$\mathcal{L}_\infty$-Norm Computation for Continuous-Time Descriptor Systems Using Structured Matrix Pencils

Peter Benner, Vasile Sima, *Senior Member, IEEE,*
and Matthias Voigt

***Abstract*—In this technical note, we discuss an algorithm for the computation of the $\mathcal{L}_\infty$-norm of transfer functions related to descriptor systems. We show how one can achieve this goal by computing the eigenvalues of certain skew-Hamiltonian/Hamiltonian matrix pencils and analyze arising problems. We also formulate and prove a theoretical result which serves as a basis for testing a transfer function matrix for properness. Finally, we illustrate our results using a descriptor system related to mechanical engineering.**

***Index Terms*—Continuous time systems, $\mathcal{H}_\infty$ control, numerical stability, transfer function matrices, singular systems. skew-Hamiltonian/Hamiltonian matrix pencils.**

I. Introduction

In many applications from industry and technology, computer simulations are performed using models which can be formulated by systems of differential equations. Often the equations underlie additional algebraic constraints which prevent the system from attaining every possible state. In this context we speak of descriptor systems (or singular systems). These systems naturally arise in a large variety of applications such as electrical circuit simulation, multibody dynamics with constraints or the semidiscretization of certain partial differential equations (see [1] and references therein). Very important characteristic values of such systems are the $\mathcal{L}_\infty$-norms of the corresponding transfer functions. These norms have found important applications in robust control or model order reduction [1]–[3].

Consider a continuous-time linear time-invariant descriptor system

$$E\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t) + Du(t), \tag{1}$$

with $E,\, A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, $D \in \mathbb{R}^{p\times m}$, descriptor vector $x(t) \in \mathbb{R}^n$, control vector $u(t) \in \mathbb{R}^m$, and output vector $y(t) \in \mathbb{R}^p$. Here, E usually is a singular matrix. By taking the Laplace transform [4] of both equations in (1) and inserting the first equation into the other one we obtain the matrix-valued *transfer function* of the system

$$G(s) := C\,(sE - A)^{-1}\,B + D \tag{2}$$

which directly maps inputs to outputs in the frequency domain. For convenience, we assume that $G \in \mathcal{RL}_\infty^{p\times m}$, where $\mathcal{RL}_\infty^{p\times m}$ denotes the Banach space of all rational $p \times m$ matrix-valued functions that are bounded on the imaginary axis. The natural norm for this space is the $\mathcal{L}_\infty$-norm which is defined by

$$\|G\|_{\mathcal{L}_\infty} := \sup_{\omega\in\mathbb{R}} \sigma_{\max}\left(G(\mathrm{i}\omega)\right),$$

where $\sigma_{\max}$ denotes the maximum singular value [2]. Note that for any transfer function $G \in \mathcal{RL}_\infty^{p\times m}$, the matrix pencil $\lambda E - A$ is regular (i.e., $\det(\lambda E - A)$ is not identical to the zero polynomial) and does not have any finite eigenvalues on the imaginary axis. However, even if $\lambda E - A$ is regular and has no finite purely imaginary eigenvalues, the transfer function could still be unbounded at infinity and hence is not an element of the space $\mathcal{RL}_\infty^{p\times m}$. This motivates the following definition. We call a transfer function G *proper* if $\lim_{\omega\to\infty} \|G(\mathrm{i}\omega)\| < \infty$ and *strictly proper* if $\lim_{\omega\to\infty} \|G(\mathrm{i}\omega)\| = 0$ for any induced matrix norm $\|\cdot\|$. Otherwise we call it *improper* [5]. As an agreement, we assign the norm value ∞ to transfer functions $G \notin \mathcal{RL}_\infty^{p\times m}$. In the sequel we do not assume that the system is stable. However, if it is stable the $\mathcal{L}_\infty$-norm is equivalent to the $\mathcal{H}_\infty$-norm [2]. The implementation based on the conceptual algorithms proposed in this paper covers the case of systems with unbounded transfer functions. The check for properness is done optionally. If the test is skipped, an error indicator is set for an improper system.

A regular matrix pencil $\lambda E - A$ can be reduced to *Weierstraß canonical form* [5]

$$E = W\begin{bmatrix} I_{n_f} & 0 \\ 0 & N \end{bmatrix} T, \quad A = W\begin{bmatrix} J & 0 \\ 0 & I_{n_\infty} \end{bmatrix} T,$$

where W and T are nonsingular, I_m is an identity matrix of order m, J and N are in Jordan canonical form and N is nilpotent with index of nilpotency ν. The number ν is also called the *algebraic index* of the descriptor system (1) and n_f and n_∞ are the dimensions of the deflating subspaces of $\lambda E - A$ corresponding to the finite and infinite eigenvalues, respectively. By using the transformation matrices W and T we can also write B and C as

$$B = W\begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad C = \begin{bmatrix} C_1 & C_2 \end{bmatrix} T.$$

