On the Computation of Particular Eigenvectors of Hamiltonian Matrix Pencils

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We discuss a structure-preserving algorithm for the accurate solution of generalized eigenvalue problems for skew-Hamiltonian/Hamiltonian matrix pencils $\lambda\mathcal{N} - \mathcal{H}$. By embedding the matrix pencil $\lambda\mathcal{N} - \mathcal{H}$ into a skew-Hamiltonian/Hamiltonian matrix pencil of double size it is possible to avoid the problem of non-existence of a structured Schur form. For these embedded matrix pencils we can compute a particular condensed form to accurately compute the simple, finite, purely imaginary eigenvalues of $\lambda\mathcal{N} - \mathcal{H}$. In this paper we describe a new method to compute also the corresponding eigenvectors by using the information contained in the condensed form of the embedded matrix pencils and associated transformation matrices.

1 Introduction

Skew-Hamiltonian/Hamiltonian matrix pencils arise in many control applications for linear time-invariant descriptor systems such as linear-quadratic optimal control, optimal $\mathcal{H}_\infty$ control, passivity check, or passivity enforcement [1]. In these applications we need different spectral information, i.e., a certain subset of eigenvalues and associated deflating subspaces or eigenvectors. In this paper we focus on the computation of the simple, finite, purely imaginary eigenvalues and associated eigenvectors which are, e.g., needed for passivity enforcement of descriptor systems. Details of that application will be reported elsewhere, for the standard system case we refer, e.g., to [2].

2 Computation of the Purely Imaginary Eigenvalues

A structure-preserving algorithm for computing the purely imaginary eigenvalues in a very accurate and reliable manner is presented in [1]. We summarize the basic ideas since we need these for the eigenvector computation as well. Let $\lambda\mathcal{N} - \mathcal{H} \in \mathbb{R}^{2n\times 2n}$ be a given skew-Hamiltonian/Hamiltonian matrix pencil (see [1] for the definition and structural properties). To avoid the problem of the non-existence of a structured Schur form in the case when simple, finite, purely imaginary eigenvalues exist, we define the double-sized matrix pencil $\lambda\mathcal{B}_\mathcal{N} - \mathcal{B}_\mathcal{H} := \lambda \left[\begin{smallmatrix} \mathcal{N} & 0 \\ 0 & \mathcal{N} \end{smallmatrix}\right] - \left[\begin{smallmatrix} \mathcal{H} & 0 \\ 0 & -\mathcal{H} \end{smallmatrix}\right]$, whose eigenvalues now all have even algebraic multiplicity. To restore the skew-Hamiltonian/Hamiltonian structure we perform an appropriate orthogonal equivalence transformation $\lambda\tilde{\mathcal{B}}_\mathcal{N} - \tilde{\mathcal{B}}_\mathcal{H} := \mathcal{X}^T \left(\lambda\mathcal{B}_\mathcal{N} - \mathcal{B}_\mathcal{H}\right)\mathcal{X}$. By using the *generalized symplectic URV decomposition* [1] of $\lambda\mathcal{N} - \mathcal{H}$ we can compute an orthogonal matrix $\mathcal{Q}$ such that

$$\lambda\bar{\mathcal{B}}_\mathcal{N} - \bar{\mathcal{B}}_\mathcal{H} := \mathcal{J}\mathcal{Q}^T\mathcal{J}^T\left(\lambda\tilde{\mathcal{B}}_\mathcal{N} - \tilde{\mathcal{B}}_\mathcal{H}\right)\mathcal{Q} = \lambda\left[\begin{array}{cc|cc} N_1 & 0 & N_2 & 0 \\ 0 & M_1 & 0 & M_2 \\ \hline 0 & 0 & N_1^T & 0 \\ 0 & 0 & 0 & M_1^T \end{array}\right] - \left[\begin{array}{cc|cc} 0 & H_{11} & 0 & H_{12} \\ -H_{22}^T & 0 & H_{12}^T & 0 \\ \hline 0 & 0 & 0 & H_{22} \\ 0 & 0 & -H_{11}^T & 0 \end{array}\right],$$

