Numerical Computation of Structured Complex Stability Radii of Large-Scale Matrices and Pencils

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***Abstract*— In this paper we discuss the problem of computing structured complex stability radii of large and sparse matrices and pencils. For this purpose we consider certain structured pseudospectra. To compute the structured complex stability radius we have to find the pseudospectrum which touches the imaginary axis. Therefore, we set up an iteration over the real part of the rightmost pseudoeigenvalue. For that we use a new fast iterative scheme which is based on certain rank-1 perturbations of the matrix or pencil. Finally, we illustrate the performance of our algorithm by using real-world example data.**

I. Introduction

One of the most important properties of dynamical systems is stability. However, even if a system is stable it might be close to instability. This situation might be dangerous, e.g., in the design of feedback controllers. The models we consider are only approximations to real-world phenomena, i.e., they contain modeling errors. The question that arises is whether the feedback controller we have designed based on the model indeed stabilizes the dynamics of the real-world problem. In other words, the question is how robust is the controller with respect to perturbations; or how much can we perturb the controller such that the closed-loop system is still stable?

We consider linear time-invariant systems of the form

$$\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t),$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, and $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the control input vector, and $y(t) \in \mathbb{R}^p$ is the output vector. We furthermore assume that all involved matrices are large and sparse and that $m, p \ll n$. Such systems are called *(asymptotically) stable* if and only if the matrix A is stable, i.e., $\Lambda(A) \subset \mathbb{C}^- := \{z \in \mathbb{C} : \operatorname{Re} z < 0\}$, where $\Lambda(A)$ denotes the spectrum of A. To measure the robustness of the system against perturbations, i.e., the "distance to instability", unstructured and structured stability radii have been introduced in [1]–[3]. In this paper we focus on structured stability radii which measure the robustness of a closed-loop system

$$\dot{x}(t) = A_K x(t) := (A + BKC)x(t),$$

with respect to perturbations in the controller K. These perturbations lead to the closed-loop system

$$\begin{aligned}\dot{x}(t) = A_{K+\Delta}x(t) :&= (A + B(K + \Delta)C)x(t)\\ &= (A_K + B\Delta C)x(t),\end{aligned}$$

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where $\Delta \in \mathbb{C}^{m\times p}$ is the perturbation matrix. Therefore, the *structured complex stability radius* of the stable matrix A (we replace A_K by A for simplicity) with respect to B and C is defined by

$$r_{\mathbb{C}}(A, B, C) := \inf\left\{\|\Delta\|_2 : \Lambda(A + B\Delta C) \cap \mathrm{i}\mathbb{R} \neq \emptyset \text{ with } \Delta \in \mathbb{C}^{m\times p}\right\}.$$

Note, that structured stability radii can be defined for arbitrary norms of the perturbation Δ. In this paper we stick to the 2-norm $\|\cdot\|_2$ as it is of largest practical relevance.

We also consider generalizations of $r_{\mathbb{C}}(A, B, C)$ for *matrix pencils* $\lambda E - A$. These pencils typically arise in the analysis of dynamical systems of the form

$$E\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t), \tag{1}$$

where $E \in \mathbb{R}^{n\times n}$ is allowed to be a singular matrix. However, we assume that $\lambda E - A$ is *regular*, i.e., $\det(\lambda E - A) \not\equiv 0$. In this context we speak about *differential-algebraic equations* or *descriptor systems* [4], [5]. Assume that $\lambda E - A$ is *stable*, i.e., $\Lambda_{\mathrm{f}}(E, A) \subset \mathbb{C}^-$, where $\Lambda_{\mathrm{f}}(E, A)$ denotes the *finite spectrum* of $\lambda E - A$. The definition of the structured complex stability radius $r_{\mathbb{C}}(E, A, B, C)$ for such matrix pencils is rather involved and will therefore be discussed in the next section.

We briefly discuss associated concepts in the frequency domain. Consider the *transfer function* of (1), defined by

$$G(z) = C(zE - A)^{-1}B. \tag{2}$$

We call $G(z)$ *proper* if $\lim_{\omega\to\infty} \|G(\mathrm{i}\omega)\|_2 < \infty$, otherwise we call it *improper*. By $\mathcal{RH}_\infty^{p\times m}$ we denote the rational Banach space of all stable and proper transfer functions of the form (2), see [6]. For this space we define the $\mathcal{H}_\infty$*-norm*, given by

$$\|G\|_{\mathcal{H}_\infty} := \sup_{z\in\mathbb{C}^+} \sigma_{\max}(G(z)) = \sup_{\omega\in\mathbb{R}} \sigma_{\max}(G(\mathrm{i}\omega)),$$

with $\mathbb{C}^+ := \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$. In [7] we show that the relation

$$\|G\|_{\mathcal{H}_\infty} = \begin{cases} 1/r_{\mathbb{C}}(E, A, B, C), & \text{if } G \not\equiv 0,\\ \infty, & \text{if } G \equiv 0\end{cases} \tag{3}$$

holds.

