The Dissipation Inequality for Differential-Algebraic Systems

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We discuss the infinite time-horizon linear-quadratic optimal control problem for differential-algebraic equations. In contrast to previous approaches we do not impose any assumptions on the system except for impulse controllability. In particular, we show that the optimal control problem is feasible if and only if a dissipation inequality has a solution. Moreover, we discuss conditions under which the problem has a minimizing (instead of only an infimizing) solution and furthermore, when this minimizing solution is unique.

Notation

$\mathbb{K}$	ether the set $\mathbb{R}$ of real numbers or the set $\mathbb{C}$ of complex numbers, respectively
$\mathbb{C}^+$	the open sets of complex numbers with positive real part
$\overline{S}$	closure of the set S
$\mathbb{K}[s]$	the rings of polynomials with coefficients in $\mathbb{K}$
$\mathbb{K}(s)$	the fields of rational functions with coefficients in $\mathbb{K}$
$\mathcal{R}^{m,n}$	the set of $m \times n$ matrices with entries in a ring $\mathcal{R}$
$\mathrm{Gl}_n(\mathcal{R})$	the groups of invertible $n \times n$ matrices with entries in $\mathcal{R}$
A^*	conjugate transpose of a matrix A
$\mathrm{im}\, A$, $\ker A$	image and kernel of a matrix $A \in \mathbb{C}^{n,m}$, respectively
$\mathrm{rank}\, A$	rank of a matrix $A \in \mathbb{C}^{n\times m}$, respectively
$A =_{\mathcal{V}} (\geq_{\mathcal{V}})\, B$	for Hermitian $A,\ B \in \mathbb{K}^{n,n}$ and a subspace $\mathcal{V} \in \mathbb{K}^n$, it holds $x^*(A-B)x = (\geq)\, 0$ for all $x \in \mathcal{V}\backslash\{0\}$
$\mathcal{L}^2(\mathcal{I};\mathbb{R}^n)$	the set of measurable and square integrable functions $f : \mathcal{I} \to \mathbb{R}^n$ on the set $\mathcal{I} \subseteq \mathbb{R}$
$\mathcal{L}^2_{\mathrm{loc}}(\mathcal{I};\mathbb{R}^n)$	the set of measurable and locally square integrable functions $f : \mathcal{I} \to \mathbb{R}^n$ on the set $\mathcal{I} \subseteq \mathbb{R}$

1 Introduction

In this article we consider linear differential-algebraic control systems of the form

$$E\dot{x}(t) = Ax(t) + Bu(t), \tag{1}$$

where $E, A \in \mathbb{K}^{n,n}$, $B \in \mathbb{K}^{n,m}$ are such that the pencil $sE - A \in \mathbb{K}[s]^{n,n}$ is *regular*, i.e., there exists a $\lambda \in \mathbb{C}$ such that $\det(\lambda E - A) \neq 0$; the set of these systems is denoted by $\Sigma_{n,m}$ and we write $[E, A, B] \in \Sigma_{n,m}$.

The function $u : \mathbb{R} \to \mathbb{K}^m$ is called *input* of the system; we call $x(t)$ the *state* of $[E, A, B]$ at time $t \in \mathbb{R}$. A trajectory $(x, u) : \mathbb{R} \to \mathbb{K}^n \times \mathbb{K}^m$ is said to be a *solution* of (1) if and only if it belongs to the *behavior* of (1):

$$\mathfrak{B}_{[E,A,B]} := \big\{(x,u) \in \mathcal{L}^2_{\mathrm{loc}}(\mathbb{R};\mathbb{K}^n) \times \mathcal{L}^2_{\mathrm{loc}}(\mathbb{R};\mathbb{K}^m) : E\dot{x} \in \mathcal{L}^2_{\mathrm{loc}}(\mathbb{R};\mathbb{K}^n) \text{ and } (x,u) \text{ solves } E\dot{x}(\cdot) = Ax(\cdot) + Bu(\cdot)\big\}.$$

Moreover, for $[E, A, B] \in \Sigma_{n,m}$ and $x_0 \in \mathbb{K}^n$, we define the *behavior with initial differential variable* Ex_0 by

$$\mathfrak{B}_{[E,A,B]}(x_0) = \big\{(x,u) \in \mathfrak{B}_{[E,A,B]} : Ex(0) = Ex_0\big\}.$$

