{D} & 0 \\ \hline \omega \hat{E} & 0 & -\hat{A} \end{array} \right]. \quad (14)$$

The application of the HIAF from (3) to  $U$  and  $V$  reveals

$$\text{In}(U) = \text{In}(\text{He}(G(j\omega))) + \text{In} \left( \begin{bmatrix} \hat{A} & \omega \hat{E} \\ \omega \hat{E} & -\hat{A} \end{bmatrix} \right), \text{ and}$$

$$\begin{aligned} \text{In}(V) &= \text{In} \left( \begin{bmatrix} \hat{A} & \hat{B} \\ \hat{B}^T & \hat{D} \end{bmatrix} \right) + \\ &\text{In} \left( -\hat{A} - \begin{bmatrix} \omega \hat{E} & 0 \end{bmatrix} \begin{bmatrix} \hat{A} & \hat{B} \\ \hat{B}^T & \hat{D} \end{bmatrix}^{-1} \begin{bmatrix} \omega \hat{E} \\ 0 \end{bmatrix} \right). \end{aligned}$$

Consider now the expression for  $\text{In}(U)$ . As a consequence of the Hurwitz assumption for  $A$ , we have  $\det((j\omega)^2 I_n - A^2) = (-1)^n \det(\omega^2 I_n + A^2) \neq 0$  for all  $\omega \in \mathbb{R}$ . Then, the observations (A1)–(A3), and (B1)–(B2) infer that  $\text{In}(\hat{A} + \omega^2 \hat{E} \hat{A}^{-1} \hat{E}) = \text{In} \hat{A}$  for all  $\omega \in \mathbb{R}$ , and HIAF immediately reveals that

$$\text{In} \left( \begin{bmatrix} \hat{A} & \omega \hat{E} \\ \omega \hat{E} & -\hat{A} \end{bmatrix} \right) = (n, n, 0), \quad \text{for all } \omega \in \mathbb{R}.$$

According to Sylvester's Law of Inertia, we have  $\text{In}(U) = \text{In}(V)$ , and specifically, at  $\omega = 0$ :

$$\begin{aligned} \text{In}(\text{He}(G(0))) + (n, n, 0) &= \text{In}(\hat{A}) + \text{In}(\hat{D} - \hat{B}^T \hat{A}^{-1} \hat{B}) \\ &\quad + \text{In}(-\hat{A}), \end{aligned}$$

where we used the explicit HIAF-expressions of  $\text{In}(U)$  and  $\text{In}(V)$ . With  $\text{In}(\hat{A}) + \text{In}(-\hat{A}) = (n, n, 0)$ , it follows hereof that  $\text{In}(\text{He}(G(0))) = \text{In}(\hat{D} - \hat{B}^T \hat{A}^{-1} \hat{B})$ , that is, in light of the condition (E1):

$$\text{In}(\hat{D} - \hat{B}^T \hat{A}^{-1} \hat{B}) = (m, 0, 0). \quad (15)$$

Hence, (E1) immediately implies the claim (F1) if we apply the matrix transformations (9) herein.

Moreover, utilizing (15), the equation  $\text{In}(U) = \text{In}(V)$  infers that for any  $\omega \in \mathbb{R}$  the following must hold true:

$$\begin{aligned} \text{In}(\text{He}(G(j\omega))) + (n - m, n, 0) &= \text{In}(\hat{A}) \\ &\quad + \text{In}(-\hat{A} - \omega^2 \hat{N}), \end{aligned} \quad (16)$$

where we applied the matrix inversion formula (2) and for notation brevity we introduce

$$\hat{N} := \hat{E} \left( \hat{A}^{-1} + \hat{A}^{-1} \hat{B} \left( \hat{D} - \hat{B}^T \hat{A}^{-1} \hat{B} \right)^{-1} \hat{B}^T \hat{A}^{-1} \right) \hat{E}.$$

Hence, the condition (E1):

$$\text{In}(\text{He}(G(j\omega))) = (m, 0, 0), \quad \omega \in \mathbb{R} \quad (17)$$

implies a constant inertia for the matrix family  $-\hat{A} - \omega^2 \hat{N}$ ,  $\omega \in \mathbb{R}$ . Referring to the observation (B2), the latter is equivalent to preventing the matrix

$$\hat{A}^{-1} \hat{N} = \hat{E}^{-1} A^{-1} N \hat{E},$$

and, thus also  $A^{-1} N$ , from possessing negative real eigenvalues, where

$$N := A^{-1} + A^{-1} B (D - C A^{-1} B)^{-1} C A^{-1}.$$

In other words, we proved that (E1) implies (F1) and (F2). On the other hand, if (F1) and (F2) hold true, then (16) directly yields (17) which implies (E1).

