$\mathcal{L}_\infty$-Norm Computation for Discrete-Time Descriptor Systems

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In this talk we discuss a numerical algorithm for the computation of the $\mathcal{L}_\infty$-norm of linear time-invariant discrete-time descriptor systems of the form

$$\begin{aligned} Ex_{k+1} &= Ax_k + Bu_k, \\ y_k &= Cx_k + Du_k, \end{aligned}$$

similarly to the continuous-time case, described in [1]. This norm plays important roles in model order reduction (as an error measure) or in optimal $\mathcal{H}_\infty$ control (as a robustness measure). We show that the $\mathcal{L}_\infty$-norm can be obtained by computing unit eigenvalues of certain matrix pencils with symplectic structure. To increase accuracy and reliablity of the eigenvalue computation we extend this eigenvalue problem to a symplectic eigenvalue problem of larger dimension. As there are no algorithms known which can compute the unit eigenvalues in a structure-preserving manner we apply the generalized Cayley transform [2] to obtain an even matrix pencil. We can transform this matrix pencil furthermore to a skew-Hamiltonian/Hamiltonian matrix pencil. Then, the unit eigenvalues of the symplectic matrix pencil are mapped to the purely imaginary eigenvalues of the skew-Hamiltonian/Hamiltonian matrix pencil which can be very accurately computed by the real-case version of the algorithm presented in [3]. Finally, we present first numerical results of our method.

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Discrete-Time Descriptor Systems

Given: Discrete-time LTI descriptor system

$$\Sigma : \quad \begin{cases} Ex_{k+1} = Ax_k + Bu_k, \\ y_k = Cx_k + Du_k, \end{cases}$$

- $E,\ A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, $D \in \mathbb{R}^{p\times m}$,
- descriptor vector $x_k \in \mathbb{R}^n$, input vector $u_k \in \mathbb{R}^m$, output vector $y_k \in \mathbb{R}^p$.
- Assumptions: $\lambda E - A$ is regular, i.e. $\det(\lambda E - A) \not\equiv 0$.

Z-Transform and Transfer Function

Apply the Z-transform

$$\mathcal{Z}\{f\}(z) = \sum_{k=0}^{\infty} f_k z^{-k}$$

to the signal sequences $x = \{x_k\}_{k=0}^{\infty}$, $u = \{u_k\}_{k=0}^{\infty}$, $y = \{y_k\}_{k=0}^{\infty}$.
This leads to

$$\mathcal{Z}(\Sigma) : \quad \begin{cases} zE\mathcal{Z}\{x\}(z) - Ex_0 = A\mathcal{Z}\{x\}(z) + B\mathcal{Z}\{u\}(z), \\ \mathcal{Z}\{y\}(z) = C\mathcal{Z}\{x\}(z) + D\mathcal{Z}\{u\}(z). \end{cases}$$

With $Ex_0 = 0$ we obtain

$$\mathcal{Z}\{y\}(z) = \underbrace{\left(C\,(zE - A)^{-1}\,B + D\right)}_{=:G(z) \text{ (transfer function)}} \mathcal{Z}\{u\}(z).$$

$\mathcal{L}_\infty$-Spaces and $\mathcal{L}_\infty$-Norm

Definition: the spaces $\mathcal{L}_\infty^{p\times m}(e^{i\omega})$ and $\mathcal{RL}_\infty^{p\times m}(e^{i\omega})$

- With $\mathcal{L}_\infty^{p\times m}(e^{i\omega})$ we denote the space of $p \times m$ matrix-valued transfer functions which are bounded on the unit circle.
- With $\mathcal{RL}_\infty^{p\times m}(e^{i\omega})$ we denote the rational subspace of $\mathcal{L}_\infty^{p\times m}(e^{i\omega})$. All its elements have a descriptor system realization of the form Σ.

Definition: $\mathcal{L}_\infty$-norm

Natural norm for the space $\mathcal{RL}_\infty^{p\times m}(e^{i\omega})$:

$$\|G\|_{\mathcal{L}_\infty} := \sup_{\omega\in[0,2\pi)} \sigma_{\max}\left(G\left(e^{i\omega}\right)\right).$$

Remark

For stable systems (all poles of $G(z)$ are inside the unit circle), the $\mathcal{L}_\infty$-norm is equivalent to the $\mathcal{H}_\infty$-norm.