In this way the system (1) is decoupled into its *slow subsystem*

$$\dot{x}_1(t) = Jx_1(t) + B_1u(t), \quad y_1(t) = C_1x_1(t),$$

which is a standard state space system and its *fast subsystem*

$$N\dot{x}_2(t) = x_2(t) + B_2u(t), \quad y_2(t) = C_2x_2(t) + Du(t),$$

which represents the algebraic constraints of system (1), see [4]. System (1) is called *C-controllable* if $\operatorname{rank}\begin{bmatrix} \alpha E - \beta A & B \end{bmatrix} = n$ for all $(\alpha, \beta) \in \mathbb{C}^2\backslash\{(0,0)\}$, and *C-observable* if $\operatorname{rank}\begin{bmatrix} \alpha E^T - \beta A^T & C^T \end{bmatrix} = n$ for all $(\alpha, \beta) \in \mathbb{C}^2\backslash\{(0,0)\}$. In particular, the slow (fast) subsystem is called C-controllable or C-observable if the corresponding condition holds for all complex pairs (α, β) with $\beta \neq 0$ ($\beta = 0$). A real matrix $M \in \mathbb{R}^{2n\times 2n}$ is called *Hamiltonian* if $M\mathcal{J} = (M\mathcal{J})^T$ and *skew-Hamiltonian* if $M\mathcal{J} = -(M\mathcal{J})^T$, where $\mathcal{J} = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$. A real matrix pencil $\lambda N - M$ is called *skew-Hamiltonian/Hamiltonian* if N is skew-Hamiltonian and M is Hamiltonian. The spectrum of such a matrix pencil has the nice property that it is symmetric with respect to both the real and the imaginary axis [6], [7].

The remainder of this article is structured as follows. In Section II we present a new method for computing the $\mathcal{L}_\infty$-norm for proper transfer functions related to descriptor systems. Algorithms for the computation of the $\mathcal{L}_\infty$-norm of standard state-space systems rely on the relationship of the singular values of a transfer function and the purely imaginary eigenvalues of a specific Hamiltonian matrix. First, Byers developed a bisection method [8] which converges linearly to the norm. A few years later, several other authors improved this algorithm [9], [10] to obtain a quadratic rate of convergence. In this article we generalize this algorithm to descriptor systems which leads to skew-Hamiltonian/Hamiltonian eigenvalue problems. We put

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Peter Benner and Matthias Voigt are with the Max Planck Institute for Dynamics of Complex Technical Systems, Sandtorstraße 1, 39106 Magdeburg, Germany, {benner, voigtm}@mpi-magdeburg.mpg.de, Phone: +49 391 6110 {450, 464}, Fax: +49 391 6110 453.

Vasile Sima is with the National Institute for Research and Development in Informatics, Advanced Research Department, 8-10 Averescu Ave., 011455 Bucharest, Romania, vsima@ici.ro, Phone: +40 21 316 07 65, Fax: +40 21 316 05 39.

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a particular emphasis on computing the eigenvalues in a structure-preserving manner to increase reliability and accuracy of the method. In Section III we state and prove a theorem which serves as a basis for an algorithm which can check if a transfer function is proper. We illustrate our theoretical results using an example from mechanical engineering in Section IV. Finally, in Section V, we give a short conclusion and state different possible directions of future research.

II. Computation of the $\mathcal{L}_\infty$-Norm

In this section we derive an algorithm for the computation of the $\mathcal{L}_\infty$-norm for proper transfer functions related to descriptor systems of the form (1) and clarify the relation of the $\mathcal{L}_\infty$-norm, generalized algebraic Riccati equations, and skew-Hamiltonian/Hamiltonian matrix pencils.

A. Preliminaries

The $\mathcal{L}_\infty$-norm of a proper transfer function (2) can be interpreted as the $\mathcal{L}_2$-gain of system (1) (see [11]), i.e., whenever x, u, y satisfy (1), $Ex(0) = 0$ and $T_f > 0$, we have

$$\int_0^{T_f} y(t)^T y(t) \mathrm{d}t \le \|G\|_{\mathcal{L}_\infty}^2 \int_0^{T_f} u(t)^T u(t) \mathrm{d}t.$$

There is also a connection between the $\mathcal{L}_\infty$-norm and the following indefinite linear-quadratic regulator problem [11]:

$$\min_{u \in \mathcal{L}_2} \int_0^\infty \left(\gamma^2 u(t)^T u(t) - y(t)^T y(t) \right) \mathrm{d}t \tag{3}$$

subject to $E\dot{x}(t) = Ax(t) + Bu(t)$, $y(t) = Cx(t) + Du(t)$, $Ex(0) = 0$ with $\gamma > 0$. First, we recall that $\|G\|_{\mathcal{L}_\infty}^2 = \max \int_0^\infty y(t)^T y(t) \mathrm{d}t$ subject to $E\dot{x}(t) = Ax(t) + Bu(t)$, $y(t) = Cx(t) + Du(t)$, $Ex(0) = 0$, and $\int_0^\infty u(t)^T u(t) \mathrm{d}t \le 1$. We see that $\|G\|_{\mathcal{L}_\infty} > \gamma$ implies that the minimum value of (3) is $-\infty$, whereas if the minimum value of (3) is greater than $-\infty$ (and hence zero), we have $\|G\|_{\mathcal{L}_\infty} \le \gamma$. One can formulate a generalized algebraic Riccati equation