Next we consider the behavior of  $\text{He}(G(j\omega))$  for  $\omega \rightarrow \infty$ . Therefore, we apply Schur's formula (1) in the identity (13), yielding:

$$\det(\text{He}(G(j\omega))) = \frac{\det(U)}{\det \left( \begin{bmatrix} \hat{A} & \omega \hat{E} \\ \omega \hat{E} & -\hat{A} \end{bmatrix} \right)}.$$

A substitution of  $\det(U) = \det(V)$ , and application of the determinant formula (1) in (14) leads to

$$\det(\text{He}(G(j\omega))) = c \cdot \frac{\det(I_n + \omega^2 A^{-1} N)}{\det(\omega^2 I_n + A^2)},$$

where  $c$  is some *positive* scalar. As a consequence we obtain

$$\begin{aligned} \lim_{\omega \rightarrow \infty} \det(\text{He}(G(j\omega))) &= c \cdot \lim_{\omega \rightarrow \infty} \frac{\det \left( \frac{1}{\omega^2} I_n + A^{-1} N \right)}{\det \left( I_n + \frac{1}{\omega^2} A^2 \right)} \\ &= c \cdot \lim_{\omega \rightarrow \infty} \det \left( \frac{1}{\omega^2} I_n + A^{-1} N \right). \end{aligned} \quad (18)$$

Now let  $\omega \rightarrow \infty$  in (16) and recall that by the theorem assumption  $\lim_{\omega \rightarrow \infty} \text{rank}(\text{He}(G(j\omega))) = \text{rank} D = r$ , yielding

$$\lim_{\omega \rightarrow \infty} \text{In}(-\hat{A} - \omega^2 \hat{N}) = \text{In}(-\hat{N}) = (*, *, m - r), \quad (19)$$

where we used (5). This reveals a zero eigenvalue of an algebraic multiplicity of at least  $(m - r)$  for the matrices  $\hat{A}^{-1}\hat{N}$  and  $A^{-1}N$ . As a consequence, from (18), we conclude that condition (E2) implies that the matrix  $A^{-1}N$  has precisely  $(m - r)$  zero eigenvalues, which is the claim (F3) of our theorem. On the other hand, the three claims (F1)–(F3) in conjunction with equation (18) guarantee that condition (E2) holds true. This completes the proof.  $\square$

**Remark 1** (SISO Systems). *With  $m = 1$ , our theorem recovers the result of [6].*

**Corollary 1.** *Let  $G(\cdot) \in \mathcal{RH}_\infty^{m \times m}$  be a symmetric matrix function with a minimal realization (7) and let  $D > 0$ . Then  $G(\cdot)$  is SPR if and only if  $A(A - BD^{-1}C)$  does not have a nonpositive real eigenvalue.*

*Proof.* As  $D$  is now invertible,  $A^{-1} + A^{-1}B(D - CA^{-1}B)^{-1}CA^{-1} = (A - BD^{-1}C)^{-1}$  and the result follows immediately since the requirement (F2) is equivalent to that for their corresponding inverse  $(A - BD^{-1}C)A$  and consequently also for  $A(A - BD^{-1}C)$ .  $\square$

**Remark 2** (Reciprocal systems). *If  $G(\cdot) \in \mathcal{RH}_\infty^{m \times m}$ , then*

$$\tilde{G}(s) := G(1/s) \stackrel{\min}{\sim} \left[ \begin{array}{c|c} A^{-1} & -A^{-1}B \\ \hline CA^{-1} & D - CA^{-1}B \end{array} \right] \quad (20)$$

*represents its reciprocal system. Obviously,  $G(\cdot)$  and  $\tilde{G}(\cdot)$  share the same stability and symmetry properties. Moreover, it can be shown that, in general,  $G(\cdot)$  is PR if and only if  $\tilde{G}(\cdot)$  is PR. To this end, simply apply  $\tilde{W} = W - LA^{-1}B$  and  $\tilde{L} = LA^{-1}$  in the PR lemma for the reciprocal system (20). However, such a relationship does not hold for the SPRness. In fact, Theorem 4 provides a related condition which we now restate as a corollary.*