Numerical methods for computing structured stability radii (and equivalently $\mathcal{H}_\infty$-norms) have been known for a long time. Most of them are based on relations between stability radii and the spectrum of certain Hamiltonian matrices or pencils. For an overview, we refer to [8]–[13].

The most general algorithm [12] iterates over certain *skew-Hamiltonian/Hamiltonian matrix pencils* and determines the finite, purely imaginary eigenvalues in each step. The drawback of the algorithm is the decision whether there are such eigenvalues. It is important to find *all* of them since otherwise the algorithm could fail. In [12] this issue was addressed by using a structure-preserving eigensolver (real-case version of [14]), which prevents the finite, purely imaginary eigenvalues to move off the imaginary axis as long as their pairwise distance is sufficiently large. However, this method computes a full structured factorization of the pencil in each step. Due to its cubic complexity it is infeasible for large-scale problems. We could consider a closely related extended *even matrix pencil* and try to compute the desired eigenvalues close to a specified shift with the method presented in [15]. However, the question remains how to reliably compute *all* finite, purely imaginary eigenvalues. Therefore, in this paper we use another approach based on pseudospectra. Recently, also a similar method based on optimization over so-called spectral value sets was developed [16]. However, it does not take descriptor systems into account and uses a different optimization technique.

II. Structured Complex Stability Radii for Matrix Pencils

We turn now to the definition of the structured complex stability radius for matrix pencils which requires the introduction of several mathematical and control theoretic terms.

A regular matrix pencil $\lambda E - A$ can be reduced to *Weierstraß canonical form* [5]

$$E = W \begin{bmatrix} I_{n_\mathrm{f}} & 0 \\ 0 & N \end{bmatrix} T, \quad A = W \begin{bmatrix} J & 0 \\ 0 & I_{n_\infty} \end{bmatrix} T, \tag{4}$$

where W and T are nonsingular, I_k is an identity matrix of order k, J and N are in Jordan canonical form and N is nilpotent with index of nilpotency ν. The number ν is also called the *algebraic index* of the descriptor system (1) and n_f and n_∞ are the dimensions of the deflating subspaces of $\lambda E - A$ corresponding to the finite and infinite eigenvalues, respectively. We say that a finite or infinite eigenvalue is *defective*, if one of the corresponding Jordan blocks in J or N is larger than one. By using the transformation matrices W and T, we can also write B and C as

$$B = W \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad C = \begin{bmatrix} C_1 & C_2 \end{bmatrix} T,$$

to obtain an equivalent descriptor system.

Furthermore we need some controllability and observability concepts [4]. The descriptor system (1) is called

- *completely controllable*, if $\operatorname{rank} \begin{bmatrix} \lambda E - A & B \end{bmatrix} = n$ for all $\lambda \in \mathbb{C}$ and furthermore $\operatorname{rank} \begin{bmatrix} E & B \end{bmatrix} = n$;
- *completely observable*, if $\operatorname{rank} \begin{bmatrix} \lambda E - A \\ C \end{bmatrix} = n$ for all $\lambda \in \mathbb{C}$ and furthermore $\operatorname{rank} \begin{bmatrix} E \\ C \end{bmatrix} = n$.

We can also define these concepts for single eigenvalues of $\lambda E - A$. A descriptor system (1) is called

- *controllable at* $\lambda \in \mathbb{C}$ if $\operatorname{rank} \begin{bmatrix} \lambda E - A & B \end{bmatrix} = n$;
- *controllable at* $\lambda = \infty$ if $\operatorname{rank} \begin{bmatrix} E & B \end{bmatrix} = n$;
- *observable* at $\lambda \in \mathbb{C}$ if $\operatorname{rank} \begin{bmatrix} \lambda E - A \\ C \end{bmatrix} = n$;
- *observable* at $\lambda = \infty$ if $\operatorname{rank} \begin{bmatrix} E \\ C \end{bmatrix} = n$;

otherwise it is called uncontrollable or unobservable at λ, respectively. Note that in the above definition, one can also consider each individual Jordan block of the Weierstraß canonical form separately in case of multiple eigenvalues. This is possible by considering the corresponding eigenvectors. Let x and y be right and left eigenvectors corresponding to a Jordan block in J or N of (4). Then

$$y^H \begin{bmatrix} \lambda E - A & B \end{bmatrix} = \begin{bmatrix} 0 & y^H B \end{bmatrix}, \quad \begin{bmatrix} \lambda E - A \\ C \end{bmatrix} x = \begin{bmatrix} 0 \\ Cx \end{bmatrix}.$$

