Computation of the $\mathcal{H}_\infty$-Norm for Large-Scale Systems

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(joint work with Peter Benner and Ryan Lowe)

In this short report we consider linear time-invariant descriptor systems

$$E\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t),$$

where E, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, $x(t) \in \mathbb{R}^n$ is the descriptor vector, $u(t) \in \mathbb{R}^m$ is the input vector, and $y(t) \in \mathbb{R}^p$ is the output vector. Assuming that the pencil $\lambda E - A$ is regular, the relationship between inputs and outputs in the frequency domain is given by the transfer function

$$G(s) := C(sE - A)^{-1}B.$$

By $\mathcal{RH}_\infty^{p\times m}$ we denote the Banach space of all rational $p \times m$ matrix-valued functions that are analytic and bounded in the open right half-plane $\mathbb{C}^+ := \{s \in \mathbb{C} : \mathrm{Re}(s) > 0\}$. For $G \in \mathcal{RH}_\infty^{p\times m}$, the $\mathcal{H}_\infty$-norm is defined by

$$\|G\|_{\mathcal{H}_\infty} := \sup_{s\in\mathbb{C}^+} \|G(s)\|_2 = \sup_{\omega\in\mathbb{R}} \|G(\mathrm{i}\omega)\|_2 .$$

The aim is to compute this norm under the assumption that all the matrices E, A, B, C are large and sparse and that m, $p \ll n$. We propose two approaches to achieve this.

The first approach is presented in [1] and considers perturbed transfer functions

$$G_\Delta(s) = C(sE - (A + B\Delta C))^{-1}B.$$

We define the structured complex stability radius by

$$r_{\mathbb{C}}(E, A, B, C) := \inf \left\{ \|\Delta\|_2 : G_\Delta \notin \mathcal{RH}_\infty^{p\times m} \text{ for some } \Delta \in \mathbb{C}^{m\times p} \right\}.$$

It can be shown that

$$r_{\mathbb{C}}(E, A, B, C) = \begin{cases} 1/\|G\|_{\mathcal{H}_\infty} & \text{if } G \not\equiv 0, \\ \infty & \text{if } G \equiv 0. \end{cases}$$

The condition that $G_\Delta \notin \mathcal{RH}_\infty^{p\times m}$ can be achieved in three ways. First, it can happen that $G_\Delta(\cdot)$ is not well-defined which is the case when the pencil $\lambda E - (A + B\Delta C)$ is singular. Second, $G_\Delta(\cdot)$ might be improper, i.e., unbounded at infinity. These two cases are treated separately. The algorithm concentrates on the third case, namely $G_\Delta(\cdot)$ has poles on the imaginary axis. This computation is based on structured ε-pseudospectra

$$\Pi_\varepsilon(E, A, B, C) = \left\{ s \in \mathbb{C} : s \text{ is a pole of } G_\Delta(\cdot) \text{ for a } \Delta \in \mathbb{C}^{m\times p} \text{ with } \|\Delta\|_2 < \varepsilon \right\}.$$

To compute $r_{\mathbb{C}}(E, A, B, C)$ we have to find the value of ε for which $\Pi_\varepsilon(E, A, B, C)$ touches the imaginary axis. This is done in a nested iteration, similarly as in [2]. In the inner iteration we compute the rightmost point of $\Pi_\varepsilon(E, A, B, C)$ for a *fixed* value of ε. This is done by computing an appropriate perturbation Δ that moves one of the poles of $G(\cdot)$ to the boundary of $\Pi_\varepsilon(E, A, B, C)$. We exploit the fact that an optimizing perturbation is of rank one, i.e., $\Delta = \varepsilon uv^H$ with $u \in \mathbb{R}^m$, $v \in \mathbb{R}^p$

and $\|u\|_2 = \|v\|_2 = 1$. In the outer iteration, ε is varied by applying Newton's method. To determine the pole that should be perturbed in the inner iteration we compute some dominant poles of $G(\cdot)$ [3]. This is particularly important to find *global* instead of local optimizers.

The second method goes back to [4] and has been generalized to descriptor systems in [5]. There we consider even matrix pencils of the form

$$\mathcal{H}_\gamma(\lambda) := \left[\begin{array}{cc|cc} 0 & -\lambda E^T - A^T & -C^T & 0 \\ \lambda E - A & 0 & 0 & -B \\ \hline -C & 0 & \gamma I_p & 0 \\ 0 & -B^T & 0 & \gamma I_m \end{array}\right].$$

If $\lambda E - A$ has no finite, purely imaginary eigenvalues and $\gamma > \min_{\omega\in\mathbb{R}} \|G(\mathrm{i}\omega)\|_2$, then $\|G\|_{\mathcal{H}_\infty} \geq \gamma$ if and only if $\mathcal{H}_\gamma(\lambda)$ has finite, purely imaginary eigenvalues. This can be used to implement an algorithm that iterates over γ and checks in every step whether $\mathcal{H}_\gamma(\lambda)$ has finite, purely imaginary eigenvalues. These eigenvalues also determine the boundary points of the components of the level-set

$$\Omega_\gamma := \{\omega \in \mathbb{R} : \|G(\mathrm{i}\omega)\|_2 > \gamma\}.$$

As discussed in [5], is important to find *all* finite, purely imaginary eigenvalues to obtain the entire level-set and to ensure global convergence to the $\mathcal{H}_\infty$-norm. However, in the large-scale setting, we cannot compute all eigenvalues of $\mathcal{H}_\gamma(\lambda)$, but we can only use iterative methods to determine *some* eigenvalues close to a number of prespecified shifts. However, heuristically the $\mathcal{H}_\infty$-norm is attained close to a dominant pole. Therefore, we use the dominant poles to determine shifts for the even eigensolver presented in [6]. In this way, we cannot ensure to find the whole level-set Ω_γ, but we can still find one of its components that contains the optimizing frequency ω. The results of this approach and a comparison to the pseudospectral method are discussed in [7]. Numerical examples show that both methods work well, even for rather difficult examples.

References

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