$$0 = \mathcal{Q} + \mathcal{A}^T \mathcal{X} + \mathcal{X}^T \mathcal{A} - \mathcal{X}^T \mathcal{G} \mathcal{X}, \quad \mathcal{E}^T \mathcal{X} = \mathcal{X}^T \mathcal{E}$$

associated to the LQR problem (3) with $\mathcal{Q} = \mathcal{Q}^T$, $\mathcal{G} = \mathcal{G}^T$ (see [12] and references therein). The matrix coefficients $\mathcal{E}$, $\mathcal{A}$, $\mathcal{Q}$, and $\mathcal{G}$ are the block entries of the skew-Hamiltonian/Hamiltonian matrix pencil $\lambda N - M_\gamma$ with

$$N = \begin{bmatrix} E & 0 \\ 0 & E^T \end{bmatrix}, \quad M_\gamma = \begin{bmatrix} A & 0 \\ 0 & -A^T \end{bmatrix} + \begin{bmatrix} B & 0 \\ 0 & -C^T \end{bmatrix} \times \begin{bmatrix} -D & \gamma I \\ \gamma I & -D^T \end{bmatrix}^{-1} \begin{bmatrix} C & 0 \\ 0 & B^T \end{bmatrix}. \tag{4}$$

The matrix M_γ can also be expressed as

$$M_\gamma = \begin{bmatrix} A - BR^{-1}D^T C & -\gamma B R^{-1} B^T \\ \gamma C^T S^{-1} C & -A^T + C^T D R^{-1} B^T \end{bmatrix}$$

with the matrices $R = D^T D - \gamma^2 I$, and $S = DD^T - \gamma^2 I$ (see [6], [7] for more details). We know that under the assumption that the system (1) is minimal [4] and $\gamma > \min_{\omega \in \mathbb{R}} \sigma_{\max}(G(\mathrm{i}\omega))$, the minimum value of (3) attains zero if and only if the generalized algebraic Riccati equation has a stabilizing solution $\mathcal{X}$, satisfying $\mathcal{E}^T \mathcal{X} = \mathcal{X}^T \mathcal{E} \succcurlyeq 0$. Following [12], for $\gamma \ne \|G\|_{\mathcal{L}_\infty}$ this is equivalent to the absence of finite purely imaginary eigenvalues of the matrix pencil $\lambda N - M_\gamma$ in (4). Using this we can formulate a theorem which connects the singular values of a transfer function matrix with the eigenvalues of the associated matrix pencil (4) as it has been done for standard systems in [11].

Theorem 1: Assume the matrix pencil $\lambda E - A$ is regular and has no finite eigenvalues on the imaginary axis, $\gamma > 0$ is not a singular value of D and $\omega_0 \in \mathbb{R}$. Then, γ is a singular value of $G(\mathrm{i}\omega_0)$ if and only if $\mathrm{i}\omega_0 N - M_\gamma$ is singular.

Proof: The argumentation follows the one of the proof of Theorem 1 in [11]. Let γ be a singular value of $G(\mathrm{i}\omega_0)$. Then there exist nonzero vectors $u \in \mathbb{C}^m$, $v \in \mathbb{C}^p$ such that $G(\mathrm{i}\omega_0)u = \gamma v$, $G(\mathrm{i}\omega_0)^H v = \gamma u$. Thus,

$$\begin{aligned} \left(C \left(\mathrm{i}\omega_0 E - A \right)^{-1} B + D \right) u &= \gamma v, \\ \left(B^T \left(-\mathrm{i}\omega_0 E^T - A^T \right)^{-1} C^T + D^T \right) v &= \gamma u. \end{aligned} \tag{5}$$

Define

$$r = \left(\mathrm{i}\omega_0 E - A \right)^{-1} Bu, \quad s = \left(-\mathrm{i}\omega_0 E^T - A^T \right)^{-1} C^T v. \tag{6}$$

Now solving for u and v in terms of r and s yields

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} -D & \gamma I \\ \gamma I & -D^T \end{bmatrix}^{-1} \begin{bmatrix} C & 0 \\ 0 & B^T \end{bmatrix} \begin{bmatrix} r \\ s \end{bmatrix}. \tag{7}$$

Note, that (7) guarantees that $\begin{bmatrix} r^T & s^T \end{bmatrix}^T \ne 0$. From (6) we get

$$\left(\begin{bmatrix} \mathrm{i}\omega_0 E & 0 \\ 0 & \mathrm{i}\omega_0 E^T \end{bmatrix} - \begin{bmatrix} A & 0 \\ 0 & -A^T \end{bmatrix} \right) \begin{bmatrix} r \\ s \end{bmatrix} = \begin{bmatrix} B & 0 \\ 0 & -C^T \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}.$$