where $\mathcal{J} := \left[\begin{smallmatrix} 0 & I_{2n} \\ -I_{2n} & 0 \end{smallmatrix}\right]$, N_1, M_1, H_{11} are upper triangular, and H_{22}^T is upper quasi triangular. Note that this kind of transformation also preserves the skew-Hamiltonian/Hamiltonian structure. Now, the spectrum of $\lambda\mathcal{N} - \mathcal{H}$ is given by $\Lambda(\mathcal{H},\mathcal{N}) = \pm\mathrm{i}\sqrt{\Lambda(N_1^{-1}H_{11}M_1^{-1}H_{22}^T)}$, where $N_1^{-1}H_{11}M_1^{-1}H_{22}^T$ is understood as a formal matrix product in the sense of [3]. The simple, finite, purely imaginary eigenvalues of $\lambda\mathcal{N} - \mathcal{H}$ are given by the 1×1 diagonal blocks of this matrix product. Hence they do not experience any error in their real parts and can be detected in a very reliable way.

3 Computation of the Corresponding Eigenvectors

To compute the eigenvectors corresponding to the purely imaginary eigenvalues we will make use of the structure of $\lambda\bar{\mathcal{B}}_\mathcal{N} - \bar{\mathcal{B}}_\mathcal{H}$. As in passivity enforcement we only need the purely imaginary eigenvalues with positive imaginary parts, we restrict ourselves to the computation of the eigenvectors corresponding to these eigenvalues. In the sequel we denote by $\mathrm{Eig}_{\Lambda_0}(\mathcal{A},\mathcal{B})$

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a matrix whose columns are the eigenvectors of the matrix pencil $\lambda\mathcal{B}-\mathcal{A}$ associated to all eigenvalues contained in the set Λ_0. As an intermediate step, we compute $\mathrm{Eig}_{\mathbb{R}+}\left(\begin{bmatrix} 0 & H_{11} \\ H_{22}^T & 0 \end{bmatrix}, \begin{bmatrix} N_1 & 0 \\ 0 & M_1 \end{bmatrix}\right)$ which is done by the following steps. First, we reorder the positive real eigenvalues of the generalized matrix product $P := N_1^{-1}H_{11}M_1^{-1}H_{22}^T$ to the top, i.e. we compute orthogonal matrices $U_i = \begin{bmatrix} U_i^{(1)} & U_i^{(2)} \end{bmatrix}$, $i = 1, \ldots, 4$, such that

$$U_2^T N_1 U_1 = \begin{bmatrix} N_1^{(11)} & N_1^{(12)} \\ 0 & N_1^{(22)} \end{bmatrix}, \quad U_2^T H_{11} U_3 = \begin{bmatrix} H_{11}^{(11)} & H_{11}^{(12)} \\ 0 & H_{11}^{(22)} \end{bmatrix},$$
$$U_4^T M_1 U_3 = \begin{bmatrix} M_1^{(11)} & M_1^{(12)} \\ 0 & M_1^{(22)} \end{bmatrix}, \quad U_4^T H_{22}^T U_1 = \begin{bmatrix} H_{22}^{(11)} & H_{22}^{(12)} \\ 0 & H_{22}^{(22)} \end{bmatrix}$$

are still in upper (quasi) triangular form, but the eigenvalues of the generalized matrix product $P^{(11)} := \left(N_1^{(11)}\right)^{-1} H_{11}^{(11)} \left(M_1^{(11)}\right)^{-1} H_{22}^{(11)}$ are the positive real ones of P [3]. However, the eigenvalues of $\lambda \begin{bmatrix} N_1^{(11)} & 0 \\ 0 & M_1^{(11)} \end{bmatrix} - \begin{bmatrix} 0 & H_{11}^{(11)} \\ H_{22}^{(11)} & 0 \end{bmatrix}$ are the positive and negative square roots of the eigenvalues of $P^{(11)}$ and thus we triangularize this matrix pencil and reorder its eigenvalues by computing orthogonal matrices $V_1 = \begin{bmatrix} V_1^{(1)} & V_1^{(2)} \end{bmatrix}$, $V_2 = \begin{bmatrix} V_2^{(1)} & V_2^{(2)} \end{bmatrix}$ such that