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Consider the quadratic functional

$$\mathcal{J}(x,u,t_0,t_1) := \int_{t_0}^{t_1} \begin{pmatrix} x(\tau) \\ u(\tau) \end{pmatrix}^* \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{pmatrix} x(\tau) \\ u(\tau) \end{pmatrix} \mathrm{d}\tau.$$

Similarly as in [1] for ODEs, we study the infinite time horizon linear-quadratic optimal control problem

Minimize

$$\mathcal{J}(x,u,0,\infty)$$

subject to $(x,u) \in \mathfrak{B}_{[E,A,B]}(x_0)$ with $\lim_{t\to\infty} Ex(t) = 0$.

Moreover, we define the *optimal value function* $V^+ : \operatorname{im} E \to \mathbb{R}$ by

$$V^+(Ex_0) = \inf \left\{ \mathcal{J}(x,u,0,\infty) : (x,u) \in \mathfrak{B}_{[E,A,B]}(x_0) \text{ and } \lim_{t\to\infty} Ex(t) = 0 \right\}. \tag{2}$$

The questions we are going to answer in this article are the following:

1. Is the optimal control problem *feasible*, i.e., does

$$-\infty < V^+(Ex_0) < \infty \tag{3}$$

   hold for all initial differential variables Ex_0?

2. Under which conditions is the infimum a *minimum*?

3. When is the minimizer *unique*?

2 Main Results

First, we need some controllability and stabilizability notions for differential-algebraic systems.

Definition 2.1 [2] The system $[E,A,B] \in \Sigma_{n,m}$ is called

a) *impulse controllable* $:\Leftrightarrow\ \forall x_0 \in \mathbb{R}^n:\ \mathfrak{B}_{[E,A,B]}(x_0) \neq \emptyset$;

b) *strongly stabilizable* $:\Leftrightarrow\ \forall x_0 \in \mathbb{K}^n\ \exists (x,u) \in \mathfrak{B}_{[E,A,B]}(x_0)$ with $\lim_{t\to\infty} Ex(t) = 0$.

In order to check the above properties, there exist algebraic characterizations in terms of the matrices E, A, and B, summarized in the next proposition.

Proposition 2.2 *Let* $[E,A,B] \in \Sigma_{n,m}$ *be given with* $r = \operatorname{rank}(E)$. *Let* $S_\infty \in \mathbb{K}^{n,n-r}$ *be a matrix with* $\operatorname{im} S_\infty = \ker E$. *Then the system* $[E,A,B]$ *is*

a) impulse controllable $:\Leftrightarrow\ \operatorname{rank}\begin{bmatrix} E & AS_\infty & B \end{bmatrix} = n$;

b) strongly stabilizable $:\Leftrightarrow\ \operatorname{rank}\begin{bmatrix} E & AS_\infty & B \end{bmatrix} = n$ *and* $\operatorname{rank}\begin{bmatrix} \lambda E - A & B \end{bmatrix} = n$ *for all* $\lambda \in \overline{\mathbb{C}^+}$.

From the definition above it is immediately clear that $V^+(Ex_0) < \infty$ for all $x_0 \in \mathbb{K}^n$ if and only if $[E,A,B]$ is strongly stabilizable.

To show that $-\infty < V^+(Ex_0)$ holds for all $x_0 \in \mathbb{K}^n$ we consider a class of *dissipation functions* $V : \operatorname{im} E \to \mathbb{R}$. That is, V is continuous, $V(0) = 0$, and it satisfies the *dissipation inequality*

$$\mathcal{J}(x,u,t_0,t_1) + V(Ex(t_1)) \geq V(Ex(t_0)) \qquad \forall\,(x,u) \in \mathfrak{B}_{[E,A,B]},\ t_0,t_1 \in \mathbb{R} \text{ with } t_0 \leq t_1.$$