With (7), this is equivalent to

$$\left(\begin{bmatrix} A & 0 \\ 0 & -A^T \end{bmatrix} + \begin{bmatrix} B & 0 \\ 0 & -C^T \end{bmatrix} \begin{bmatrix} -D & \gamma I \\ \gamma I & -D^T \end{bmatrix}^{-1} \times \begin{bmatrix} C & 0 \\ 0 & B^T \end{bmatrix} \right) \begin{bmatrix} r \\ s \end{bmatrix} = \mathrm{i}\omega_0 \begin{bmatrix} E & 0 \\ 0 & E^T \end{bmatrix} \begin{bmatrix} r \\ s \end{bmatrix}. \tag{8}$$

Thus we get $M_\gamma \begin{bmatrix} r \\ s \end{bmatrix} = \mathrm{i}\omega_0 N \begin{bmatrix} r \\ s \end{bmatrix}$. This proves one direction of Theorem 1.

Now we prove the converse direction. Suppose that the matrix pencil $\lambda N - M_\gamma$ has the eigenvalue $\mathrm{i}\omega_0$, that is, (8) holds for some $\begin{bmatrix} r^T & s^T \end{bmatrix}^T \ne 0$. Defining u and v by (7) clearly yields $\begin{bmatrix} u^T & v^T \end{bmatrix}^T \ne 0$. (Otherwise $\begin{bmatrix} r^T & s^T \end{bmatrix}^T$ would be zero, following from (6).) Then from (7) and (8), we conclude (5), which establishes that γ is a singular value of $G(\mathrm{i}\omega_0)$. ■

Next we state and prove a modified version of Theorem 2 from [11].

Theorem 2: Let $\gamma > \min_{\omega \in \mathbb{R}} \sigma_{\max}(G(\mathrm{i}\omega))$ be not a singular value of D. Then $\|G\|_{\mathcal{L}_\infty} \ge \gamma$ if and only if $\lambda N - M_\gamma$ has finite purely imaginary eigenvalues (i.e., at least one).

Proof: Assume first $\|G\|_{\mathcal{L}_\infty} \ge \gamma$. From the definition of the $\mathcal{L}_\infty$-norm and the continuity of $\sigma_{\max}(G(\mathrm{i}\omega))$ it follows that there exists $\omega_0 \in \mathbb{R}$ such that $\sigma_{\max}(G(\mathrm{i}\omega_0)) = \gamma$. Together with Theorem 1 we obtain that $\mathrm{i}\omega_0 N - M_\gamma$ is singular, so the matrix pencil $\lambda N - M_\gamma$ has at least one finite purely imaginary eigenvalue.

If we assume on the other hand that $\lambda N - M_\gamma$ has finite purely imaginary eigenvalues, e.g., $\mathrm{i}\omega_0$, Theorem 1 yields that γ is a singular value of $G(\mathrm{i}\omega_0)$, hence $\|G\|_{\mathcal{L}_\infty} \ge \gamma$. ■

B. The Algorithm

Using the two theorems from above we are able to state an iterative method which iterates over γ and checks in each step if the matrix pencil $\lambda N - M_\gamma$ has finite purely imaginary eigenvalues. In Algorithm 1 we illustrate the generalization of the algorithm proposed in [9] to descriptor systems with *proper* transfer functions. We remark that Algorithm 1 does not converge for improper transfer functions since we cannot achieve the absence of imaginary eigenvalues of $\lambda N - M_\gamma$ during the iteration. A graphical interpretation of the method can also be found in [9]. As the algorithm for standard