Applications

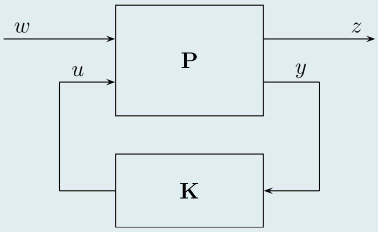
Model order reduction

Let $\hat{\Sigma} = (\lambda\hat{E} - \hat{A}, \hat{B}, \hat{C}, \hat{D})$ be a reduced order model of the system Σ. The transfer function of the error system $\Sigma^{err} := \Sigma - \hat{\Sigma}$ is given by

$$G^{err}(z) = \begin{bmatrix} C & -\hat{C} \end{bmatrix} \left(z \begin{bmatrix} E & 0 \\ 0 & \hat{E} \end{bmatrix} - \begin{bmatrix} A & 0 \\ 0 & \hat{A} \end{bmatrix} \right)^{-1} \begin{bmatrix} B \\ \hat{B} \end{bmatrix} + D - \hat{D}.$$

$\|G^{err}\|_{\mathcal{H}_\infty}$ is the size of the worst-case approximation error.

$\mathcal{H}_\infty$-control [Green, Limebeer '95]



- Plant **P**, dynamic compensator **K**,
- noise w, estimation error z,
- $\mathcal{H}_\infty$-norm of the transfer function from w to z displays the worst-case influence of the disturbances w on the output z.

Basic Theorems

Theorem [Hinrichsen, Son '91, Genin, Van Dooren, Vermaut '98, Voigt' 10]

Assume that the matrix pencil $\lambda E - A$ is regular and has no unitary eigenvalues, $\gamma > 0$ is not a singular value of D and $\omega_0 \in [0, 2\pi)$. Then, γ is a singular value of $G(e^{i\omega_0})$ if and only if $e^{i\omega_0}$ is an eigenvalue of the matrix pencil

$$\lambda N_\gamma - M_\gamma := \begin{bmatrix} \lambda E - A + BD^T S^{-1} C & -BB^T + BD^T S^{-1} DB^T \\ \lambda C^T S^{-1} C & E^T - \lambda A^T + \lambda C^T S^{-1} DB^T \end{bmatrix}$$

with $S := DD^T - \gamma^2 I_p$.

Corollary

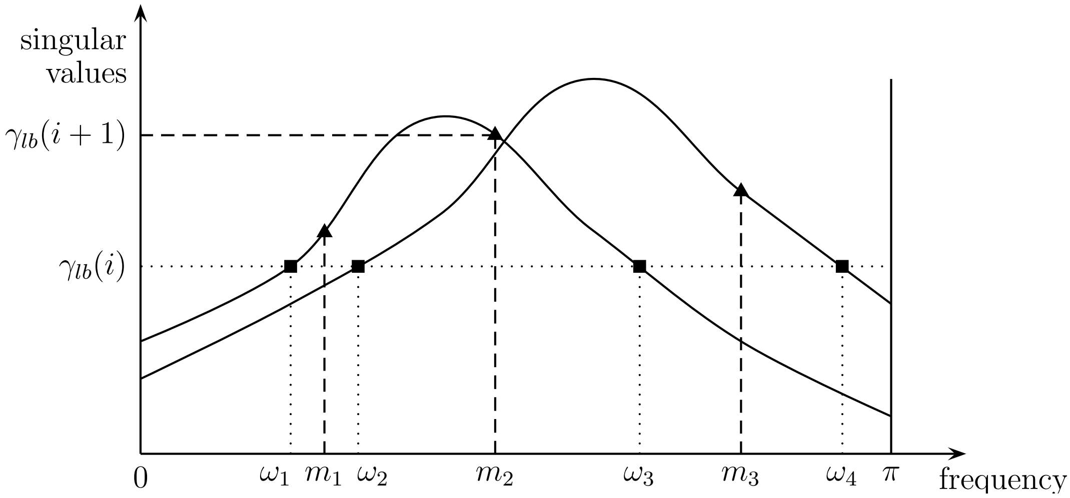
Let $\gamma > \min_{\omega \in [0,2\pi)} \sigma_{\max}\left(G\left(e^{i\omega}\right)\right)$ be not a singular value of D. Then, $\|G\|_{\mathcal{L}_\infty} \geq \gamma$ if and only if $\lambda N_\gamma - M_\gamma$ has unitary eigenvalues.