We say that such a Jordan block is controllable if $B^T y \neq 0$, and observable if $Cx \neq 0$, otherwise we call it uncontrollable or unobservable. We use this block-wise definition when talking about controllability and observability of eigenvalues.

Now we define the numbers

$$\begin{aligned} r_{\mathbb{C}}^{\mathrm{f}}(E, A, B, C) := \inf \big\{ &\|\Delta\|_2 : \\ &\Lambda_{\mathrm{f}}(E, A + B\Delta C) \cap \mathrm{i}\mathbb{R} \neq \emptyset \text{ with } \Delta \in \mathbb{C}^{m \times p} \big\}, \\ r_{\mathbb{C}}^{\infty}(E, A, B, C) := \inf \big\{ &\|\Delta\|_2 : \\ &\Lambda_{\infty}(E, A + B\Delta C) \text{ with } \Delta \in \mathbb{C}^{m \times p} \text{ has} \\ &\text{controllable and observable defective eigenvalues} \\ &\text{or } \lambda E - (A + B\Delta C) \text{ is a singular pencil} \big\}, \end{aligned}$$

where $\Lambda_\infty(E, A)$ is the set of infinite eigenvalues of $\lambda E - A$ (counting multiplicity). Then we define the *structured complex stability radius* by

$$\begin{aligned} r_{\mathbb{C}}(E, A, B, C) := \min \big\{ &r_{\mathbb{C}}^{\mathrm{f}}(E, A, B, C), \\ &r_{\mathbb{C}}^{\infty}(E, A, B, C) \big\}. \end{aligned}$$

The value of $r_{\mathbb{C}}^{\mathrm{f}}(E, A, B, C)$ is the size of the smallest structured perturbation that makes the system unstable. The interpretation of $r_{\mathbb{C}}^{\infty}(E, A, B, C)$ is more involved. *Defective* infinite eigenvalues do not directly make the system unstable. However, if there are controllable and observable ones we can construct an arbitrarily small structured perturbation such that the system will be unstable. This means that systems with controllable and observable defective infinite eigenvalues are on the "boundary to instability". If the perturbed matrix pencil becomes singular, then all complex numbers are eigenvalues and the system dynamics is not well-defined.

It is easy to see that $r_{\mathbb{C}}(A, B, C) = r_{\mathbb{C}}(I_n, A, B, C)$. Therefore, we stick to the more general concept in this paper.

III. Derivation of the Algorithm

In this section we derive the theoretical basis of our algorithm. Therefore, we compute both $r_{\mathbb{C}}^{\infty}(E, A, B, C)$ and $r_{\mathbb{C}}^{\mathrm{f}}(E, A, B, C)$ independent of each other. The computation of $r_{\mathbb{C}}^{\infty}(E, A, B, C)$ can be done by employing the following lemma [7]:

Lemma 1. *It holds that*

$$r_{\mathbb{C}}^{\infty}(E,A,B,C) = \begin{cases} 1/\lim\limits_{\omega\to\infty} \sigma_{\max}(G(\mathrm{i}\omega)) & \text{if } G \not\equiv 0, \\ \infty & \text{if } G \equiv 0. \end{cases}$$

For the computation of $r_{\mathbb{C}}^{\mathrm{f}}(E,A,B,C)$ we need some further mathematical terms. The *structured ε-pseudospectrum* of the pencil $\lambda E - A$ with respect to B and C is defined by

$$\Lambda_\varepsilon(E,A,B,C) = \{z \in \mathbb{C} : z \in \Lambda_{\mathrm{f}}(E, A + B\Delta C) \text{ for some } \Delta \in \mathbb{C}^{m\times p} \text{ with } \|\Delta\|_2 < \varepsilon\}.$$

Furthermore, we need the *structured ε-pseudospectral abscissa*, given by

$$\alpha_\varepsilon(E,A,B,C) := \max\{\operatorname{Re} z : z \in \Lambda_\varepsilon(E,A,B,C)\}.$$

From these definitions it is clear that $\alpha_{r_{\mathbb{C}}^{\mathrm{f}}}(E,A,B,C) = 0$. So the main idea of our algorithm is to find the (unique) root of the function $\alpha(\varepsilon) := \alpha_\varepsilon(E,A,B,C)$. To get an efficient algorithm we need to evaluate $\alpha(\varepsilon)$ for different values of ε in a cheap way.