Algorithm 1 Algorithm for Computing the $\mathcal{L}_\infty$-Norm

Input: Continuous-time linear time-invariant descriptor system $(E; A, B, C, D)$ with *proper* transfer function G, tolerance ε.
Output: $\|G\|_{\mathcal{L}_\infty}$.

```
Compute the eigenvalues of the matrix pencil λE − A. If there
  are finite purely imaginary eigenvalues, set ||G||_{L∞} = ∞ and return.
Compute an initial value γ_lb < ||G||_{L∞}.
repeat
    Set γ := (1 + 2ε)γ_lb.
    Compute the eigenvalues of the matrix pencil λN − M_γ.
    if no finite purely imaginary eigenvalues then
        γ_ub = γ, break.
    else
        Set {iω_1, ..., iω_k} = finite purely imaginary eigenvalues of
          λN − M_γ with ω_i ≥ 0 for i = 1, ..., k.
        Set m_j = sqrt(ω_j ω_{j+1}), j = 1, ..., k − 1.
        Compute the largest singular value of G(im_j) for j = 1, ..., k − 1.
        Set γ_lb = max_{1≤j≤k−1} σ_max(G(im_j)).
    end if
until break
Set ||G||_{L∞} = (1/2)(γ_lb + γ_ub).
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state-space systems, our generalization is still monotonically and quadratically converging and the relative error of the computed $\mathcal{L}_\infty$-norm is at most ε, see [9], [10] for details. Since Algorithm 1 has to check if the matrix pencils $\lambda N - M_\gamma$ have finite purely imaginary eigenvalues, a special emphasis should to given to the accuracy of the eigenvalue computation. This is necessary to ensure reliable results as inaccuracy in the eigenvalues could force the algorithm to produce wrong results. Therefore we apply a new structure-exploiting and -preserving approach to compute the eigenvalues of the arising skew-Hamiltonian/Hamiltonian matrix pencils as described in [6], [7]. By applying this method, simple finite purely imaginary eigenvalues do not experience any error in their real parts. This follows from the fact that the spectrum of every skew-Hamiltonian/Hamiltonian matrix pencil is symmetric with respect to the imaginary axis. That is, eigenvalues occur in pairs $(\lambda, -\lambda)$ if they are real, or in quadruples $\left(\lambda, -\lambda, \bar{\lambda}, -\bar{\lambda}\right)$ if they are complex. Furthermore, the new eigenvalue solver only allows structured perturbations, i.e., skew-Hamiltonian/Hamiltonian ones. So if we perturb a simple imaginary eigenvalue λ in its real part there does not exist its symmetric counterpart $-\bar{\lambda}$, in other words, only an error in the imaginary part is possible. In Fig. 1 we show the finite purely imaginary eigenvalues computed by the new structure-preserving method compared with the output of the QZ algorithm [13] for an example matrix pencil which confirms the theoretical results. Note that in this example, using the QZ algorithm for computing the imaginary eigenvalues might force the $\mathcal{L}_\infty$-norm algorithm to fail, since an error of about $2 \cdot 10^{-12}$ in the real parts is too large to be considered as zero.

C. Choice of the Initial Lower Bound

In Algorithm 1 it is necessary to determine an appropriate initial value $\gamma_{lb} < \|G\|_{\mathcal{L}_\infty}$. This is important as a good choice of this value might drastically decrease the computational costs of the algorithm. Note that some systems attain their $\mathcal{L}_\infty$-norm at the frequencies $\omega = 0$ or $\omega = \infty$ (e.g., models for RC circuits [15]). There are also several heuristic approaches to evaluate the transfer function (using the given realization) at more test frequencies in order to get better initial approximations of the norm. We

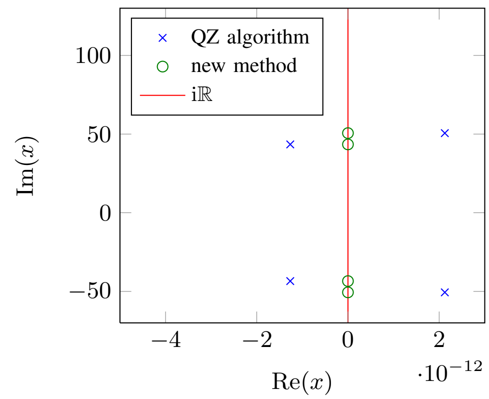

Fig. 1. Computed finite purely imaginary eigenvalues of a skew-Hamiltonian/Hamiltonian example matrix pencil set up by the system matrices of a semidiscretized Stokes equation [1], [14]

propose to apply the method given in [16]. Finally, we choose $\gamma_{lb} := \max\{\sigma_{\max}(G(0)), \sigma_{\max}(G(\mathrm{i}\omega_p)), \sigma_{\max}(G(\infty))\}$, where ω_p is the test frequency that gives the maximum singular value. The computation of $\sigma_{\max}(G(\mathrm{i}\omega))$ $(\omega \neq \infty)$ is rather simple but in contrast to the standard system case, evaluating $\sigma_{\max}(G(\infty))$ is a more difficult task. To achieve this goal we separate finite and infinite eigenvalues in $\lambda E - A$ using the QZ algorithm with eigenvalue reordering and subsequently solve a particular generalized Sylvester equation [17], [18]. This leads to an additive decomposition of the transfer function (2) into a strictly proper part G_{sp} and a polynomial part $P(s)$. Therefore we can drop G_{sp} if we consider $\lim_{\omega\to\infty} \sigma_{\max}(G(\mathrm{i}\omega))$. Since we assume that G is proper, P has to be a constant polynomial and hence $\sigma_{\max}(G(\infty)) = \sigma_{\max}(P(0))$. We refer to [19] for more details.