**Corollary 2.** *Let a symmetric  $G(\cdot) \in \mathcal{RH}_\infty^{m \times m}$  be given by (7) and consider its reciprocal system*

$$\tilde{G}(\cdot) \stackrel{\min}{\sim} \left[ \begin{array}{c|c} \tilde{A} & \tilde{B} \\ \hline \tilde{C} & \tilde{D} \end{array} \right].$$

*Then  $G(\cdot)$  is SPR if and only if*

- (G1)  $\tilde{G}(\cdot) \in \mathcal{RH}_\infty^{m \times m}$ ;
- (G2)  $\tilde{D} > 0$ ; and
- (G3)  $\tilde{A}(\tilde{A} - \tilde{B}\tilde{D}^{-1}\tilde{C})$  has no negative real eigenvalues and a zero eigenvalue of algebraic multiplicity  $(m - r)$ , where  $r = \text{rank}(D)$ .

**Remark 3** (SSPRness). *A consequence of this statement is the following:  $G(\cdot)$  and  $\tilde{G}(\cdot)$  are simultaneously SPR if and only if they are both SSPR, that is if  $D > 0$  and  $\tilde{D} > 0$ . Moreover, it is a well-known fact that  $G(\cdot)$  is SSPR if and only if its inverse  $G^{-1}(\cdot)$  is SSPR. Hence, if  $G(\cdot)$  is SSPR, then all  $\tilde{G}(\cdot)$ ,  $G^{-1}(\cdot)$ , and  $\tilde{G}^{-1}(\cdot)$  are simultaneously SSPR.*

**Remark 4** (Passivity test). *Corollary 1 first appeared in [5] as a non-Hamiltonian approach for passivity assessment. However, [5] does not consider the case with singular  $D$ . As suggested in [4], principally, in view of Remark 2, for such cases it is sufficient to run Corollary 2 on the reciprocal*

*system  $\tilde{G}(\cdot)$ . Yet the evaluation of the conditions (F2) and (F3) remains an issue, as the explicit matrix inversion can be numerically unstable. Recently we found a way to circumvent this difficulty by regularizing the feedthrough term  $D$  and turning over to a particular descriptor system realization. (Due to the space constraints this will be worked out in a forthcoming paper.)*

#### IV. SPECTRAL CONDITIONS FOR STRICT NEGATIVE IMAGINARINESS

**Definition 2** ([16], [17]). *A function  $G(\cdot) \in \mathcal{R}^{m \times m}$  from (6) is said to be negative imaginary (abbr. NI) if*

- (H1)  $G(\cdot)$  is analytic in  $\mathbb{C}^+$ ;
- (H2)  $D = D^T$ ;
- (H3) for  $\omega \in (0, \infty)$  such that  $j\omega$  is not a pole of  $G(\cdot)$  it holds  $j \text{SHe}(G(j\omega)) \geq 0$ ; and
- (H4) if  $j\omega_0$  is a pole of  $G(\cdot)$ , then it is at most simple with the residue matrix  $K_0 = \lim_{s \rightarrow j\omega_0} (s - j\omega_0)s(G(s) - D) \geq 0$ .

**Definition 3** ([16]). *A function  $G(\cdot) \in \mathcal{R}^{m \times m}$  is said to be strictly negative imaginary (abbr. SNI) if*

- (I1)  $G(\cdot) \in \mathcal{RH}_\infty^{m \times m}$ ;
- (I2)  $D = D^T$ ;
- (I3)  $j \text{SHe}(G(j\omega)) > 0$  for all  $\omega \in (0, \infty)$ .

Similarly as in Theorem 3 we have the following spectral test for (I3), see [18].

**Theorem 5.** *Let  $A$  be Hurwitz. Then (I3) is equivalent to the existence of an  $\omega_0 \in (0, \infty)$  such that  $j \text{SHe}(G(j\omega_0)) > 0$  and the non-existence of nonzero, finite, purely imaginary eigenvalues of the pencil*

$$\left[ \begin{array}{ccc} 0 & \lambda I_n - A & -B \\ \lambda I_n + A^T & 0 & C^T \\ B^T & -C & D - D^T \end{array} \right].$$

PR and NI systems are closely related by the fact that  $G(\cdot)$  is NI if and only if

$$R(s) := s(G(s) - D) \quad (21)$$

is PR [17]. This does not hold for the SNI and SPR properties due to the violation of the SPRness by the blocking zero at the origin of  $R(\cdot)$  [16]. Motivated by this difficulty, we now introduce a *strong negative imaginary* concept, which will enable a direct link to the SPRness of the corresponding reciprocal system  $\tilde{R}(s) := R(1/s)$  (see below).

