Solution of Descriptor Lur’e Equations via Even Matrix Pencils

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Descriptor Lur’e equations are an important tool for the solution of linear-quadratic optimal control problems for differential-algebraic systems. In this article we discuss how one can construct all solutions of these equations by the deflating subspaces of associated even matrix pencils.

1 Introduction

Assume that we have given a descriptor system

\[ Ex'(t) = Ax(t) + Bu(t), \]  

where \( E, A \in \mathbb{K}^{n,n}, B \in \mathbb{K}^{n,m} \) are such that the pencil \( sE - A \in \mathbb{K}[s]^{n,n} \) is regular, i.e., there exists a \( \lambda \in \mathbb{C} \) such that \( \det(\lambda E - A) \neq 0 \). For \( Q = Q^* \in \mathbb{K}^{n,n}, S = S^* \in \mathbb{K}^{n,m}, R = R^* \in \mathbb{K}^{m,m} \) we consider descriptor Lur’e equations of the form

\[
\begin{align*}
A^* X + X^* A + Q &= K^* K + H^* \Sigma H, \\
X^* B + S &= K^* L + H^* \Sigma J, \\
E^* X &= X^* E, \\
R &= L^* L + J^* \Sigma J,
\end{align*}
\]

which play an important role for optimal control problems of systems of the form (1), see \cite{1}. Let \( r = \text{rank} E \) and define the system space of (1) by

\[ \mathcal{V} := \left\{ \left( \begin{array}{c} x \\ u \end{array} \right) \in \mathbb{K}^{n+m} : Ax + Bu \in \text{im} E \right\} \subseteq \mathbb{K}^{n+m}. \]

Then a sextuple \( (X, K, L, H, J, \Sigma) \) consisting of some

a) matrices \( X \in \mathbb{K}^{n,n}, K \in \mathbb{K}^{p,n}, L \in \mathbb{K}^{p,m} \) for some \( p \in \mathbb{N}_0 \) with \( \ker \begin{bmatrix} K \\ L \end{bmatrix} = \mathcal{V}^\perp; \)
b) matrices \( H \in \mathbb{K}^{n-r,n}, J \in \mathbb{K}^{n-r,m} \) with \( \ker \begin{bmatrix} H \\ J \end{bmatrix} = \mathcal{V}; \)
c) a signature matrix \( \Sigma \in \mathbb{R}^{n-r,n-r} \) (that is, \( \Sigma = \text{diag}(-I_{p_1}, I_{p_2}) \) for some \( p_1, p_2 \in \mathbb{N}_0 \))

is called solution of the descriptor Lur’e equation (2), if it fulfills (2) and \( \text{rank}_{\mathbb{C}(s)} \left[ \begin{array}{c}
-sE + A \\
K \\
B \\
S \\
\end{array} \right] = n + p. \)

In this article we study the relationship of (2) to even matrix pencils of the form

\[
\begin{bmatrix}
0 & -sE + A & B^* \\
-sE^* + A^* & Q & S \\
B & S^* & R
\end{bmatrix},
\]

namely we will show how to construct solutions of (2) via deflating subspaces of \( sE - A. \)

2 Main Result

Definition 2.1 a) A matrix \( Y \) is called a basis matrix for a subspace \( \mathcal{Y} \subseteq \mathbb{K}^N \) if it has full column rank and \( \text{im} \ Y = \mathcal{Y}. \)
b) A subspace \( \mathcal{Y} \subseteq \mathbb{C}^N \) is called deflating subspace for the pencil \( sE - A \in \mathbb{C}[s]^{N,N} \) if, for a basis matrix \( Y \in \mathbb{C}^{N,k} \) of \( \mathcal{Y} \), there exists some \( l \in \mathbb{N}_0 \), a matrix \( Z \in \mathbb{C}^{N,l} \) and a pencil \( s\mathcal{E} - \mathcal{A} \in \mathbb{C}[s]^{l,k} \) with \( \text{rank}_{\mathbb{C}(s)} (s\mathcal{E} - \mathcal{A}) = l \) such that