D. Improving the Accuracy and Reliability of the Eigenvalue Computation

Naively computing the matrix M_γ in (4) could be very ill-advised because it contains a lot of matrix products and inverses. The matrices R and S could be ill-conditioned and even if they are not, forming "matrix-times-its-transpose" products like $BR^{-1}B^T$ suffers from the same kind of numerical instability as forming the normal equations to solve linear least square problems (see Example 5.3.2 in [13]). When explicitly computing the blocks of M_γ this could easily corrupt the entries of the matrix by rounding errors and hence highly perturb the eigenvalues of the matrix pencil $\lambda N - M_\gamma$. In particular, finite purely imaginary eigenvalues can be easily moved away from the imaginary axis by this kind of errors which forces our algorithm for computing the $\mathcal{L}_\infty$-norm to produce wrong results. Therefore it is desirable to work directly on the original data without explicitly forming matrix products and inverses. This can be achieved by applying an extension strategy similar to the one described in [6]. We extend the matrix pencil (4) to

$$\lambda\mathcal{N} - \mathcal{M}_\gamma = \left[\begin{array}{cc|cc} \lambda E - A & 0 & -B & 0 \\ 0 & \lambda E^T + A^T & 0 & C^T \\ \hline -C & 0 & -D & \gamma I_p \\ 0 & -B^T & \gamma I_m & -D^T \end{array}\right], \quad (9)$$

which can be shown to have the same finite eigenvalues as the original matrix pencil (4) [19]. However, we loose the skew-Hamiltonian/Hamiltonian structure by this operation which we can recover by performing the following steps. First we observe that the dimension of the matrix pencil (9) is $2n+m+p$ which is odd if and only if $m+p$ is odd. As every skew-Hamiltonian/Hamiltonian matrix pencil has an even dimension we append a zero column to both B and D and define $\tilde{B} := \begin{bmatrix} B & 0 \end{bmatrix}$, $\tilde{D} := \begin{bmatrix} D & 0 \end{bmatrix}$, $\tilde{m} = m+1$ if $m+p$ is odd. This step is equivalent to the introduction of an artificial input with no influence on the system's behavior and its $\mathcal{L}_\infty$-norm. If $m+p$ is even we simply set $\tilde{B} := B$, $\tilde{D} := D$, $\tilde{m} := m$. By permuting and scaling the block rows and columns of (9) we obtain the *even matrix pencil* [20]

$$\lambda\hat{\mathcal{N}} - \hat{\mathcal{M}}_\gamma = \left[\begin{array}{cc|cc} 0 & -\lambda E^T - A^T & -C^T & 0 \\ \lambda E - A & 0 & 0 & -\tilde{B} \\ \hline -C & 0 & \gamma I_p & -\tilde{D} \\ 0 & -\tilde{B}^T & -\tilde{D}^T & \gamma I_{\tilde{m}} \end{array}\right].$$

Now we exploit the symmetries of the matrix $\hat{\mathcal{M}}_\gamma$ and repartition its blocks. We get

$$\begin{array}{c} \\ n \\ n \end{array} \begin{array}{c} \begin{array}{cc} p & \tilde{m} \end{array} \\ \begin{bmatrix} C^T & 0 \\ 0 & \tilde{B} \end{bmatrix} \end{array} =: \begin{array}{c} \\ n \\ n \end{array} \begin{array}{c} \begin{array}{cc} k & k \end{array} \\ \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \end{array},$$

$$\begin{array}{c} \\ p \\ \tilde{m} \end{array} \begin{array}{c} \begin{array}{cc} p & \tilde{m} \end{array} \\ \begin{bmatrix} -\gamma I_p & \tilde{D} \\ \tilde{D}^T & -\gamma I_{\tilde{m}} \end{bmatrix} \end{array} =: \begin{array}{c} \\ k \\ k \end{array} \begin{array}{c} \begin{array}{cc} k & k \end{array} \\ \begin{bmatrix} S_{11} & S_{12} \\ S_{12}^T & S_{22} \end{bmatrix} \end{array}$$

with $k = \frac{\tilde{m}+p}{2}$, $S_{11} = S_{11}^T$, and $S_{22} = S_{22}^T$. By again permuting and scaling the block rows and columns we obtain the extended skew-Hamiltonian/Hamiltonian matrix pencil

$$\lambda\bar{\mathcal{N}} - \bar{\mathcal{M}} = \left[\begin{array}{cc|cc} \lambda E - A & -R_{21} & 0 & -R_{22} \\ -R_{12}^T & -S_{12}^T & -R_{22}^T & -S_{22} \\ \hline 0 & R_{11} & \lambda E^T + A^T & R_{12} \\ R_{11}^T & S_{11} & R_{21}^T & S_{12} \end{array}\right]$$

which can now be treated by the structure-preserving approach for computing the eigenvalues. We stress that (near) singularity of the matrix E has no influence on the quality of the results.

III. Testing Properness of a Transfer Function Matrix

When calculating the $\mathcal{L}_\infty$-norm we have to ensure that the corresponding transfer function is proper. Often one knows from the modeling that the transfer function is proper. Then it is not necessary to run the following testing procedure and one can directly execute Algorithm 1. However, if one does not know about this property one can check this by the procedure following from the next theorem. If the transfer function turns out to be improper, it is not an element of the space $\mathcal{RL}_\infty^{p\times m}$ and hence the $\mathcal{L}_\infty$-norm is infinite.