Sketch of the Algorithm [Bruinsma, Steinbuch '90]

Input: Discrete-time linear time-invariant descriptor system with transfer function $G \in \mathcal{RL}_\infty^{p \times m}\left(e^{i\omega}\right)$, tolerance ε.
Output: $\|G\|_{\mathcal{L}_\infty}$.

```
Compute an initial value γ_lb < ||G||_L∞.
repeat
  Set γ := (1 + ε)γ_lb.
  Compute the unitary eigenvalues of the matrix pencil λN_γ − M_γ.
  if no unitary eigenvalues then
    break.
  else
    Set {e^{iω_1}, ..., e^{iω_k}} = unitary eigenvalues with ω_j ∈ [0, π),
    j = 1, ..., k − 1.
    Set m_j = 1/2 (ω_j + ω_{j+1}), j = 1, ..., k − 1.
    Compute the largest singular value of G(e^{im_j}), j = 1, ..., k − 1.
    Set γ_lb = max_j σ_max(G(e^{im_j})).
  end if
until break
Set ||G||_L∞ = γ_lb.
```

Graphical Interpretation



Properties

Properties of the algorithm

- Monotonically converging,
- quadratic rate of convergence,
- relative error is at most ε (assuming exact arithmetic).
- The computing time is affected by number of frequency points in each step $\Longrightarrow$ generally, the last step takes less time than the first, good choice of initial value can reduce CPU time drastically.
- Care must be taken of the eigenvalue computation as it is important to catch all unitary eigenvalues $\Longrightarrow$ use certain matrix pencil transformations and a structure-preserving algorithm to compute the eigenvalues.

Choice of the Initial Value γ_{lb}

1. Evaluate $G(z)$ at the boundary frequencies $\omega = 0$ and $\omega = \pi$, i.e. compute $\sigma_{\max}\left(G\left(e^{i\cdot 0}\right)\right)$ and $\sigma_{\max}\left(G\left(e^{i\pi}\right)\right)$.
2. Evaluate $G(z)$ at more heuristic test frequencies: [SIMA '06]
   1. Compute the logarithms of the finite eigenvalues $\lambda_j := r_j e^{i\omega_j}$ of $\lambda E - A$, i.e., $\ln \lambda_j = \ln r_j + i\omega_j =: \nu_j + i\omega_j =: \psi_j$. Eigenvalues close to the unit circle are mapped to eigenvalues close to the imaginary axis!
   2. Compute $\sigma_{\max}\left(G\left(e^{i\mu_j}\right)\right)$, where

$$\mu_j := \max\left\{\frac{1}{4}\left|\psi_j\right|^2, \omega_j^2 - \nu_j^2\right\}^{\frac{1}{2}} \quad \text{for } \omega_j \in (0,\pi)$$

$$= |\psi_j| \max\left\{\frac{1}{4}, 1 - 2s_j^2\right\}^{\frac{1}{2}} \quad \text{with } s_j := \frac{\nu_j}{|\psi_j|} \quad \text{for } \omega_j \in (0,\pi).$$

   3. Set $\omega_p := \operatorname{argmax} \sigma_{\max}\left(G\left(e^{i\mu_j}\right)\right)$.
3. Set $\gamma_{lb} := \max\left\{\sigma_{\max}\left(G\left(e^{i\cdot 0}\right)\right), \sigma_{\max}\left(G\left(e^{i\omega_p}\right)\right), \sigma_{\max}\left(G\left(e^{i\pi}\right)\right)\right\}$.