**Remark 5.** ([16]) *As a result of missing link between the SNIness and SPRness, the approaches based on SPR synthesis cannot be used for control of NI systems (irrespectively whether strict or non-strict).*

**Definition 4** (Strong Negative Imaginariness). *A function  $G(\cdot) \in \mathcal{R}^{m \times m}$  is said to be strongly negative imaginary (abbr. SSNI) if it is SNI and*

$$\lim_{\omega \rightarrow 0} \frac{1}{\omega} j \text{SHe}(G(j\omega)) > 0. \quad (22)$$

Note that under the restriction (I3), the condition (22) is equivalent to

$$\lim_{\omega \rightarrow 0} \frac{1}{\omega^m} \det(j \text{SHe}(G(j\omega))) > 0. \quad (23)$$

The main result in this section reads as follows.

**Theorem 6.** *Let  $G(\cdot) \in \mathcal{RH}_\infty^{m \times m}$  be a symmetric matrix function with a state-space realization given in (7). Then  $G(\cdot)$  is SNI if and only if:*

(J1)  $CB > 0$ ; and

(J2)  $A(I_n - B(CB)^{-1}C)A$  has no negative real eigenvalues.

The function  $G(\cdot)$  is SSNI if and only if, additionally,

(J3)  $A(I_n - B(CB)^{-1}C)A$  has a zero eigenvalue of algebraic multiplicity  $m$ .

*Proof.* For the first part of the theorem, we only need to consider condition (I3), as the rest holds by theorem assumptions. Following the analogous technique as in the proof of Theorem 4 for the SPR case, we have:

$$j \text{SHe}(G(j\omega)) = \begin{bmatrix} \hat{B}^T & 0 \end{bmatrix} \begin{bmatrix} -\omega E & \hat{A} \\ \hat{A} & \omega E \end{bmatrix}^{-1} \begin{bmatrix} \hat{B} \\ 0 \end{bmatrix}$$

which represents the Schur complement of

$$U = \left[ \begin{array}{cc|c} -\omega \hat{E} & \hat{A} & \hat{B} \\ \hat{A} & \omega \hat{E} & 0 \\ \hline \hat{B}^T & 0 & 0 \end{array} \right]$$

with respect to the leading  $2 \times 2$  block. Again, we introduce  $V$  using a permutation matrix  $\Pi$ , such that  $U = \Pi^* V \Pi$ , and

$$V = \left[ \begin{array}{cc|c} -\omega \hat{E} & \hat{B} & \hat{A} \\ \hat{B}^T & 0 & 0 \\ \hline \hat{A} & 0 & \omega \hat{E} \end{array} \right].$$

Applying the HIAF for  $\omega \in (0, \infty)$  yields

$$\begin{aligned} \ln(U) &= \ln(j \text{SHe}(G(j\omega))) + \ln \left( \begin{bmatrix} -\omega \hat{E} & \hat{A} \\ \hat{A} & \omega \hat{E} \end{bmatrix} \right) \\ &= \ln(j \text{SHe}(G(j\omega))) + (n, n, 0), \text{ and} \\ \ln(V) &= \ln(-\omega \hat{E}) + \ln \left( \frac{1}{\omega} \hat{B}^T \hat{E}^{-1} \hat{B} \right) \\ &\quad + \ln \left( \omega \hat{E} + \frac{1}{\omega} \hat{N} \right), \end{aligned}$$

where, for convenience, we introduce the abbreviation

$$\hat{N} := \hat{A} \left( \hat{E}^{-1} - \hat{E}^{-1} \hat{B} \left( \hat{B}^T \hat{E}^{-1} \hat{B} \right)^{-1} \hat{B}^T \hat{E}^{-1} \right) \hat{A}.$$