Theorem 3: Let $(E; A, B, C, D)$ be a descriptor system with C-controllable and C-observable fast subsystem [4] and transfer function G. Let furthermore

$$U(\lambda E - A)V = \lambda \begin{bmatrix} T & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \tag{10}$$

be a decomposition of the matrix pencil $A - \lambda E$ by a generalized state-space transform with nonsingular matrices $U, V \in \mathbb{R}^{n\times n}$ and a full-rank matrix $T \in \mathbb{R}^{r\times r}$, and $A_{11} \in \mathbb{R}^{r\times r}$, $A_{12} \in \mathbb{R}^{r\times n-r}$, $A_{21} \in \mathbb{R}^{n-r\times r}$, $A_{22} \in \mathbb{R}^{n-r\times n-r}$. Then G is proper if and only if A_{22} is invertible.

Proof: Define

$$UB = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad CV = \begin{bmatrix} C_1 & C_2 \end{bmatrix}$$

with $B_1 \in \mathbb{R}^{r\times m}$, $B_2 \in \mathbb{R}^{n-r\times m}$, $C_1 \in \mathbb{R}^{p\times r}$, $C_2 \in \mathbb{R}^{p\times n-r}$. The system $(E; A, B, C, D)$ is assumed to have a C-controllable fast subsystem, so it follows that

$$\operatorname{rank} \begin{bmatrix} E & B \end{bmatrix} = \operatorname{rank} \begin{bmatrix} T & 0 & B_1 \\ 0 & 0 & B_2 \end{bmatrix} = n$$

which means that the matrix B_2 must have full row rank. By a similar argument,

$$\operatorname{rank} \begin{bmatrix} E \\ C \end{bmatrix} = \operatorname{rank} \begin{bmatrix} T & 0 \\ 0 & 0 \\ C_1 & C_2 \end{bmatrix} = n$$

holds and hence C_2 has to be a full column rank matrix. Now we write the transfer function of our descriptor system in terms of the transformed matrices, that is,

$$G(s) = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \underbrace{\begin{bmatrix} sT - A_{11} & -A_{12} \\ -A_{21} & -A_{22} \end{bmatrix}^{-1}}_{:=K(s)} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} + D.$$

Applying the Banachiewicz inversion formula for the inverse of a nonsingular partitioned matrix [21] yields the following result:

$$K(s) = \begin{bmatrix} Q(s) + Q(s)A_{12}S^{-1}(s)A_{21}Q(s) & Q(s)A_{12}S^{-1}(s) \\ S^{-1}(s)A_{21}Q(s) & S^{-1}(s) \end{bmatrix}$$

with $Q(s) = (sT - A_{11})^{-1}$ and the Schur complement $S(s) = -A_{22} - A_{21}Q(s)A_{12}$. Now we consider $\lim_{s\to\infty} K(s)$. First of all, we observe that $\lim_{s\to\infty} Q(s) = 0$ because the matrix pencil $sT - A_{11}$ does not have any infinite eigenvalues.

Assume the matrix A_{22} is invertible. Then $\lim_{s\to\infty} S^{-1}(s) = -A_{22}^{-1}$ holds and consequently

$$\lim_{s\to\infty} K(s) = \begin{bmatrix} 0 & 0 \\ 0 & -A_{22}^{-1} \end{bmatrix} \text{ and thus}$$
$$\lim_{s\to\infty} G(s) = D - C_2 A_{22}^{-1} B_2 \tag{11}$$

which means that G is proper.

Let now A_{22} be a singular matrix. The expression $Q(s)$ can be expanded into a Laurent series at $s = \infty$ (see, e.g., [5]) which yields $Q(s) = \sum_{i=-\infty}^{\infty} Q_i s^i$ for constant coefficients Q_i. Since $\lim_{s\to\infty} Q(s) = 0$ the matrices Q_i for $i \geq 0$ have to be zero, i.e., $Q(s) = \sum_{i=-\infty}^{-1} Q_i s^i$. Also, $S(s)$ can be expanded into a Laurent series at the expansion point $s = \infty$, i.e.,

$$S(s) = \sum_{i=-\infty}^{0} S_i s^i = -A_{22} - \sum_{i=-\infty}^{-1} A_{21} Q_i A_{12} s^i. \tag{12}$$

The expression on the right-hand side of (12) tends to $-A_{22}$ for $s \to \infty$. Since A_{22} is assumed to be singular, $\lim_{s\to\infty} \lambda_{\min}(S(s)) = 0$, where $\lambda_{\min}$ denotes the smallest eigenvalue in magnitude. Hence $\lim_{s\to\infty} \left|\lambda_{\max}\left(S^{-1}(s)\right)\right| = \infty$ where $\lambda_{\max}$ denotes the largest eigenvalue in magnitude. So $S^{-1}(s)$ has a Laurent series representation at the expansion point $s = \infty$. That is,

$$S^{-1}(s) = \sum_{i=-\infty}^{\infty} \tilde{S}_i s^i = \sum_{i=-\infty}^{d} \tilde{S}_i s^i. \tag{13}$$

Since $\lim_{s\to\infty} \left\|S^{-1}(s)\right\| = \infty$, the polynomial part of the Laurent series expansion (13) is non-zero, i.e., the degree $d \geq 1$. Consequently, the entries of $K(s)$ at the block positions (1, 1), (1, 2), and (2, 1) have a lower degree than the entry at block position (2, 2) because they contain $Q(s)$ (which has a degree less than zero) as a factor. Since the matrices B_2 and C_2 have full row and column rank, respectively, the product $C_2 S^{-1}(s) B_2$ has the same degree as $S^{-1}(s)$. So we can write the transfer function as $G(s) = H(s) + C_2 S^{-1}(s) B_2 + D$ where $H(s) := C_1\left(Q(s) + Q(s)A_{12}S^{-1}(s)A_{21}Q(s)\right)B_1 + C_1\left(Q(s)A_{12}S^{-1}(s)\right)B_2 + C_2\left(S^{-1}(s)A_{21}Q(s)\right)B_1$ contains only terms that have lower degree than S^{-1}. Since for $s \to \infty$ the maximum eigenvalue of S^{-1} tends to infinity in modulus, the maximum