The Eigenvalue Problem

Reminder: original matrix pencil

$$\lambda N_\gamma - M_\gamma := \begin{bmatrix} \lambda E - A + BD^T S^{-1} C & -BB^T + BD^T S^{-1} DB^T \\ \lambda C^T S^{-1} C & E^T - \lambda A^T + \lambda C^T S^{-1} DB^T \end{bmatrix},$$
$$S = DD^T - \gamma^2 I_p$$

Structural property

Symplectic eigensymmetry, i.e., if λ is an eigenvalue, also $\bar{\lambda}^{-1}$ is an eigenvalue (and therefore $\bar{\lambda}$ and λ^{-1}).

Further remarks

- If γ is close to a singular value of D, S is ill-conditioned,
- forming "matrix-times-its-transpose" products numerically unstable.
  $\Longrightarrow$ Explicitly forming $\lambda N_\gamma - M_\gamma$ must be avoided!

An Extended Eigenvalue Problem

Multiply $\lambda N_\gamma - M_\gamma$ by $P = \begin{bmatrix} I_n & 0 \\ 0 & -\gamma I_n \end{bmatrix}$ from the left and by $Q = \begin{bmatrix} I_n & 0 \\ 0 & \frac{1}{\gamma} I_n \end{bmatrix}$ from the right in order to get the matrix pencil

$$\lambda \tilde{N}_\gamma - \tilde{M}_\gamma = \begin{bmatrix} \lambda E - A & 0 \\ 0 & \lambda A^T - E^T \end{bmatrix} - \begin{bmatrix} B & 0 \\ 0 & -\lambda C^T \end{bmatrix} \begin{bmatrix} -D & \gamma I_p \\ \gamma I_m & -D^T \end{bmatrix}^{-1} \begin{bmatrix} C & 0 \\ 0 & B^T \end{bmatrix}.$$

Exploit Schur complement structure and get the extended matrix pencil

$$\lambda \mathcal{N} - \mathcal{M}_\gamma = \lambda \left[\begin{array}{cc|cc} E & 0 & 0 & 0 \\ 0 & A^T & 0 & C^T \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] - \left[\begin{array}{cc|cc} A & 0 & B & 0 \\ 0 & E^T & 0 & 0 \\ \hline C & 0 & D & -\gamma I_p \\ 0 & B^T & -\gamma I_m & D^T \end{array}\right]$$

which has the same eigenvalues as $\lambda N_\gamma - M_\gamma$ with additional infinite eigenvalues.

Recovery of Structure

By performing signed block row and column permutations we get the D-type matrix pencil

$$\lambda \hat{\mathcal{N}} - \hat{\mathcal{M}}_\gamma = \lambda \left[\begin{array}{c|ccc} 0 & -E^T & 0 & 0 \\ \hline A & 0 & 0 & 0 \\ C & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] - \left[\begin{array}{c|ccc} 0 & -A^T & -C^T & 0 \\ \hline E & 0 & 0 & B \\ 0 & 0 & -\gamma I_p & D \\ 0 & B^T & D^T & -\gamma I_m \end{array}\right].$$

D-type matrix pencils [Xu '06]

- have the general form $\lambda \mathcal{E}_D - \mathcal{A}_D = \lambda \begin{bmatrix} 0 & F \\ -G^T & 0 \end{bmatrix} - \begin{bmatrix} 0 & G \\ -F^T & H \end{bmatrix}$ with symmetric H,
- have symplectic eigenstructure + additional infinite eigenvalues.

Generalized Cayley Transforms

Definition: generalized Cayley transform

$$\mathbf{c}(\mathcal{A},\mathcal{E}) := \lambda\,(\mathcal{A}+\mathcal{E}) - (\mathcal{A}-\mathcal{E})\,.$$

Application to D-type matrix pencils [Xu '06]

$$\lambda\tilde{\mathcal{E}} - \tilde{\mathcal{A}} := \mathbf{c}(\mathcal{A}_D,\mathcal{E}_D) = \lambda \begin{bmatrix} 0 & G+F \\ -G^T-F^T & H \end{bmatrix} - \begin{bmatrix} 0 & G-F \\ G^T-F^T & H \end{bmatrix}.$$

Properties of $\lambda\tilde{\mathcal{E}} - \tilde{\mathcal{A}}$:

- Hamiltonian eigensymmetry, i.e., if λ is an eigenvalue, also $-\bar{\lambda}$ is an eigenvalue (and therefore $\bar{\lambda}$ and $-\lambda$), and additional eigenvalues 1,
- unitary eigenvalues of $\lambda\mathcal{E}_D - \mathcal{A}_D$ are mapped to purely imaginary eigenvalues of $\lambda\tilde{\mathcal{E}} - \tilde{\mathcal{A}}$.