The first two summands in the latter expression for  $\ln(V)$  are obviously constant for all  $\omega \in (0, \infty)$ . As a consequence, due to  $\ln(U) = \ln(V)$ , we have

$$\ln(j \text{SHe}(G(j\omega))) = \text{const.} \quad (24)$$

if and only if  $\ln(\omega \hat{E} + \frac{1}{\omega} \hat{N}) = \text{const.}$  Moreover, using (4) for sufficiently large  $\omega$  we have  $\ln(-\omega \hat{E}) + \ln(\omega \hat{E} + \frac{1}{\omega} \hat{N}) =$

$(n, n, 0)$ . Assuming (24) this relation holds of all for  $\omega \in (0, \infty)$ , implying:

$$\ln(j \text{SHe}(G(j\omega))) = \ln \left( \frac{1}{\omega} \hat{B}^T \hat{E}^{-1} \hat{B} \right), \quad \omega \in (0, \infty). \quad (25)$$

Thus, in order for condition (I3) to hold, it is necessary and sufficient that

$$\hat{B}^T \hat{E}^{-1} \hat{B} = CB > 0 \text{ and } \ln \left( \omega \hat{E} + \frac{1}{\omega} \hat{N} \right) = \text{const.}$$

With regard to (B1)–(B2), the constant inertia of the matrix family  $\omega \hat{E} + \frac{1}{\omega} \hat{N}$  with  $\omega \in (0, \infty)$  is equivalent to  $\det(\omega^2 I_n + \hat{N} \hat{E}^{-1}) \neq 0$ , which excludes negative real eigenvalues of the matrix

$$A(I_n - B(CB)^{-1}C)A = \hat{N} \hat{E}^{-1}. \quad (26)$$

This completes the proof of the SNI part of theorem.

Now consider the condition (J3) of the theorem. First, note that the matrix (26) has a zero eigenvalue with an algebraic multiplicity of at least  $m$ , since  $I_n - B(CB)^{-1}C$  has  $m$  zero eigenvalues. Then, by following similar arguments as in the proof of Theorem 4, one can show that:

$$\det(j \text{SHe}(G(j\omega))) = c \cdot \frac{\det(\omega^2 I_n + \hat{N} \hat{E}^{-1})}{\omega^m \det(\omega^2 I_n + A^2)}$$

where  $c$  is some positive scalar. Consequently, we obtain

$$\lim_{\omega \rightarrow 0} \det(j \text{SHe}(G(j\omega))) = \tilde{c} \cdot \lim_{\omega \rightarrow 0} \frac{1}{\omega^m} \det(\omega^2 I_n + \hat{N} \hat{E}^{-1})$$

with a positive constant  $\tilde{c}$ . Now similarly as in the proof of Theorem 4, we can now conclude the equivalence of (23) and (J1)–(J3) which completes the proof of the second part of the theorem.  $\square$

The following statement establishes a link between the SSNI and SPR properties for symmetric systems by combining Theorems 4 and 6.

**Corollary 3.** *Let  $G(\cdot) \in \mathcal{RH}^{m \times m}$  be a symmetric transfer function with a minimal realization (7). Then  $G(\cdot)$  is SSNI in the sense of Definition 4 if and only if  $\tilde{R}(\cdot)$ , defined via (21), is SPR.*

*Proof.* The matrix functions  $R(\cdot)$  and its reciprocal system  $\tilde{R}(\cdot)$  have the realizations

$$R(\cdot) \stackrel{\min}{\sim} \left[ \begin{array}{c|c} A & B \\ \hline CA & CB \end{array} \right], \quad \tilde{R}(\cdot) \stackrel{\min}{\sim} \left[ \begin{array}{c|c} A^{-1} & -A^{-1}B \\ \hline C & 0 \end{array} \right],$$

respectively. Hence, spectral conditions (J1)–(J3) of Theorem 6 for the SSNI of  $G(\cdot)$  formally reflect the conditions (G2)–(G3) in Corollary 2. According to the same corollary, its reciprocal function  $\tilde{R}(\cdot)$  is SPR if and only if  $R(\cdot) \in \mathcal{RH}_\infty^{m \times m}$ ,  $CB > 0$  and  $A(A - B(CB)^{-1}CA)$  has no negative real eigenvalues and a zero eigenvalue of algebraic multiplicity  $m$ , as  $r = \text{rank}(0_{m \times m}) = 0$ . Finally, note that  $R(\cdot) \in \mathcal{RH}_\infty^{m \times m}$  if and only if  $G(\cdot) \in \mathcal{RH}_\infty^{m \times m}$  which completes the proof.  $\square$