A. Computation of the Structured ε-Pseudospectral Abscissa

In this subsection we derive a fast algorithm for computing $\alpha(\varepsilon)$. We assume that the pseudospectra of interest are not the whole complex plane, i.e., ε is smaller than the smallest norm of a structured perturbation that makes the pencil singular. The following fundamental result is a generalization of the corresponding one in [1] and is proven in [7].

Theorem 1. *Let $z \in \mathbb{C}\backslash\Lambda_{\mathrm{f}}(E,A)$ be given and $\varepsilon > 0$. Then the following statements are equivalent:*

(a) $z \in \Lambda_\varepsilon(E,A,B,C)$.
(b) $\sigma_{\max}(G(z)) > \varepsilon^{-1}$.
(c) There exist vectors $u \in \mathbb{C}^m$ and $v \in \mathbb{C}^p$ with $\|u\|_2 < 1$ and $\|v\|_2 < 1$ such that $z \in \Lambda_{\mathrm{f}}\left(E, A + \varepsilon B u v^H C\right)$.

From Theorem 1 we can conclude that

$$\Lambda_\varepsilon(E,A,B,C) = \left\{z \in \mathbb{C} : \sigma_{\max}\left(G(z)\right) > \varepsilon^{-1}\right\},$$

including the poles of $G(z)$. The boundary is given by

$$\partial\Lambda_\varepsilon(E,A,B,C) = \left\{z \in \mathbb{C} : \sigma_{\max}\left(G(z)\right) = \varepsilon^{-1}\right\}. \quad (5)$$

In other words, also the rightmost pseudoeigenvalue is arbitrarily close to the curve $\partial\Lambda_\varepsilon(E,A,B,C)$. Thus, our strategy consists of computing a sequence of suitable structured rank-1 perturbed pencils $\lambda E - (A + \varepsilon B u v^H C)$ such that one of the perturbed eigenvalues converges to the rightmost pseudoeigenvalue of $\lambda E - A$. A similar technique has already been successfully applied to compute the pseudospectral abscissa of a matrix, see [17]. We need the following result for the first order perturbation theory of matrix pencils.

Theorem 2. *[18] Let $\lambda E - A \in \mathbb{C}^{n\times n}$ be a given matrix pencil and let $x, y \in \mathbb{C}^n$ be right and left eigenvectors corresponding to a simple finite eigenvalue $\lambda = \frac{y^H A x}{y^H E x}$. Let $\lambda E - (A + tBuv^HC)$ be a perturbed matrix pencil with eigenvalue $\tilde{\lambda}$. Then it holds*

$$\tilde{\lambda} = \lambda + t\frac{y^H B u v^H C x}{y^H E x} + \mathcal{O}\left(t^2\right).$$

Furthermore, Theorem 2 directly yields

$$\left.\frac{\mathrm{d}\tilde{\lambda}(t)}{\mathrm{d}t}\right|_{t=0} = \frac{y^H B u v^H C x}{y^H E x}.$$

Now, we describe how such rank-1 perturbations can be constructed in an optimal way. Therefor, let $\lambda E - A$ with a simple finite eigenvalue λ and corresponding right and left eigenvectors x, y with $y^H E x > 0$ be given. Furthermore let $u \in \mathbb{C}^m$ and $v \in \mathbb{C}^p$ with $\|u\|_2 = \|v\|_2 = 1$ be given vectors. Then, it holds

$$\begin{aligned} \operatorname{Re}\left(\left.\frac{\mathrm{d}\tilde{\lambda}(t)}{\mathrm{d}t}\right|_{t=0}\right) &= \frac{\operatorname{Re}\left(y^H B u v^H C x\right)}{y^H E x} \\ &\le \frac{\left\|y^H B\right\|_2 \|Cx\|_2}{y^H E x}. \end{aligned} \quad (6)$$