Assume moreover that V is differentiable. Then the inequality is equivalent to

$$\begin{pmatrix} x(t) \\ u(t) \end{pmatrix}^* \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{pmatrix} x(t) \\ u(t) \end{pmatrix} \geq -V'(Ex(t))E\dot{x}(t) = -V'(Ex(t))(Ax(t) + Bu(t))$$
$$\forall\ (x,u) \in \mathfrak{B}_{[E,A,B]} \text{ and almost all } t \in \mathbb{R}, \tag{4}$$

where $V'(Ex(t)) \in \mathbb{K}^{1,n}$ is the Jacobian of V in $Ex(t)$. For quadratic dissipation functions, we can make the ansatz

$$V(Ex_0) = x_0^* X^* E x_0,$$

where $X \in \mathbb{K}^{n,n}$ is a matrix with $X^*E = E^*X$ (the latter property makes $V(Ex_0)$ well-defined). Using that

$$(V'(Ex_0))z = 2x_0^* X^* z \qquad \forall\, z \in \mathbb{K}^n,$$

the dissipation inequality (4) is now equivalent to the property that for all $(x,u) \in \mathfrak{B}_{[E,A,B]}$ holds

$$\begin{pmatrix} x(t) \\ u(t) \end{pmatrix}^* \begin{bmatrix} A^*X + X^*A + Q & X^*B + S \\ B^*X + S^* & R \end{bmatrix} \begin{pmatrix} x(t) \\ u(t) \end{pmatrix} \geq 0 \text{ for almost all } t \in \mathbb{R}.$$

This relation is in fact equivalent to a linear matrix matrix equality on the *system space*

$$\mathcal{V} := \left\{ \begin{pmatrix} x \\ u \end{pmatrix} \in \mathbb{K}^{n+m} : Ax + Bu \in \operatorname{im} E \right\} \subseteq \mathbb{K}^{n+m}.$$

Note that with $(x,u) \in \mathfrak{B}_{[E,A,B]}$ we have $\begin{pmatrix} x(t) \\ u(t) \end{pmatrix} \in \mathcal{V}$ for all $t \geq 0$. Therefore, we obtain

$$\begin{bmatrix} A^*X + X^*A + Q & X^*B + S \\ B^*X + S^* & R \end{bmatrix} \geq_{\mathcal{V}} 0,$$

or equivalently

$$\begin{bmatrix} A^*X + X^*A + Q & X^*B + S \\ B^*X + S^* & R \end{bmatrix} =_{\mathcal{V}} \begin{bmatrix} K^* \\ L^* \end{bmatrix} \begin{bmatrix} K & L \end{bmatrix} \tag{5}$$

for some $K \in \mathbb{K}^{p,n}$ and $L \in \mathbb{K}^{p,m}$. Now assume that $(x,u) \in \mathfrak{B}_{[E,A,B]}(x_0)$ and $\lim_{t\to\infty} Ex(t) = 0$. Then we obtain

$$\begin{aligned}
x_0^* X^* E x_0 &= -\int_0^\infty \frac{\mathrm{d}}{\mathrm{d}\tau} x(\tau)^* X^* E x(\tau) \mathrm{d}\tau \\
&= -\int_0^\infty \dot{x}(\tau)^* E^* X x(\tau) + x(\tau)^* X^* E \dot{x}(\tau) \mathrm{d}\tau \\
&= -\int_0^\infty (Ax(\tau) + Bu(\tau))^* X x(\tau) + x(\tau)^* X^* (Ax(\tau) + Bu(\tau)) \mathrm{d}\tau \\
&= \int_0^\infty \begin{pmatrix} x(\tau) \\ u(\tau) \end{pmatrix}^* \begin{bmatrix} -A^*X - X^*A & -X^*B \\ -B^*X & 0 \end{bmatrix} \begin{pmatrix} x(\tau) \\ u(\tau) \end{pmatrix} \mathrm{d}\tau \\
&= \int_0^\infty \begin{pmatrix} x(\tau) \\ u(\tau) \end{pmatrix}^* \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{pmatrix} x(\tau) \\ u(\tau) \end{pmatrix} \mathrm{d}\tau - \int_0^\infty \begin{pmatrix} x(\tau) \\ u(\tau) \end{pmatrix}^* \begin{bmatrix} K^*K & K^*L \\ L^*K & L^*L \end{bmatrix} \begin{pmatrix} x(\tau) \\ u(\tau) \end{pmatrix} \mathrm{d}\tau \\
&= \mathcal{J}(x,u,0,\infty) - \int_0^\infty \|Kx(\tau) + Lu(\tau)\|_2^2 \,\mathrm{d}\tau,
\end{aligned}$$

and hence

$$x_0^* X^* E x_0 + \|Kx + Lu\|^2_{\mathcal{L}^2([0,\infty);\mathbb{R}^p)} = \mathcal{J}(x,u,0,\infty).$$