TABLE I
CONVERGENCE HISTORY OF THE $\mathcal{L}_\infty$-NORM ALGORITHM FOR THE CONSTRAINED MASS-SPRING SYSTEM

iteration i	iterate $\gamma_{lb}(i)$	rel. error
1	0.15756820450368172	0.008983922316548
2	0.15899264646186639	$2.497728868261070 \cdot 10^{-5}$
3	0.15899661773662260	$1.865781701059065 \cdot 10^{-10}$
4	0.15899661776628779	$6.982683281081451 \cdot 10^{-16}$

singular value of S^{-1} tends to infinity, thus also $\lim_{s\to\infty} \sigma_{\max}(G(s)) = \infty$ which means that G is improper. ■

Note that when performing the properness test, we can directly compute $G(\infty)$ using formula (11). A numerical algorithm for testing a transfer function for properness consists of the following basic steps. First, we remove all uncontrollable or unobservable infinite poles of the system using the method from [22]. With this procedure it is always possible to fulfill the assumptions of Theorem 3. Second, we perform a URV decomposition [13] to transform the matrix E to the compressed form in (10), followed by updating the matrices A, B, and C. Finally we use an RRQR decomposition [23], to determine the rank of the submatrix A_{22} in (10). Other possible factorizations of the matrix pencil $\lambda E - A$ can also be used for the properness check.

IV. NUMERICAL EXAMPLE

For numerical tests we consider a damped mass-spring system with holonomic constraints with 5 masses which is described in [1]. The associated second-order equations can be linearized to a descriptor system with algebraic index 3. The results of the $\mathcal{L}_\infty$-norm computation are as follows. We obtain $\sigma_{\max}(G(0)) = 9.72222222222221683 \cdot 10^{-2}$, $\sigma_{\max}(G(\infty)) = 1.77630918782653439 \cdot 10^{-17}$, and $\sigma_{\max}(G(\mathrm{i}\omega_p)) = 0.15756820450368172$, where ω_p is chosen as suggested in Subsection II-C. In this way we already obtain a very good initial estimation of the $\mathcal{L}_\infty$-norm which is $\|G\|_{\mathcal{L}_\infty} \approx 0.15899661776628790$. In Table I we illustrate the convergence history and the relative error of the method for a relative accuracy of $\tau = 1000\varepsilon$. After already four iterations we obtain convergence. We also compared our results with those obtained by the MATLAB® R2010b function `norm`. Both results perfectly match for our example. We also compared the runtimes of the two algorithms. As shown in [19], the new eigenvalue solver is usually faster than the QZ algorithm and hence our implemetation outperforms the one of MATLAB®. For the properness test we obtain the following results. When removing all uncontrollable or unobservable infinite poles using the method from [22], the system's order reduces from $n = 11$ to $n_r = 6$ and we obtain a reduced system with a nonsingular descriptor matrix E. Therefore the matrix A_{22} in (10) is empty and Theorem 3 holds vacuously.

V. CONCLUSIONS AND OUTLOOK

We have presented an extended method for the computation of the $\mathcal{L}_\infty$-norm for descriptor systems with special emphasis on exploiting the structure of the involved skew-Hamiltonian/Hamiltonian matrix pencils. We have also shown how these matrix pencils can be extended to ensure reliability and improve the accuracy of the eigenvalue computation. In this way we could also improve the results of the $\mathcal{L}_\infty$-norm computation. We have also introduced a theoretical result which can be used to check whether a transfer function matrix is proper or improper. There are still open problems concerning the norm computation. First, we remark that we did not have a look on discrete-time systems in this article. In principle our method can also be applied to these but we then would have to deal with pencils with symplectic structure. However, there exist possibilities to transform these to more convenient structures, see e.g., [19], [24]. Second, one could still improve the convergence order of the algorithm using the method explained in [25]. Finally, as our algorithm has computational costs of $\mathcal{O}(n^3)$, it is not reasonable to apply it to large sparse systems. There exist iterative schemes to estimate the $\mathcal{L}_\infty$-norm for large sparse standard state-space systems [26]. Such a method is still unknown for descriptor systems.

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