Equality in (6) holds for $u = \frac{B^T y}{\|B^T y\|_2}$, $v = \frac{Cx}{\|Cx\|_2}$. Hence, local maximal growth in $\operatorname{Re}\left(\tilde{\lambda}(t)\right)$ as t increases from 0 is achieved for this choice of u and v. In this way we generate the initial perturbation. Next, we consider subsequent perturbations. For this, let $\lambda E - \hat{A} := \lambda E - (A + \varepsilon B\hat{u}\hat{v}^H C)$ with a simple finite eigenvalue $\hat{\lambda}$ and associated right and left eigenvectors $\hat{x}$, $\hat{y}$ with $\hat{y}^H E \hat{x} > 0$ be the perturbed matrix pencil. Let in addition the vectors $u \in \mathbb{C}^m$, $v \in \mathbb{C}^p$ with $\|u\|_2 = \|v\|_2 = 1$ be given. We consider the family of perturbations of the matrix pencil $\lambda E - \hat{A}$ of the form

$$\lambda E - \left(\hat{A} + tB\left(uv^H - \hat{u}\hat{v}^H\right)C\right),$$

which are structured ε-norm rank-1 perturbations of $\lambda E - A$ for $t = 0$ and $t = \varepsilon$. For the perturbed eigenvalue, for simplicity called again $\tilde{\lambda}$, we obtain

$$\begin{aligned} \operatorname{Re}\left(\left.\frac{\mathrm{d}\tilde{\lambda}(t)}{\mathrm{d}t}\right|_{t=0}\right) &= \frac{\operatorname{Re}\left(\hat{y}^H B\left(uv^H - \hat{u}\hat{v}^H\right)C\hat{x}\right)}{\hat{y}^H E \hat{x}} \\ &\le \frac{\left\|\hat{y}^H B\right\|_2 \|C\hat{x}\|_2 - \operatorname{Re}\left(\hat{y}^H B\hat{u}\hat{v}^H C\hat{x}\right)}{\hat{y}^H E\hat{x}}. \end{aligned} \quad (7)$$

Similarly as above, equality in (7) holds for $u = \frac{B^T\hat{y}}{\|B^T\hat{y}\|_2}$, $v = \frac{C\hat{x}}{\|C\hat{x}\|_2}$. Then the right-hand-side of (7) is larger or equal than zero. Thus, a nonnegative local maximal growth in the real part of $\tilde{\lambda}(t)$ is achieved as t increases from 0. So, the basic algorithm consists of successively choosing an eigenvalue and constructing the perturbations described above by using the corresponding eigenvectors. However, an important question is how to actually choose these eigenvalues. This will be discussed in the next subsection.

B. Choice of the Eigenvalues

Recall that we want to construct structured ε-norm rank-1 perturbations of the pencil $\lambda E - A$ such that one of the perturbed eigenvalues converges to the rightmost ε-pseudoeigenvalue. Intuitively, in each step one would choose the rightmost eigenvalue of the perturbed pencil to construct the next perturbation. However, that might not be a good choice. Note that the perturbability of an eigenvalue λ with right and left eigenvectors x and y highly depends on $\left\|B^T y\right\|_2$ and $\|Cx\|_2$. If these values are small, no large perturbation is possible. We remark that these values are also related to the classical definitions of controllability and observability of λ in a dynamical system of the form (1). Roughly speaking, the "larger" $\left\|B^T y\right\|_2$ is, the "larger" is the distance of the system to uncontrollability at λ. So, large values of $\left\|B^T y\right\|_2$ indicate a good controllability at λ. Similar considerations can also be made for observability.

Consequently, for our algorithm we look for eigenvalues that have both sufficiently large real part and a high controllability and observability. An algorithm which unites both concepts and can compute the desired eigenvalues is the *(subspace accelerated MIMO) dominant pole algorithm (SAMDP)*, introduced by Rommes and Martins [19]–[22]. The algorithm is actually designed to find the poles which have the highest influence on the frequency response of the transfer function $G(z)$. Assume that $\lambda E - A$ has only simple finite eigenvalues λ_k with left and right eigenvectors y_k and x_k, normalized such that $y_k^H E x_k = 1$. Then it holds

$$G(z) = \sum_{k=1}^{n} \frac{R_k}{z - \lambda_k} + R_\infty$$

with the *residues*

$$R_k = C x_k y_k^H B, \text{ and } R_\infty = \lim_{\omega \to \infty} G(\mathrm{i}\omega).$$