This gives rise to

$$x_0^* X^* E x_0 \leq \mathcal{J}(x,u,0,\infty) \quad \forall\, (x,u) \in \mathfrak{B}_{[E,A,B]}(x_0) \text{ with } \lim_{t\to\infty} Ex(t) = 0.$$

Therefore, if the dissipation inequality has a solution we obtain

$$-\infty < x_0^* X^* E x_0 \leq V^+(Ex_0) \quad \forall x_0 \in \mathbb{K}^n. \tag{6}$$

In fact, also the converse is true. Namely, if (3) holds for all $x_0 \in \mathbb{K}^n$, then the system is impulse controllable and (5) has a stabilizing solution, as defined below. The proof of this statement, however, is rather involved, see [3].

In order to obtain the characterize the solution of the optimal control problem we consider again the equation (5).

Definition 2.3 Assume that the triple $(X^+, K^+, L^+) \in \mathbb{K}^{n,n} \times \mathbb{K}^{n,p} \times \mathbb{K}^{m,p}$ solves (5). Then, (X^+, K^+, L^+) is called *stabilizing*, if

$$\operatorname{rank} \begin{bmatrix} -\lambda E + A & B \\ K^+ & L^+ \end{bmatrix} = n + p$$

for all $\lambda \in \mathbb{C}^+$.

The following has been shown in [3].

Theorem 2.4 *If*

$$\operatorname{rank}\begin{bmatrix} -\lambda E + A & B \\ K^+ & L^+ \end{bmatrix} = n + p$$

for all $\lambda \in \mathbb{C}^+$*, then there exists a sequence* $((x_k, u_k))_{k\in\mathbb{N}}$ *with* $(x_k, u_k) \in \mathfrak{B}_{[E,A,B]}(x_0)$ *and* $\lim_{t\to\infty} Ex_k(t) = 0$ *such that*

$$\lim_{k\to\infty} \left\| K^+ x_k + L^+ u_k \right\|_{\mathcal{L}^2([0,\infty);\mathbb{R}^p)} = 0.$$

Therefore, the optimal control costs are given by

$$V^+(Ex_0) = x_0^*(X^+)^* E x_0.$$

Furthermore. we can show the following results concerning existence and uniqueness of a minimizer.

Theorem 2.5 *a) The infimum in* (2) *is a minimizer if and only if*

$$\operatorname{rank}\begin{bmatrix} -\lambda E + A & B \\ K^+ & L^+ \end{bmatrix} = n + p$$

for all $\lambda \in \overline{\mathbb{C}^+}$ *and, moreover, the index of the pencil* $\begin{bmatrix} -sE + A & B \\ K^+ & L^+ \end{bmatrix}$ *is at most one.*

b) The minimizer is furthermore unique if and only if

$$\begin{bmatrix} -\lambda E + A & B \\ K^+ & L^+ \end{bmatrix} \in \mathrm{Gl}_{n+m}(\mathbb{C})$$

for all $\lambda \in \overline{\mathbb{C}^+}$.

3 Summary and Outlook

In this article we have discussed the linear-quadratic optimal control problem for differential-algebraic equations. As a main result we have shown that under the weak condition of impulse controllability, the optimal control problem is feasible if and only if a dissipation inequality has a stabilizing solution. Moreover, we have shown conditions for the existence and the uniqueness of a minimizer of the optimal control problem in terms of a closed-loop matrix pencil.

It remains to actually construct a solution of the optimal control problem, or equivalently to the equation (5). This construction is related to the deflating subspaces of the even matrix pencil

$$s\mathcal{E} - \mathcal{A} = \begin{bmatrix} 0 & -sE + A & B \\ sE^* + A^* & Q & S \\ B^* & S^* & R \end{bmatrix} \in \mathbb{K}[s]^{2n+m,2n+m}$$

is studied in [3, 4].

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