Transformation to an Even Matrix Pencil

Problem

$\lambda\tilde{\mathcal{E}} - \tilde{\mathcal{A}}$ has a structure that we cannot exploit.

Solution: Additional drop/add transformation [Xu '06]

$$\begin{aligned}\lambda\mathcal{E}_C - \mathcal{A}_C :&= \mathbf{d}\,(\tilde{\mathcal{A}},\tilde{\mathcal{E}}) \\ &= \begin{bmatrix} (1-\lambda)I & 0 \\ 0 & I \end{bmatrix} (\lambda\tilde{\mathcal{E}} - \tilde{\mathcal{A}}) \begin{bmatrix} I & 0 \\ 0 & \frac{1}{1-\lambda}I \end{bmatrix} \\ &= \lambda \begin{bmatrix} 0 & G+F \\ -G^T-F^T & 0 \end{bmatrix} - \begin{bmatrix} 0 & G-F \\ G^T-F^T & H \end{bmatrix}.\end{aligned}$$

Properties of the C-type matrix pencil $\lambda\mathcal{E}_C - \mathcal{A}_C$:

- even structure (i.e., $\mathcal{E}_C = -\mathcal{E}_C^T$, $\mathcal{A}_C = \mathcal{A}_C^T$), Hamiltonian spectrum,
- **d**-transformation is λ-dependent with poles for $\lambda = 1, \infty$
  $\Longrightarrow$ multiplicities of these eigenvalues may have changed.

Application to Our Problem

Application of $\mathbf{d}(\mathbf{c}(\cdot))$ to our problem

$$\lambda\bar{\mathcal{N}} - \bar{\mathcal{M}}_\gamma = \lambda \left[\begin{array}{c|ccc} 0 & -A^T - E^T & -C^T & 0 \\ \hline A+E & 0 & 0 & 0 \\ C & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] - \left[\begin{array}{c|ccc} 0 & -A^T + E^T & -C^T & 0 \\ \hline -A+E & 0 & 0 & B \\ -C & 0 & -\gamma I_p & D \\ 0 & B^T & D^T & -\gamma I_m \end{array}\right].$$

We transform the eigenvalue problem into skew-Hamiltonian/Hamiltonian structure to use a structure-preserving algorithm.

Transformation to sH/H Structure [Voigt '10]

Introduce artificial inputs or outputs to the descriptor system to get the same number of inputs and outputs. Set $q = \max\{m, p\}$ and append B, C, D, by an appropriate amount of zero rows or columns. We denote the extended matrices by $\tilde{B}$, $\tilde{C}$, $\tilde{D}$ and get

$$\lambda\tilde{\mathcal{N}} - \tilde{\mathcal{M}}_\gamma = \lambda \left[\begin{array}{c|ccc} 0 & -A^T - E^T & -\tilde{C}^T & 0 \\ \hline A+E & 0 & 0 & 0 \\ \tilde{C} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] - \left[\begin{array}{c|ccc} 0 & -A^T + E^T & -\tilde{C}^T & 0 \\ \hline -A+E & 0 & 0 & \tilde{B} \\ -\tilde{C} & 0 & -\gamma I_q & \tilde{D} \\ 0 & \tilde{B}^T & \tilde{D}^T & -\gamma I_q \end{array}\right].$$

Transformation to sH/H Structure [Voigt '10]

Perform signed block row and column transformations to get

$$\lambda\tilde{\mathcal{N}} - \tilde{\mathcal{M}}_\gamma = \lambda \left[\begin{array}{cc|cc} A+E & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & \tilde{C}^T & A^T+E^T & 0 \\ -\tilde{C} & 0 & 0 & 0 \end{array}\right] - \left[\begin{array}{cc|cc} -A+E & 0 & 0 & \tilde{B} \\ 0 & \tilde{D}^T & \tilde{B}^T & -\gamma I_q \\ \hline 0 & \tilde{C}^T & A^T-E^T & 0 \\ \tilde{C} & \gamma I_q & 0 & -\tilde{D} \end{array}\right].$$

$\lambda\tilde{\mathcal{N}} - \tilde{\mathcal{M}}_\gamma$ is a skew-Hamiltonian/Hamiltonian matrix pencil!