Thus, $\|R_k\|_2 = \|Cx_k\|_2 \left\|B^T y_k\right\|_2$ is a measure for simultaneous controllability and observability of λ_k. We observe that if λ_j is close to the imaginary axis and $\|R_j\|_2$ is large, then for $\omega \approx \mathrm{Im}(\lambda_j)$ it holds

$$G(\mathrm{i}\omega) \approx \frac{R_j}{-\mathrm{Re}(\lambda_j)} + \sum_{\substack{k=1 \\ k \neq j}}^{n} \frac{R_k}{\mathrm{i}\omega - \lambda_k} + R_\infty,$$

and therefore $\|G(\mathrm{i}\omega)\|_2$ is large, too. These considerations give the motivation for the following definition. We call an eigenvalue $\lambda_j \in \Lambda_{\mathrm{f}}(E, A)$ *dominant pole* of $G(z)$, if

$$\frac{\|R_k\|_2}{|\mathrm{Re}(\lambda_k)|} < \frac{\|R_j\|_2}{|\mathrm{Re}(\lambda_j)|}, \quad k = 1, \ldots, n, \quad k \neq j. \tag{8}$$

The most dominant poles can be determined by SAMDP and are essentially what we are looking for. However, we also deal with positive structured pseudospectral abscissae. By using the definition (8), the eigenvalues tend to loose dominance as soon as they have crossed the imaginary axis into the right half-plane. Then, in subsequent iterations, eigenvalues in the left half-plane tend to be determined as most dominant. This is of course an undesired behavior since this could lead to convergence problems when the rightmost pseudoeigenvalue is "far" in the right half-plane. Therefore, we use an alternative dominance measure which does not have this drawback. We call an eigenvalue $\lambda_j \in \Lambda_{\mathrm{f}}(E, A)$ *exponentially dominant pole* of $G(z)$, if

$$\|R_k\|_2 \exp(\beta\,\mathrm{Re}(\lambda_k)) < \|R_j\|_2 \exp(\beta\,\mathrm{Re}(\lambda_j)), \quad k = 1, \ldots, n, \quad k \neq j. \tag{9}$$

The parameter β is a weighting factor which defines the trade-off between the influence of the residue and real part of the eigenvalues. In our numerical experiments it turned out that $\beta = 100$ is a good choice for many examples (high weight on the real part). Since SAMDP delivers the poles which have the highest influence on the frequency response of a system and due to the relation (3), we can determine very good initial estimates for $r_{\mathbb{C}}^{\mathrm{f}}(E, A, B, C)$. We compute some of the dominant poles λ_k, $k = 1, \ldots, \ell$, and determine an estimate $r_{\mathbb{C}}^{\mathrm{est}}(E, A, B, C)$ as

$$r_{\mathbb{C}}^{\mathrm{est}}(E, A, B, C) = 1 / \max_{1 \leq k \leq \ell} \sigma_{\max}\left(G\left(\mathrm{i}\omega_k\right)\right) \tag{10}$$

with $\omega_k = \mathrm{Im}(\lambda_k)$, $k = 1, \ldots, \ell$.

C. Algorithmic Details

In this subsection we present some pseudocode of the algorithms that have been derived. Algorithm 1 summarizes the procedure for the computation of the structured ε-pseudospectral abscissa. In our implementation of the algorithm, we use estimates of the eigenvectors x and y that are used to construct the optimal perturbation. This is possible since we can take the eigenvectors returned by the previous $\alpha(\varepsilon)$-evaluations of the root-finding algorithm. This accelerates the computation drastically since in the final root-finding steps the eigenvectors also converge to x and y.

We mention the drawback that the algorithm not necessarily converges to the globally rightmost value on the boundary of the ε-pseudospectrum $\partial\Lambda_\varepsilon$ in (5). Mostly it does but in some rare situations the algorithm converges only to a local maximizer. This especially happens in the first iteration of the root-finding algorithm when no good estimates of the optimal eigenvectors are available. Therefore, sometimes one has to try several dominant poles to find the global maximizer in the beginning. To find the root of $\alpha(\varepsilon)$, we use a simple secant method which has a superlinear convergence with convergence rate $\frac{1+\sqrt{5}}{2} \approx 1.618$. A comparison of different standard root-finding methods and standard software to compute structured complex stability radii is presented in [23]. As the numerical examples show, we only need very few (3-5) iterations of this method in most cases to find the root with a sufficient accuracy. When computing the dominant poles of the system we already obtain a very good estimate of $r_{\mathbb{C}}^{\mathrm{est}}(E, A, B, C)$ as in (10) which is used as an initial value for the root-finder.