**Remark 6.** Our definition of SSNIness is weaker as compared to the one in the SSNI Lemma [16, Theorem 3.3] in the sense that we do not additionally require

$$\lim_{\omega \rightarrow \infty} j\omega \text{SHe}(G(j\omega)) > 0,$$

which is easily shown to be equivalent to the condition  $\lim_{\omega \rightarrow \infty} (R(j\omega) + R^H(j\omega)) > 0$ . According to our derivations the latter is obsolete as it is implied by the condition (I3) of SSNIness, more precisely, by  $CB > 0$  in (J1).

## V. NUMERICAL EXAMPLES

### A. A SISO Example

This example is taken from [2]. Consider the minimal realization (7) of a matrix function  $G(\cdot)$  with  $D = 0$  and

$$A = \begin{bmatrix} -3.5 & -8.5 & -8.5 & -2.5 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \\ B^T = [2.5 \quad -3 \quad 1 \quad 0], \quad C = [0 \quad 0 \quad 0 \quad 1].$$

This is not SNI, as for  $\omega = 1$  it holds that  $j \text{SHe}(G(j\omega)) = 0$ , see [2], which is in agreement with Theorem 6, since  $CB = 0$  violates the first condition (J1). Moreover, according to Corollary 3,  $\tilde{R}(\cdot)$  cannot be SPR, which is also true as (F1) in Theorem 4 is violated by the same condition. Now, modify  $C = [1 \quad 0 \quad 0 \quad 0]$ . Then,  $CB = 2.5$  and the eigenvalues of the matrix  $A(I_4 - B(CB)^{-1}C)A$  in the condition (J2) are:  $2.93 \pm j1.56$ ,  $0.225$  and  $0$ . Hence,  $G(\cdot)$  is SSNI. According to Theorem 3,  $\tilde{R}(\cdot)$  must be SPR. To demonstrate this, we solve the non-strict LMI (11) for the problem data of the strictly proper reciprocal system  $\tilde{R}(\cdot)$ , yielding the solution

$$P = \begin{bmatrix} 1.1569 & 2.5904 & 2.9192 & 0.5986 \\ 2.5904 & 8.2584 & 9.5643 & 2.4363 \\ 2.9192 & 9.5643 & 19.904 & 4.0342 \\ 0.5986 & 2.4363 & 4.0342 & 3.2020 \end{bmatrix} > 0.$$

### B. A MIMO Example

Consider the minimal realization (8) of a  $2 \times 2$  transfer function  $G(\cdot)$  with internally symmetric realization satisfying  $C = B^T$  and

$$A = \begin{bmatrix} -22 & -9 & -2 \\ -9 & -11 & 3 \\ -2 & 3 & -4 \end{bmatrix}, \quad B = \begin{bmatrix} 2.5 & 1 \\ -3 & 0 \\ 1 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

The matrix  $D - CA^{-1}B$  is strictly positive and the eigenvalues of  $A^{-1}(A^{-1} + A^{-1}B(D - CA^{-1}B)^{-1}CA^{-1})$  are computed to be  $0$ ,  $0.2422$  and  $0.0048276$ . Thus according to Theorem 4,  $G(\cdot)$  is SPR, as it has no negative real eigenvalues and exactly one eigenvalue equal to zero.

The result is supported by looking at the eigenvalues of the even pencil (12) which are  $\pm 14.3925$  and  $\pm 2.0320$ , together with 4 infinite eigenvalues. The absence of purely imaginary eigenvalues of the pencil indicates that the eigenvalue curves of  $\text{He}(G(j\omega))$  do not intersect the zero level.

## VI. CONCLUSIONS AND FUTURE RESEARCH

Non-Hamiltonian spectral conditions for strict positive realness and strict negative imaginarity of symmetric linear systems are discussed. In particular, we have shown that it is only necessary to check certain matrices for the absence of negative eigenvalues and a certain number of zero eigenvalues. This paper summarizes some preliminary theoretical results. In a forthcoming paper we will extend these into several directions. The results will be generalized to the class of reciprocal systems allowing symmetry of the transfer function with respect to an external signature and we drop the minimality realization condition. Furthermore we will discuss the numerical issues regarding the evaluation of our spectral conditions and we explore the quadratic stability of a class of associated switched systems.

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