\[ (sE - A)Y = Z(s\mathcal{E} - \mathcal{A}). \]
Theorem 2.2 Let the system \([E, A, B]\), the even matrix pencil \(sE - A\) as in (4) and Popov function

\[
\Phi(s) = \begin{bmatrix} (-sE - A)^{-1}B \end{bmatrix}^* \begin{bmatrix} Q & S \end{bmatrix} \begin{bmatrix} (sE - A)^{-1}B \\ I_m \end{bmatrix}
\]

be given. Consider the following statements:

1) The descriptor Lur’ e equation (2) is solvable.

2) It holds \(\Phi(\pm \infty) \geq 0\) for all \(\infty \in \sigma(E, A)\), and there exist \(Y_1, Y_2 \in K^{n+n+m}, Y_3 \in K^{m,n+m}, Z_1, Z_2 \in K^{n,n+p}, Z_3 \in K^{n,n+p}\) such that for

\[
Y = \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix}, \quad Z = \begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \end{bmatrix}, \quad (sE - A)Y = Z(sE - A).
\]

Then the following holds true:

i) the space \(\text{im} \ Y\) is \(n + m\)-dimensional and \(E\)-neutral;

ii) \(V \subset \text{im} \begin{bmatrix} Y_2 \\ Y_3 \end{bmatrix}\) with \(V\) as in (3);

iii) \(\text{rank} \ EY_2 = r\);

iv) there exist \(\tilde{E}, \tilde{A} \in K^{n+p,n+m}\), such that

\[
(sE - A)Y = Z(sE - \tilde{A}).
\]

Theorem 2.3 Assume that \([E, A, B]\) is impulse controllable and let the descriptor Lur’ e equation (2) with associated even matrix pencil \(sE - A\) as in (4) be solvable. Let an \(n + m\)-dimensional \(E\)-neutral space \(Y\) with \(Y\) as in (5) be given such that

\[
X = Y^* Y^{-}. \quad (7)
\]

It remains to check under which conditions on \(sE - \tilde{A}\) and \(Y\), the deflating subspace \(\text{im} \ Y\) indeed defines a solution of the descriptor Lur’ e equation, in particular we check whether iii) is fulfilled. This is summarized in the next theorem.

Theorem 2.3 Assume that \([E, A, B]\) is impulse controllable and let the descriptor Lur’ e equation (2) with associated even matrix pencil \(sE - A\) as in (4) be solvable. Let an \(n + m\)-dimensional \(E\)-neutral space \(Y\) with \(Y\) as in (5) be given such that

\[
(sE - A)Y = Z(sE - \tilde{A}). \quad (6)
\]

Then the following implications hold:

a) Statement 1) implies 2).

b) If \([E, A, B]\) is impulse controllable, then 2) also implies 1).

In the case where 2) holds true, a solution of the descriptor Lur’ e equations exists with \(\text{rank}_{K(s)} \Phi(s) = p\) and there exists a subspace \(\text{im} \ Y\) with \(\text{rank} Y_2 = n\) and a matrix \(Y_x^* \in K^{n+m,n}\) with \(Y_x Y_x^{-} = I_n\) such that

\[
X = Y^* Y^{-}. \quad (7)
\]

3 Summary and Outlook

In this paper we have discussed relations between the solutions of a descriptor Lur’ e equation and the deflating subspaces of an associated even matrix pencil. In particular, we have given equivalent conditions for the solvability of the descriptor Lur’ e equation and the existence of a deflating subspace. Moreover, we have given a sufficient condition on the subspace that allows to construct a solution from it. In [2], we also give more details on how to construct the solution by a transformation of the even pencil to even Kronecker canonical form [3] and investigate the relation to the optimal control problem. Moreover, the numerical solution of descriptor Lur’ e equations is currently under investigation.

References