IV. Numerical Results

In this section be present some numerical results of our implementations. The tests have been performed on a 2.6.32-23-generic-pae Ubuntu machine with Intel® Core™2 Duo CPU

Algorithm 1 Computation of the structured pseudospectral abscissa

Input: System $\Sigma = (\lambda E - A, B, C)$, perturbation level ε, tolerance on relative change τ.

Output: $\alpha_\varepsilon(E, A, B, C)$.

1: Compute the exponentially dominant pole λ_0 of $(\lambda E - A, B, C)$ with left and right eigenvectors y_0 and x_0.
2: Compute the perturbation $\hat{A} = A + \varepsilon \frac{BB^T y_0 x_0^H C^T C}{\|B^T y_0\|_2 \|C x_0\|_2}$.
3: **for** $j = 1, 2, \ldots$ **do**
4: Compute the exponentially dominant pole λ_j of $(\lambda E - \hat{A}, B, C)$ with left and right eigenvectors y_j and x_j.
5: **if** $|\mathrm{Re}\,(\lambda_j) - \mathrm{Re}\,(\lambda_{j-1})| < \tau\,|\mathrm{Re}\,(\lambda_j)|$ **then**
6: Set $k = j$.
7: Break.
8: **end if**
9: Compute the perturbation $\hat{A} = A + \varepsilon \frac{BB^T y_j x_j^H C^T C}{\|B^T y_j\|_2 \|C x_j\|_2}$.
10: **end for**
11: $\alpha_\varepsilon(E, A, B, C) = \mathrm{Re}\,(\lambda_k)$.

with 3.00GHz and 2GB RAM. The algorithms have been implemented and tested in MATLAB 7.11.0.584 (R2010b). To compute the (exponentially) dominant poles we use a (modified) MATLAB implementation by Rommes[1]. The data for the numerical examples was taken from [21], [24]–[26], and most of it can also be downloaded from Rommes' website. All systems, except the PISection examples (# 2–17 in Table I) have stable matrix pencils $\lambda E - A$. The PISection pencils have zero eigenvalues, therefore we slightly shift the spectrum into the left half-plane by using the modified pencils $\lambda E - (A - 10^{-8}E)$. For the dominance measure in (9) we use $\beta = 100$. The tolerance on the relative change of the iterates in Algorithm 1 is set to $\tau = 10^{-3}$. We also abort the iteration when the iterates are smaller than $1000\mathbf{u}$, where $\mathbf{u}$ denotes the machine precision, or when the iterates start to cycle, which typically happens when they are approaching zero. In the root finder we abort this iteration when the relative change between two subsequent iterates is below 10^{-6}. To obtain $r_{\mathbb{C}}^{\mathrm{est}}(E, A, B, C)$, we compute 20 dominant poles using the original version of SAMDP. For every further outer iteration we compute only 5 dominant poles using the modified algorithm with the exponential dominance measure.

In Table I we summarize the results of 28 numerical tests. The first 9 examples are standard or generalized state space systems whereas the other 19 ones are descriptor systems. By n_{outer} we denote the number of outer iterations, i.e., the number of $\alpha(\varepsilon)$-evaluations to find the root. By n_{inner} we refer to the total number of inner iterations, i.e., the total number of steps needed by Algorithm 1.

For all tests, the correct value of $r_{\mathbb{C}}(E, A, B, C)$ is found. In 25 tests, the first outer iteration returns a positive value. However, for 3 tests (`M10PI_n1`, `M10PI_n`, `bips07_1693`), a negative value is returned and therefore we have to try more dominant poles (one for each `M10PI_n1`, `M10PI_n` and three for `bips07_1693`) to converge to the correct initial value. Note also that the convergence speed of the algorithm highly depends on the properties of the examples, e.g., for `bips07_1998`, the convergence is significantly slower than for the rest in the `bips07` group, although they are comparable in size. Finally, in Figure 1 the transfer function for the `M20PI_n` example is evaluated on the imaginary axis including the computed $\mathcal{H}_\infty$-norm (see (3)). Our algorithm computes the correct value, even though there are lots of close-by peaks of similar height. Unfortunately, there are also examples that our algorithm cannot solve. These include systems whose transfer functions only have real poles or those where the structured pseudospectra have lots of very close locally rightmost points on the boundary, see [7] for more details.