$$V_1^T \left(\lambda \begin{bmatrix} N_1^{(11)} & 0 \\ 0 & M_1^{(11)} \end{bmatrix} - \begin{bmatrix} 0 & H_{11}^{(11)} \\ H_{22}^{(11)} & 0 \end{bmatrix} \right) V_2 = \lambda \begin{bmatrix} R_{11} & R_{12} \\ 0 & R_{22} \end{bmatrix} - \begin{bmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{bmatrix},$$

where $\Lambda\left(S_{11}, R_{11}\right) \subset \mathbb{R}^+$ and $\Lambda\left(S_{22}, R_{22}\right) \subset \mathbb{R}^-$. Now, we can compute the eigenvectors of $\lambda R_{11} - S_{11}$, i.e., we compute a matrix W such that $S_{11}W = R_{11}WD$, where D is an appropriate diagonal matrix composed of the eigenvalues of $\lambda R_{11} - S_{11}$. By collecting the information contained in the relevant columns of the transformation matrices, we obtain

$$X := \begin{bmatrix} X^{(1)} \\ X^{(2)} \end{bmatrix} := \begin{bmatrix} U_1^{(1)} & 0 \\ 0 & U_3^{(1)} \end{bmatrix} V_2^{(1)} W = \mathrm{Eig}_{\mathbb{R}+}\left(\begin{bmatrix} 0 & H_{11} \\ H_{22}^T & 0 \end{bmatrix}, \begin{bmatrix} N_1 & 0 \\ 0 & M_1 \end{bmatrix} \right).$$

It turns out that

$$\tilde{X} := \begin{bmatrix} -\mathrm{i}X^{(1)} \\ X^{(2)} \end{bmatrix} = \mathrm{Eig}_{\mathrm{i}\mathbb{R}+}\left(\begin{bmatrix} 0 & H_{11} \\ -H_{22}^T & 0 \end{bmatrix}, \begin{bmatrix} N_1 & 0 \\ 0 & M_1 \end{bmatrix} \right), \quad \text{and} \quad \begin{bmatrix} \tilde{X} \\ 0 \end{bmatrix} = \mathrm{Eig}_{\mathrm{i}\mathbb{R}+}(\bar{\mathcal{B}}_{\mathcal{H}}, \bar{\mathcal{B}}_{\mathcal{N}}).$$

Furthermore we have

$$Y := \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \mathcal{X}\mathcal{Q} \begin{bmatrix} \tilde{X} \\ 0 \end{bmatrix} = \mathrm{Eig}_{\mathrm{i}\mathbb{R}+}\left(\mathcal{B}_{\mathcal{H}}, \mathcal{B}_{\mathcal{N}}\right).$$

Due to the structure of $\lambda\mathcal{B}_{\mathcal{N}} - \mathcal{B}_{\mathcal{H}}$ we can retrieve the seeked eigenvectors from the first half of the rows of Y, i.e., Y_1.

4 Numerical Example

To test our method we use a 28×28 skew-Hamiltonian/Hamiltonian matrix pencil with 4 simple, finite, purely imaginary eigenvalues with positive imaginary parts. The results are listed in Table 1. They show that we can compute the eigenvectors slightly more accurate than standard methods. It can be also shown that the computational costs of the new method are slightly lower than those of existing algorithms.

eigenvalues	residuals new method	residuals MATLAB `eig`
0.310929i	1.2932e-15	2.1913e-15
0.325141i	8.8045e-16	1.8314e-15
0.378019i	6.7260e-16	1.5523e-15
0.630326i	1.1161e-15	1.5523e-15

Table 1 Relative residuals of the new method compared to `eig` from MATLAB

References

[1] P. Benner, R. Byers, V. Mehrmann, and H. Xu, Numerical Computation of Deflating Subspaces of Embedded Hamiltonian Pencils, Tech. rep., Chemnitz University of Technology, Faculty of Mathematics, Germany, June 1999, SFB393-Preprint 99-15.
[2] S. Grivet-Talocia and A. Ubolli, IEEE Trans. Adv. Pack. **29**(1), 39–54 (2006).
[3] R. Granat, B. Kågström, and D. Kressner, BIT Numerical Mathematics **43**(1), 1–18 (2003).