Skew-Hamiltonian/Hamiltonian Matrix Pencils

Definition and properties

Let $\mathcal{J} := \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$. The matrix pencil $\lambda\mathcal{S} - \mathcal{H} \in \mathbb{R}^{2n\times 2n}$ is skew-Hamiltonian/Hamiltonian if $\mathcal{S}$ is skew-Hamiltonian ($(\mathcal{S}\mathcal{J})^T = -\mathcal{S}\mathcal{J}$) and $\mathcal{H}$ is Hamiltonian ($(\mathcal{H}\mathcal{J})^T = \mathcal{H}\mathcal{J}$).

Block structure: $\lambda\mathcal{S} - \mathcal{H} = \lambda \begin{bmatrix} F & G \\ H & F^T \end{bmatrix} - \begin{bmatrix} R & S \\ T & -R^T \end{bmatrix}$ with skew-symmetric G, H, and symmetric S, T.

Structure-preserving method [Benner, Byers, Mehrmann, Xu '99]

- uses special orthogonal transformations on an embedded matrix pencil to get a skew-Hamiltonian/Hamiltonian Schur-like form,
- simple, finite, purely imaginary eigenvalues stay on the imaginary axis $\Longrightarrow$ robust detection of interesting eigenvalues!

Example Description and Results

Example description

Example from modeling a mass-spring-damper system with:

- $n = 11$ descriptor variables, $m = 1$ input, $p = 1$ output,
- $\lambda E - A$ has 8 finite and 3 infinite eigenvalues.

Computation of the initial value

- $\sigma_{\max}\left(G\left(e^{i\cdot 0}\right)\right) = 4.14620127792975600 \cdot 10^{-3}$,
- $\sigma_{\max}\left(G\left(e^{i\pi}\right)\right) = 5.50049659572314498 \cdot 10^{-3}$.

Other test frequencies:

ω	$\sigma_{\max}\left(G\left(e^{i\omega}\right)\right)$
1.572768	$5.23947523458074063 \cdot 10^{-3}$
1.250212	$4.91055265529189832 \cdot 10^{-3}$
1.276012	$4.93766502456469050 \cdot 10^{-3}$
1.250212	$4.91055265529190613 \cdot 10^{-3}$

γ-Iteration

Desired accuracy: $\varepsilon = 1000\mathbf{u}$ ($\mathbf{u}$ = machine precision).

iteration	γ_{lb}	imag. eig. of $\lambda\tilde{\mathcal{N}} - \tilde{\mathcal{M}}_\gamma$
1	$5.50049659572314498 \cdot 10^{-3}$	1.431624i 633862.192397i
2	$5.57395403422781491 \cdot 10^{-3}$	1.791283i 3.177910i
3	$5.59298056369935456 \cdot 10^{-3}$	2.202331i 2.319735i
4	$5.59316018833811313 \cdot 10^{-3}$	2.258739i 2.259758i
5	$5.59316020191237984 \cdot 10^{-3}$	none

Results

- $\|G\|_{\mathcal{L}_\infty} = 5.59316020191237984 \cdot 10^{-3}$ (last 4 digits are insecure),
- peak frequency: $\omega_p = 2.3081845795498692$.

Conclusions and Outlook

What we have done

- $\mathcal{L}_\infty$-norm computation for discrete-time descriptor systems by using structured matrix pencils,
- matrix pencil transformations in order to use a structure-preserving algorithm to compute the eigenvalues.

Future work: $\mathcal{L}_\infty$-norm computation for large-scale systems

Idea: We compute a reduced system via balanced truncation and use the algorithm presented here on the reduced system to get initial values for an iterative scheme.

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