[1] `http://sites.google.com/site/rommes/software`

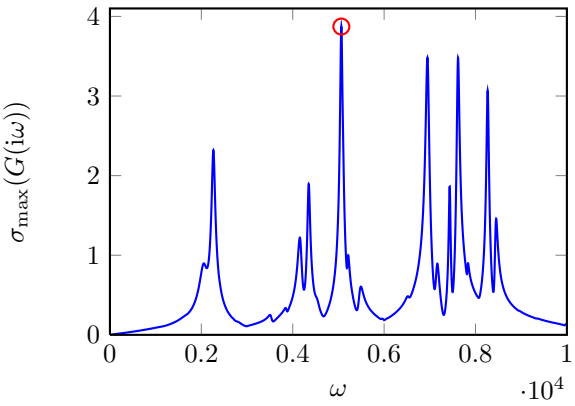

Fig. 1. Transfer function plot for the `M20PI_n` test example with computed $\mathcal{H}_\infty$-norm (red circle)

V. Conclusions and Future Research Perspectives

In this paper we have introduced a new iterative scheme for computing the structured complex stability radius of a matrix or a pencil, based on the method introduced in [17]. The algorithm computes a sequence of structured pseudospectral abscissae. This is done by computing an optimal rank-1 perturbation of the system such that one of the eigenvalues of the perturbed matrix or pencil converges to the rightmost pseudoeigenvalue. This algorithm can be seen as a basis to solve certain related problems. Open questions concern the computation of unstructured stability radii. Another interesting problem is the computation of *real* stability radii where the perturbation matrix is assumed to be real. Finally, we mention the passivity radius of a dynamical system which might also be put into this framework.

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TABLE I

NUMERICAL RESULTS FOR 28 TEST EXAMPLES

#	example	n	m	p	$r_{\mathbb{C}}(E,A,B,C)$	$\alpha_{r_{\mathbb{C}}}(E,A,B,C)$	n_{outer}	n_{inner}	time in s
1	CDplayer	120	2	2	4.31068e-07	-3.0943e-15	3	9	2.92
2	S10PI_n1	528	1	1	2.51602e-01	1.8451e-12	4	14	3.22
3	S20PI_n1	1028	1	1	2.90430e-01	6.3894e-12	4	15	5.16
4	S40PI_n1	2028	1	1	2.98747e-01	1.3026e-10	4	18	9.43
5	S80PI_n1	4028	1	1	2.96722e-01	4.1804e-13	4	14	15.74
6	M10PI_n1	528	3	3	2.46511e-01	3.9342e-13	5	18	4.64
7	M20PI_n1	1028	3	3	2.58224e-01	1.0025e-13	4	15	6.19
8	M40PI_n1	2028	3	3	2.61940e-01	8.8036e-13	4	14	8.71
9	M80PI_n1	4028	3	3	2.62898e-01	3.0728e-13	4	15	16.69
10	S10PI_n	682	1	1	2.51602e-01	0	4	14	3.81
11	S20PI_n	1182	1	1	2.90430e-01	9.2875e-12	4	15	5.54
12	S40PI_n	2182	1	1	2.98747e-01	1.2982e-10	4	18	10.26
13	S80PI_n	4182	1	1	2.96722e-01	3.9808e-13	4	14	16.70
14	M10PI_n	682	3	3	2.46511e-01	0	5	18	5.82
15	M20PI_n	1182	3	3	2.58224e-01	4.0637e-13	4	16	7.22
16	M40PI_n	2182	3	3	2.61940e-01	2.2161e-13	4	15	10.05
17	M80PI_n	4182	3	3	2.62898e-01	0	4	15	16.92
18	bips98_606	7135	4	4	4.95158e-03	2.3928e-12	4	16	45.86
19	bips98_1142	9735	4	4	6.62334e-03	5.5157e-13	5	34	105.98
20	bips98_1450	11305	4	4	5.06614e-03	5.0261e-13	5	26	98.96
21	bips07_1693	13275	4	4	4.89792e-03	1.1406e-13	8	49	185.36
22	bips07_1998	15066	4	4	5.07449e-03	1.1421e-13	4	28	138.02
23	bips07_2476	16861	4	4	5.27485e-03	1.2511e-11	4	57	282.77
24	bips07_3078	21128	4	4	4.77453e-03	7.1606e-14	5	19	134.46
25	xingo_afonso_itaipu	13250	1	1	2.46546e-01	-3.0780e-14	3	9	47.82
26	mimo8x8_system	13309	8	8	1.87164e+01	2.3477e-13	4	24	109.42
27	mimo28x28_system	13251	28	28	8.43042e+00	-2.1586e-14	3	11	106.23
28	mimo46x46_system	13250	46	46	4.86309e-03	9.3221e-14	3	11	166.15

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